An Equivalent of the Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s)>1$

In this article we prove an equivalent of the Riemann Hypothesis.

2 Proof

The non trivial zeroes of the Riemann Zeta function lie in the

critical strip

$$0 < Re(s) < 1$$

Riemann's Xi function is defined as [4, p.1],

$\epsilon(s) \ = \ s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2$

$\epsilon(s)$ *satisfies the functional equation* $\epsilon(s) = \epsilon(1-s)$

The zero of $(s-1)$ *cancels the pole of* $\zeta(s)$, *and the real zeroes of* $s\,\zeta(s)$ *are cancelled by the simple poles of* $\Gamma(s/2)$ *which never vanishes.*

Thus, $\epsilon(s)$ *is an entire function whose zeroes are the non trivial zeroes of* $\zeta(s)([3, p.2])$

The Riemann Hypothesis states that all the non trivial zeroes of the Riemann Zeta function lie on the critical line with real part equal to 1/2

The Riemann Xi function [1, p. 39] is defined as

$$\epsilon(s) = \epsilon(0)\prod_{\rho}(1 - \frac{s}{\rho}) \qquad ... \qquad (1)$$

where ρ *ranges over all the roots* ρ *of* $\epsilon(\rho) = 0$.

If we combine the factors

$(1-\frac{s}{\rho})$ *and* $(1-\frac{s}{1-\rho})$, *then* $\epsilon(s)$ *is Absolutely convergent infinite product* [1, p.42].

Also, $\epsilon(0) = 1/2$

From [1, *p.* 42, *section* 2.5],

$$\prod_{\rho}(1 - \frac{s}{\rho}) = \prod_{Im(\rho)>0} 1 - \frac{s(1-s)}{\rho(1-\rho)} \qquad ... \qquad (2)$$

$$\epsilon(s) = \epsilon(0)\prod_{\rho}(1 - \tfrac{s}{\rho}).$$

$$\epsilon(s) = \epsilon(0)\prod_{Im(\rho)>0} 1 - \frac{s(1-s)}{\rho(1-\rho)}.$$

We split the above product into 3 factors corresponding to

Re$(\rho) < 1/2$, $Re(\rho) > 1/2$ *and* $Re(\rho) = 1/2$.

$$\epsilon(s) = \epsilon(0) \prod_{Im(\rho)>0, Re(\rho)<1/2} 1 - \frac{s(1-s)}{\rho(1-\rho)} \prod_{Im(\rho)>0, Re(\rho)>1/2} 1 - \frac{s(1-s)}{\rho(1-\rho)}$$

$$(1 - \frac{s(1-s)}{(1/2+it)(1-(1/2+it))}).$$

$Substituting\ \rho' = 1 - \rho\ in\ the\ second\ factor\ of\ the\ above\ product,$

$Im(\rho) > 0 \Rightarrow Im(\rho') < 0.$

$Re(\rho) > 1/2 \Rightarrow Re(\rho') < 1/2.$

$$\epsilon(s) = \epsilon(0) \prod_{Im(\rho)>0, Re(\rho)<1/2} 1 - \frac{s(1-s)}{\rho(1-\rho)} \prod_{Im(\rho')<0, Re(\rho')<1/2} 1 - \frac{s(1-s)}{\rho'(1-\rho')}$$

$$(1 - \frac{s(1-s)}{(1/2+it)(1/2-it))}).$$

$$\epsilon(s) = \epsilon(0) \prod_{Im(\rho)\neq 0, Re(\rho)<1/2} [1 - \frac{s(1-s)}{\rho(1-\rho)}][1 - \frac{s(1-s)}{(1/2+it)(1/2-it))}].$$

$$\epsilon(s) = \epsilon(0) \prod_{Im(\rho)\neq 0, Re(\rho)<1/2} [1 - \frac{s(1-s)}{\rho(1-\rho)}][1 - \frac{s(1-s)}{(1/2+it)(1/2-it))}].$$

Let, ρ_0 *be a zero of the Riemann's Xi function.*

$\Rightarrow \epsilon(\rho_0) = 0.$

$\epsilon(0) = 1/2$ [2, *p.* 37 *Theorem*2.11],

$$\Rightarrow \prod_{Im(\rho)\neq 0, Re(\rho)<1/2} [1 - \frac{\rho_0(1-\rho_0)}{\rho(1-\rho)}][1 - \frac{\rho_0(1-\rho_0)}{(1/2+it)(1/2-it))}] = 0. \quad ... \qquad (3)$$

$$Claim : [1 - \frac{\rho_0(1-\rho_0)}{(1/2+it)(1/2-it))}] = [1 - \frac{\rho_0}{1/2+it}][1 - \frac{\rho_0}{1/2-it}].$$

$$1-\frac{\rho_0(1-\rho_0)}{(1/2+it)(1/2-it))} = 1-\frac{\rho_0}{(1/2+it)(1/2-it)}+\frac{\rho_0^2}{(1/2+it)(1/2-it)}$$
$$= 1-\frac{\rho_0}{1/2+it}-\frac{\rho_0}{1/2-it}+\frac{\rho_0^2}{(1/2+it)(1/2-it)}$$
$$= (1-\frac{\rho_0}{1/2+it})-\frac{\rho_0}{1/2-it}(1-\frac{\rho_0}{1/2+it})$$
$$= [1-\frac{\rho_0}{1/2+it}][1-\frac{\rho_0}{1/2-it}]$$

Using the above result in Equation (3),

$$\prod_{Im(\rho)\neq 0, Re(\rho)<1/2}[1-\frac{\rho_0(1-\rho_0)}{\rho(1-\rho)}][1-\frac{\rho_0}{1/2+it}][1-\frac{\rho_0}{1/2-it}] = 0. \quad ... \qquad (4)$$

Now if, $\prod_{Im(\rho)\neq 0, Re(\rho)<1/2}[1-\frac{\rho_0(1-\rho_0)}{\rho(1-\rho)}] \neq 0.$

Then,

$$(1-\frac{\rho_0}{1/2+it})(1-\frac{\rho_0}{1/2-it}) = 0.$$
$$\Rightarrow 1-\frac{\rho_0}{1/2+it} = 0 \; or \; 1-\frac{\rho_0}{1/2-it} = 0$$
$$\Rightarrow 1/2+it-\rho_0 = 0 \; or \; 1/2-it-\rho_0 = 0$$

Let, $\rho_0 = \sigma_0 + it_0.$

$$\Rightarrow 1/2+it-\sigma_0-it_0 = 0 \; or \; 1/2-it-\sigma_0-it_0 = 0$$

$$\Rightarrow (1/2-\sigma_0)+i(t-t_0) = 0 \; or \; (1/2-\sigma_0)-i(t+t_0) = 0$$

In either of the above cases we get, $\sigma_0 = 1/2.$

$Re(\rho_0) = 1/2.$

$So,\ for\ an\ arbitrary\ zero\ \rho_0\ of\ Riemann\ Xi\ function\ ,$

$$if \prod_{Im(\rho)\neq 0, Re(\rho)<1/2} [1-\frac{\rho_0(1-\rho_0)}{\rho(1-\rho)}] \neq 0,\ then\ Re(\rho_0)=1/2.$$

$So,\ Riemann\ Hypothesis\ is\ equivalent\ to$

$$\prod_{Im(\rho)\neq 0, Re(\rho)<1/2} [1-\frac{\rho_0(1-\rho_0)}{\rho(1-\rho)}] \neq 0.$$

3 References:-

1. H.M Edwards - Riemann's Zeta function- Academic Press (1974).

2. Kevin Broughan - Equivalents of the Riemann Hypothesis : Arithmetic Equivalents Cambridge University Press (2017) .

3. A Monotonicity of Riemann's Xi function and a reformulation of the Riemann Hypothesis, Periodica Mathematica Hungarica - May 2010.

4. E. C. Titchmarsh, D. R. Heath-Brown - The theory of the Riemann Zeta function [2nd ed] Clarendon Press; Oxford University Press (1986).

5.Stanislaw Saks, Antoni Zygmund Analytic Functions, 2nd Edition Hardcover(1965) .

6. Tom M. Apostol - Introduction to Analytical Number Theory (1976).

7. 4. Lars Ahlfors - Complex analysis [3 ed.] McGraw -Hill (1979).