On the Zeros of Riemann Ξ-function

Wiroj Homsup

Abstract: Based on functional equations of the Riemann Ξ-function and an application of the L'Hopital rule, it is shown that the Riemann Ξ-function has only real zeros.

1. Introduction

The Riemann ξ-function is the entire function defined by the formula

$$\xi(s) = \frac{1}{2}\, s(s-1)\pi^{-s/2}\,\Gamma(\frac{s}{2})\zeta(s) \qquad (1)$$

Its zeros are exactly the non-trivial zeros of the Riemann zeta function ζ(s), those that lie in the critical strip 0 < Re(s) < 1. The Riemann Ξ-function Ξ(z) = $\xi(\frac{1}{2}+iz)$, obtained using the variable change s = $\frac{1}{2} + iz$ which send the critical line Re(s) = $\frac{1}{2}$ to the real z-axis, has the Fourier integral representation

$$\Xi(z) = 2\int_0^\infty \Phi(t) cos ztdt \qquad (2)$$

in which

$$\Phi(t) = \sum_1^\infty (4\pi^2 n^4 e^{\frac{9t}{2}} - 6\pi n^2 e^{\frac{5t}{2}})\exp(-\pi n^2 e^{2t}), \quad 0 < t < \infty,$$

is a rapidly decreasing function [1]. The Riemann Ξ-function satisfies the functional equations Ξ(z) = Ξ(-z) and $\Xi(z) = \overline{\Xi(\bar{z})}$.

2. Zeros of the Riemann Ξ-function

Let us denote non-trivial zeros as $z_0 = \sigma_0 + it_0$. Its conjugate $\bar{z}_0 = \sigma_0 - it_0$. Then

$$\Xi(z_0) = 2\int_0^\infty \Phi(t) cos z_0 tdt = 0,$$

and

$$\Xi(\bar{z}_0) = 2\int_0^\infty \Phi(t) cos \bar{z}_0 tdt = 0$$

From the functional equation $\Xi(z) = \overline{\Xi(\bar{z})}$, it means that $\frac{\Xi(z0)}{\Xi(\bar{z}_0)} = \frac{0}{0} = 1$

By the L'Hopital rule [2],

$$\lim_{u\to\infty} \frac{\frac{d}{du}\ 2\int_0^u \Phi(\mathrm{t}) cos z_0 t dt}{\frac{d}{du}\ 2\int_0^u \Phi(\mathrm{t}) \cos \bar{z}_0 t dt} = 1 \tag{3}$$

Thus, by the Leibniz's rule for differentiation under the integral sign,

$$\lim_{u\to\infty} \frac{cos z_0 u}{\cos \bar{z}_0 u} = 1 \tag{4}$$

Eq. (4) is satisfied if $t_0 = 0$.

3. Conclusion

Based on functional equations of the Riemann Ξ-function, all zeros of the Ξ-function are real.

References

[1] Lagarias, Jeffrey and Montague, David, The integral of the Riemann ξ-function. arXiv:1106.4348v3 [math.NT] 18 Aug 2011.

[2] Gunes, Bahattin, The Analysis of the Riemann Hypothesis, 2019, hal-02077752v2.

Email: wiroj5637@gmail.com