Using the set of relative integers in order to find the upper bounds for prime gaps

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ABSTRACT. In this article we present a procedure for the determination of the upper bounds for prime gaps different from the most famous and known approaches. The proposed method analyzes the distribution of prime numbers using the set of relative integers $\mathbb{Z}$. Using negative numbers too, it becomes intuitive to understand that that the arrangement of 2P+1 consecutive numbers that goes -P to P, is the only arrangement that minimizes the distance between two powers having the same absolute value of the base D, with $|D| \leq P$. This arrangement is considered important because by increasing the number of powers of the prime numbers within a range of consecutive numbers, it is presumed to decrease the overlap between the prime numbers considered. Therefore, by reducing these overlaps, we suppose to obtain an arrangement, in which the prime numbers less than and equal to P and their multiples occupy the greatest possible number of positions within a range of 2P+1 consecutive numbers. Consequently, the maximum gap between two consecutive prime numbers $P_{n+1} - P_n$ can never be greater than $2P_n$. If this result could be demonstrated, would imply the resolution of the Legendre's conjecture.

Introduction

Assuming that the arrangement of the prime numbers and their multiples, in which the greatest number of consecutive positions are occupied, is the arrangement where the prime numbers considered overlap each other as little as possible; we will analyze the arrangement that minimizes the distance between two powers having the same absolute value of the base D, with $|D| \leq P$. This type of analysis is done using the set of relative integers $\mathbb{Z}$. We can use the set of relative integers because we exploit the fact that, given prime number P, the prime numbers less than and equal to P create a pattern, in which all the possible arrangements of the considered prime numbers are present, which is repeated with a period F = 2 · 3 ·… .P. Therefore given a prime number P, the period F will never be infinite, so we can develop a modular arithmetic of modulus F, in which the first terms are consecutive to the last terms. In practice the first 10 terms of this pattern go from 1 to 10, instead the last 10 terms go from -9 to 0. So the number zero represents F the last term of this pattern, in which all the prime numbers considered overlap. In this way we can pass from the set of natural numbers $\mathbb{N}$ to the set of relative integers $\mathbb{Z}$. Consequently, the minimum distance between two powers, having the same absolute value of the base D, is not $D - D^2$ but 2D (the distance between –D and D). So the arrangement of 2P+1 consecutive numbers in which two powers, having the same absolute value of the base D with $|D| \leq P$, are at the minimum distance is the one that goes from –P to P. The sequence going from –P to P is particularly interesting because it also contains the -1 and 1, two numbers that are not multiples of any prime number. Therefore, if this arrangement is the arrangement where the prime numbers, less than and equal to P and their multiples, occupy the maximum number of positions on an interval containing 2P+1 consecutive numbers, the maximum gap between two consecutive prime numbers $P_{n+1} - P_n$ can never be greater than $2P_n$. Consequently, given a number N there is always a prime number P greater than N and less than $N + 2\sqrt{N} + 1$, therefore the Legendre's conjecture is true.

Analysis of the distribution of prime numbers using the set of relative integers

As anticipated in the introduction, the prime numbers less than and equal to P generate a pattern, in which are present all the possible arrangements of the considered prime numbers, which is repeated with period F = 2 * 3 * .... P obtained by multiplying P by the prime numbers less than P. This pattern is fundamental because it also contains the arrangement, in which the prime numbers less than and equal to P and their multiples occupy the maximum number of consecutive positions.

Since the period F is not infinite, we can develop a modular arithmetic of modulus F, in which the first terms are consecutive to the last ones. The first 10 terms of this pattern go from 1 to 10, instead the last 10 terms go from -9 to 0. It is interesting to note that the number zero represents F the last term of this pattern, in which all the prime numbers considered overlap. Therefore, we consider relevant to study the distribution of prime numbers using the set of relative integers ℤ. Using also negative numbers we can define the following sequence.

$$-P \dots\dots\dots\dots\dots\dots.. -1\ 0\ 1 \dots\dots\dots\dots\dots\dots.. P \tag{1}$$

In which it is intuitive to understand how this sequence minimizes the distance between two powers having the same absolute value of the base D, with $|D| \le P$.

In this arrangement the minimum distance between two powers, having the same absolute value of the base D with $|D| \le P$, is not $D - D^2$ but 2D. Indeed –D and D are two powers that have the same absolute value of the base, therefore their distance is 2D, the least possible. The study of the distribution of powers is very important, because we want to find the arrangement in which the numbers less than or equal to P overlap each other as little as possible.

So the next step is to try to demonstrate that the arrangement (1) is also the arrangement, in which the prime numbers, less than and equal to P, occupy the maximum number of positions in an interval that contains 2P+1 consecutive numbers. In order to solve this important problem we will present a procedure that we believe is very promising.

Let us start by changing the arrangement (1) considering only the odd numbers. We thus obtain the following arrangement of P+1 odd consecutive numbers.

$$-P \dots\dots\dots\dots\dots\dots.. -3 -1\ \ 1\ \ \ 3 \dots\dots\dots\dots\dots\dots.. P \tag{2}$$

We define two groups of odd numbers: D_{ma} and D_m.

$$P/2 < D_{ma} \le P$$
$$1 < D_m < P/2$$

Now we only consider the odd numbers D_{ma}, these numbers can at most be present twice inside the arrangement (2), which we know contain P+1 odd consecutive numbers.

Taking into consideration only the odd numbers D_{ma} we try to find the arrangement, in a range consisting of P+1 odd consecutive numbers, in which the greatest possible number of positions are occupied. The arrangement that solves this problem is the arrangement (2).

The reason is that this arrangement is the only arrangement, in which all the odd numbers considered occupy two positions. Indeed, the prime number P, in order to occupy two positions within a range consisting of P+1 odd consecutive numbers, must occupy the first and last positions. Consequently, the odd number equal to P-1, must occupy the second and penultimate positions. Continuing iteratively for the other odd numbers, it is shown that the arrangement (2) is the arrangement of P+1 odd consecutive numbers, in which the largest number of positions are occupied considering the odd numbers less than or equal to P and greater than P/2 .

Now we take into consideration the odd numbers D_m, in this case different arrangements can exist compared to (2), in which these numbers occupy an extra position. Therefore, we try to understand what happens when we translate an odd number D_m so that it occupies an extra position. In this case, there will always be a position occupied in the range from $D_m\lfloor P/D_m \rfloor$ to P or in the range from $-D_m\lfloor P/D_m \rfloor$ to -P. The reason is that the arrangement that goes from $-D_m\lfloor P/D_m \rfloor$ to $D_m\lfloor P/D_m \rfloor$ is the arrangement of $D_m + 1$ odd consecutive numbers, where D_m occupies the maximum number of positions. Therefore, the extra position occupied must be in the range from $D_m\lfloor P/D_m \rfloor$ to P or in the range from $-D_m\lfloor P/D_m \rfloor$ to -P. Since the value $D_m\lfloor P/D_m \rfloor$ always greater than P/2, this implies that the extra position occupied by an odd number D_m overlaps with an odd number D_{ma}. At this point, in order to keep the gain of the extra position, we will have to move the odd number D_{ma}, however, as shown above, there is only one arrangement in which each odd number D_{ma} occupies two positions. Consequently, moving the odd number D_m implies that the new arrangement, of the odd numbers D_{ma}, occupies one position less than the case of the arrangement (2).

So a D_m number in order to occupy an additional position must necessarily occupy at least one position occupied by a D_{ma} number, consequently the length calculated by the ends not occupied in the arrangement (2) is reduced, therefore a D_{ma} number will occupy one position less. The reason is that the D_{ma} numbers cannot occupy two positions if the distance, between the unoccupied ends in a range of odd consecutive numbers, is less than $2D_{ma}$.

We report the following example: if the last three positions in the arrangement (2) are occupied by the translation of the numbers D_m, the numbers: P, P-1 and P-2 will never occupy two positions, so we will lose three positions. Consequently, the translation of the numbers D_m has as final result an arrangement where an equal or lesser number of positions will be occupied with respect to the sequence (2).

The argument just made applies to every odd number D_m, therefore we can presume that there is no other arrangement, of P+1 odd consecutive numbers, in which the odd numbers less than or equal to P occupy one position more than the arrangement (2).

Conclusion

In this article, we have analyzed a procedure for the determination of the upper bounds for prime gaps different from the more famous and known approaches [1], [2] and [3]. The proposed method analyzes the distribution of prime numbers using the set of relative integers ℤ. Using negative numbers, it becomes intuitive to understand that the arrangement (1) is the only arrangement, of 2P+1 consecutive numbers, which minimizes the distance between two powers having the same absolute value of the base D, with $|D| \leq P$.

The arrangement (1) is considered important because by increasing the number of powers of the prime numbers within a range of consecutive numbers, it is presumed to decrease the overlap between the prime numbers considered. Therefore, by reducing these overlaps, we suppose to obtain an arrangement, in which the prime numbers less than and equal to P and their multiples occupy the greatest possible number of positions within a range of 2P+1 consecutive numbers. Consequently, the maximum gap between two consecutive prime numbers $P_{n+1} - P_n$ can never be greater than $2P_n$. If this result could be demonstrated, would imply the resolution of the Legendre's conjecture [4].

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