Three-Dimensional Flow Impinging Obliquely on a Rigid Cylinder

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An exact solution of the Navier-Stokes equation is given which represents steady three-dimensional flow of a viscous fluid impinging on Rigid Cylinder obliquely. Numerical discussions of the relevant functions as well as the structure of the flow field are made. A comparison with an existing theory is also given.

Key words. Navier-Stokes equations, exact solutions.
AMS subject classifications. 76D05, 35Q30, 74F10

Due to the inherent nonlinearity of the Navier-Stokes equation, only three true exact three-dimensional solutions are known. Namely:
- Homan flow [2], modified by Karman [3] for the case of a rotating disk;
- the conical jet of Slezkin [4], generalized to the case of swirling flow by Holstein [5] and Yih [6];
- Himenz flow [1], generalized to the case of an oblique flow by Stuart [7] and Dowgialo [8].

This note presents a new exact solution to the Navier-Stokes equation, which belongs to the same class as the three listed above. This is the case of a spatial flow obliquely running onto a rigid cylinder.

To construct a solution of this class, the corresponding ideal fluid flow is used as a basis, which is at the same time a solution of the Navier-Stokes equation, which is nonlinear, and a simpler, linear equation of the vortex-free flow

(1) $$\vec{\Omega} = \vec{\nabla} \wedge \vec{u}^{\,0} = \vec{0}\ ,$$

in which the velocity field, represented as a vector product of the gradients of its integral surfaces $\psi_i^0, i = 1,2$

(2) $$\vec{u}^{\,0} = \vec{\nabla}\psi_1^0 \wedge \vec{\nabla}\psi_2^0$$

With a special view of surfaces ψ_i^0

(3) $$\psi_1^0 = f_0^0(x_1) + f_1^0(x_1) f_2^0(x_2), \psi_2^0 = x_3 - \int \frac{f_3^0(x_1)}{f_1^0(x_1)} dx_1$$

the variables in equation (1) are separated, which makes it possible to reduce it to a system of ordinary differential equations.

Further, in order to extend the ideal solution (3) to the case of a viscous flow, the form of the function $f_2^0(x_2)$ is preserved, and the remaining functions are searched again, assuming their asymptotic desire for their “ideal” analogues:

(4) $$\psi_1 = f_0(x_1) + f_1(x_1) f_2^0(x_2), \psi_2 = x_3 - \int \frac{f_3(x_1)}{f_1(x_1)} dx_1$$

We choose the Cartesian coordinates (x, y) in the plane of the cylinder section and the coordinate z in the direction of its axis. A non-viscous version of the current stream given in terms $\psi_i^0, i = 1,2$ of the coordinates of the source function, $[x, y, z] \to [l, o, z]$, where

$$l = \frac{\ln(x^2 + y^2)}{2}, o = \arctan(y, x)$$

after substituting (3) into the vortex-free flow equation (1), the differential equations for

$f_0^0(l), f_1^0(l), f_2^0(o), f_3^0(l)$ take the form

(5) $$f_0^{0''} = 0, f_1^{0''} = 0, f_3^{0'} = 2f_3^0, \frac{d^2 f_2^0}{do^2} = 0$$

As a result, for the flow of an ideal fluid, we obtain

(3') $$\psi_1^0 = al + (l+b)o, \psi_2^0 = z - c\int \frac{e^{2l}}{l} dl$$

where, a, b and c are scale constants. The ideal flow functions are shown in Fig. 1 & 2. The velocity field of the ideal flow in this case has the form

(2') $$\vec{u}^0 = \left[((a+o)\sin(o) + (l+b)\cos(o))e^{-l}, ((b+l)\sin(o) - (a+o)\cos(o))e^{-l}, c\right]$$

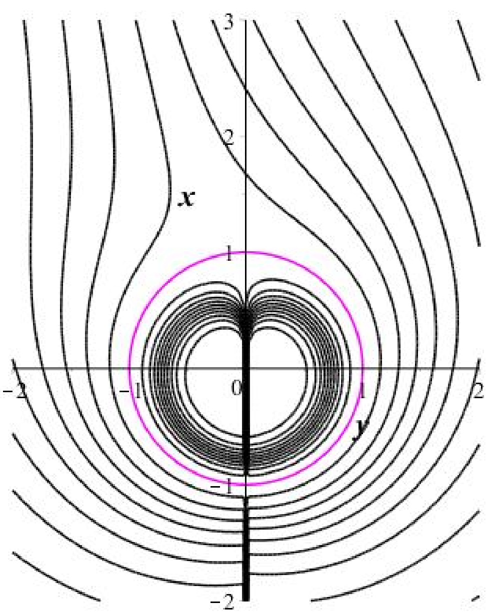

Fig. 1. $\psi_1^0, z = 0$

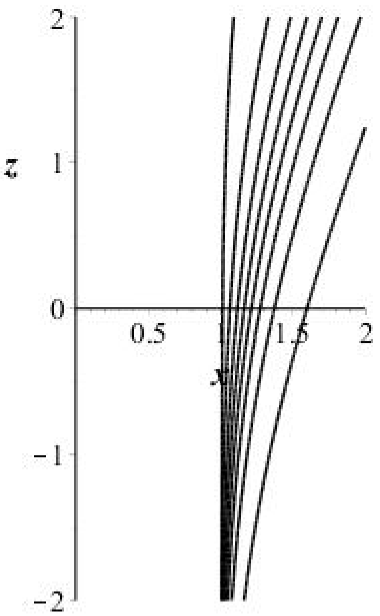

Fig. 2. $\psi_2^0, y = 0$.

If fluid viscosity is taken into account, a boundary layer appears along the wall. We assume a generalization of (3 ') in the form of (4), assuming $f_2^0(o) = o$:

(4') $$\mathrm{Re}\cdot\psi_1 = f_0(l) + f_1(l)\cdot o, \psi_2 = z - \int \frac{f_3(l)}{f_1(l)} dl$$

Where $\mathrm{Re} = \frac{u_s d}{\nu}$ is the Reynolds number.

Then the stationary Navier-Stokes equation

$$\vec{\nabla} \wedge \left(\mathrm{Re}\cdot \vec{\Omega} \wedge \vec{u} + \vec{\nabla} \wedge \vec{\Omega}\right) = \vec{0},$$

gives ordinary differential equations for $f_0(l), f_1(l), f_3(l)$:

(6) $$f_1'''' - (f_1 + 4)f_1''' + (f_1' + 2f_1 + 4)f_1'' = 0,$$

(7) $$f_0''' - (f_1 + 1)f_0'' - f_0' + (f_1'' + f_1 + 1)f_0 = 0,$$

(8) $$f_2''' - (f_1 + 2)f_2'' - (f_1' - 2f_1)f_2' = 0,$$

where the dashes denote differentiation by l . Suitable boundary conditions follow from the expression for the flow rate

(9) $$\mathrm{Re}\cdot\vec{u} = \left[\left((of_1' + e^l f_0)\sin(o) + f_1\cos(o)\right)e^{-l}, \left(f_1\sin(o) - (of_1' + e^l f_0)\cos(o)\right)e^{-l}, f_3\right],$$

and have the form:

$$f_0(0) = 0, f_1(0) = 0, f_1'(0) = 0, f_3(0) = 0$$

$$f_0(\infty) = 0, f_1(\infty) \to 0, f_1'(\infty) = 1, f_3(\infty) = v_3$$

The solutions of equations (6) - (8) and the components of the velocity field are presented in Figs. 3 & 4.

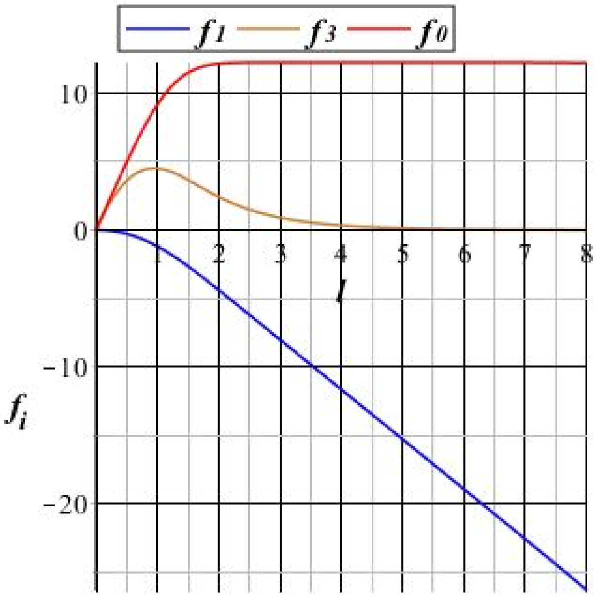

Fig.3.

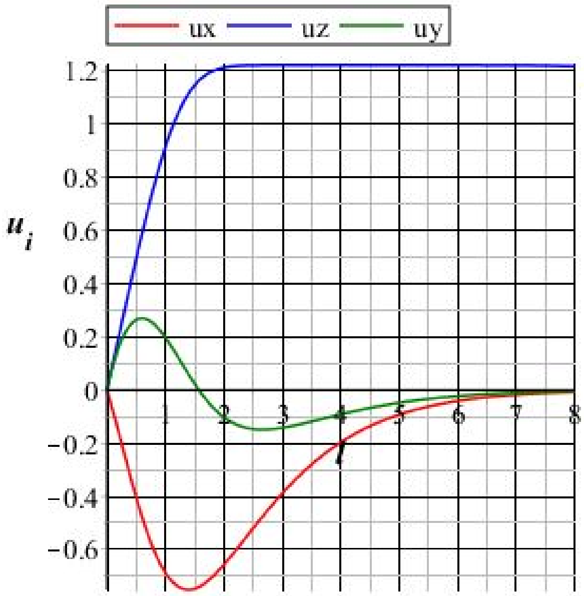

Fig.4.

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