Division by Zero Calculus in Ford Circles

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March 16, 2020

Abstract: We will refer to an application of the division by zero calculus in Ford circles that have the relations to some criteria of irrational numbers as covering problems and to the Farey sequence F_n for any positive integer n.

Key Words: Division by zero, division by zero calculus, $1/0 = 0/0 = z/0 = \tan(\pi/2) = 0, [(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} = 1$, Ford circle, Farey series, Farey intermediate number, packing by circle, criteria of irrational number.

AMS Mathematics Subject Classifications: 00A05, 00A09, 42B20, 30E20.

1 Introduction - definitions of Ford circles and division by zero calculus

First we will recall Ford circles. Consider any two relatively prime integers h and k, then the circle $C(h, k)$ of radius $1/(2k^2)$ centered at $(h/k, 1/(2k^2))$ is known as a Ford circle. Let d be the distance between the centers of the circles with $C(h,k)$ and $C(h',k')$

$$d^2 = \left(\frac{h'}{k'} - \frac{h}{k}\right)^2 + \left(\frac{1}{2k'^2} - \frac{1}{2k^2}\right)^2$$

and s be the sum of the radii

$$s = r_1 + r_2 = \frac{1}{2k^2} + \frac{1}{2k'^2}.$$

Then

$$d^2 - s^2 = \frac{(h'k - hk')^2 - 1}{k^2 k'^2}.$$

From $d^2 - s^2 \geq 0$, $(h'k - k'h)^2 \geq 1$, and so the two circles are touching (tangency) if and only if

$$|h'k - k'h| = 1. \tag{1.1}$$

See ([8]).

Ford circles are related to the Farey sequence ([4], Conway and Guy 1996).

The Farey sequence F_n for any positive integer n is the set of irreducible rational numbers a/b with $0 \leq a \leq b \leq n$ and $(a,b) = 1$ arranged in increasing order. The first few are

$$F_1 = \{0/1, 1/1\}$$

$$F_2 = \{0/1, 1/2, 1/1\}$$

$$F_3 = \{0/1, 1/3, 1/2, 2/3, 1/1\}$$

$$F_4 = \{0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1\}$$

$$F_5 = \{0/1, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1/1\}$$

and so on.

Except for F_1, each F_n has an odd number of terms and the middle term is always $1/2$.

Let p/q, p'/q', and p''/q'' be three successive terms in a Farey series. Then

$$qp' - pq' = 1$$

and

$$\frac{p'}{q'} = \frac{p + p''}{q + q''}. \tag{1.2}$$

This is the intermediate number of Farey.

If $h_1/k_1, h_2/k_2$, and h_3/k_3 are three consecutive terms in a Farey sequence, then the circles $C(h_1, k_1)$ and $C(h_2, k_2)$ are tangent at

$$\alpha_1 = \left(\frac{h_2}{k_2} - \frac{k_1}{k_2(k_2^2 + k_1^2)}, \frac{1}{k_1^2 + k_2^2} \right) \tag{1.3}$$

and the circles $C(h_2, k_2)$ and $C(h_3, k_3)$ intersect in

$$\alpha_2 = \left(\frac{h_2}{k_2} + \frac{k_3}{k_2(k_2^2 + k_3^2)}, \frac{1}{k_2^2 + k_3^2} \right).$$

Moreover, α_1 lies on the circumference of the semicircle with diameter $(h_1/k_1, 0) - (h_2/k_2, 0)$ and α_2 lies on the circumference of the semicircle with diameter $(h_2/k_2, 0) - (h_3/k_3, 0)$ ([1], Apostol 1997, p. 101).

Division by zero and division by zero calculus were, indeed, very simple. For the basic references on the division by zero and the division by zero calculus, see the references cited in the reference.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{1.4}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense. However, we gave several definitions for $1/0 = 0$ and $0/0 = 0$ with many examples and applications. See, for example, [27].

In addition, when the function $f(x)$ is not differentiable, by many meanings of zero, we **should define** as

$$\frac{f(x)}{x-a}|_{x=a} = 0,$$

for example, since 0 represents impossibility. In particular, the value of the function $y = |x|/x$ at $x = 0$ is zero. For this paper, we need only the definition of the division by zero calculus.

The aim of this paper is to consider the special circle $C(h, 0)$ from the view point of the division by zero calculus. For this purpose, we will consider the group of the circles $C(h, k)$ for real numbers h, k (we do not consider the conditions of rational numbers and of co-primeness (h,k)=1).

The division by zero calculus is to consider the case $k = 0$ in the fractional h/k.

Then, how to consider h for $h/0 = 0$? On the above line and from the primeness $(h, k) = 1$, we would like to consider the case $h = 1$. Indeed, we would like to show that **the irruducible fraction of $h/0$ may be considered as $1/0$; that is $h = 1$.**

In this case, we will consider the property of Ford circles from the viewpoint of the division by zero calculus.

2 3 circles appear as the circle $C(1, 0)$

We will show that 3 circles appear as the circle $C(1, 0)$ from the division by zero calculus view point. We write $C(h, k)$ as follows:

$$\left(x - \frac{h}{k}\right)^2 + \left(y - \frac{1}{2k^2}\right)^2 = \left(\frac{1}{2k^2}\right)^2;$$

that is,

$$x^2 - 2\frac{h}{k}x + \left(\frac{h}{k}\right)^2 + y^2 - \frac{1}{k^2}y = 0. \tag{2.1}$$

Hence, by the division by zero calculus, we have, for $k = 0$, $x = y = 0$; this means that the circle $C(1, 0)$ is the point circle and it is the origin

$$C(1, 0) = \{0\}. \tag{2.2}$$

Next, from

$$x^2k - 2hx + \frac{h^2}{k} + y^2k - \frac{1}{k}y = 0, \tag{2.3}$$

we obtain, similarly

$$C(1,0) = \{x = 0\}. \tag{2.4}$$

Finally, from

$$x^2k^2 - 2hkx + h^2 + y^2k^2 - y = 0, \tag{2.5}$$

we obtain, similarly

$$C(1,0) = \{y = 1\}. \tag{2.6}$$

In the sequel, we will consider these three cases.

3 Case I

This point circle is a very natural case. In particular, note that a point circle may be considered as zero radius and zero curvature ([26, 12, 27]). Firstly, it may be considered as touching with the real line. Secondly, the condition (1.1) is valid for $k = 0, h = 1$ and note that in the case $k' = 0$; that means that there is no non-degenerate circles $C(h', k')$ touching with the origin point circle. The third condition (1.2) also is satisfied with the sense that the three circles all have to be the origin point circle.

α_1 property (1.3) is valid in the degenerated sense of $k_1, k_2 = 0$ and $\alpha_1 = 0$.

4 Case II

Firstly, note that $\tan(\pi/2) = 0$ and for some natural sense we can consider that the y axis and the x axis are orthogonal, however, they are, at the same time, touching each other; that is the gradients of the both lines are zero and the same. This property appeared in many cases, already. See, for example, ([12, 14, 17, 18, 20, 19, 27]).

Any circle $C(h,k)$ touching with the x and y axes can be represented by the relation

$$h = \frac{1}{2k}.$$

Then, of course, we have

$$\frac{h}{k} = \frac{1}{2k^2}.$$

Therefore, with the parameter $k > 0$, when we consider two circles $C(1,0)$ and $C(1/(2k),k)$, the property (1.1) is valid only with $k = 1$.

The reasons are on the facts that the center and radius of a line are the origin and zero, respectively, when we consider a line as a circle.

The property (1.2) is not valid.

α_1 property (1.3) is not valid.

5 Case III

In this case, we can consider that the both lines $y = 1$ and $y = 0$ are touching each other at the point at infinity. In this case, the situation is clear, because any circle $C(h,k)$ touching with the both lines is represented by

$$k^2 = 1.$$

Therefore, we see that in this case all the properties are valid.

α_1 property (1.3) is also valid.

In particular, note that, even this case, the center of the circle $C(1,0) = \{y = 1\}$ is the origin.

6 Remarks

The Ford circles have deep properties for some criteria of irrational numbers with covering problems as follows:

Theorem: *For a real number α, it is an irrational number if and only if there exist infinitely many numbers h/k; irruducible rational numbers satisfying the inequality*

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{2k^2}.$$

See, for example, ([4, 6, 9]).

As a general circle group of the Ford circles, we will consider

$$(x-\xi)^2 + (y - f(\xi))^2 = f(\xi)^2$$

with a differentiable function $f(\xi)$ around the origin. Then, by the same logic we obtain the three cases, similarly for $\xi = 0$

$$(I): \qquad x^2 + y^2 - 2f(0)y = 0,$$

$$(II): \qquad x + f'(0)y = 0$$

and

$$(III): \qquad 1 - f''(0)y = 0.$$

Acknowledgments

The author wishes to express his sincere thanks to Professor Tosihiro Nakanishi for his introduction to the very interesting topics. Indeed, he published the survey paper in Japanese with the title of Geometry of Complex Numbers and Circles ([15]) and gave valuable related information to the author.

The author also expresses his thanks to Hiroshi Okumura for his good suggestions for the representation of the paper with great pleasures, because we have the same feelings on the topics.

References

[1] T. M. Apostol, Ford Circles, Section 5.5 in Modular Functions and Dirichlet Series in Number Theory, 2nd ed. New York: Springer-Verlag, (1997), 99-102.

[2] A. Bogomolny, Farey Series, A Story. http://www.cut-the-knot.org/blue/FareyHistory.shtml.

[3] C. B. Boyer, An early reference to division by zero, The Journal of the American Mathematical Monthly, **50** (1943), (8), 487- 491. Retrieved March 6, 2018, from the JSTOR database.

[4] J. H. Conway and R. K. Guy, Farey Fractions and Ford Circles. The Book of Numbers. New York: Springer-Verlag, (1996), 152-154.

[5] W. W. Däumler, H. Okumura, V. V. Puha and S. Saitoh, Horn Torus Models for the Riemann Sphere and Division by Zero, viXra:1902.0223 submitted on 2019-02-12 18:39:18.

[6] R. Devaney, The Mandelbrot Set and the Farey Tree, and the Fibonacci Sequence, Amer. Math. Monthly **106**(1999), 289-302.

[7] J. Farey, On a Curious Property of Vulgar Fractions, London, Edinburgh and Dublin Phil. Mag. 47, 385 (1816).

[8] L. R. Ford, Fractions, Amer. Math. Monthly **45**(1934), 586-601.

[9] H. H. Hardy and E. M. Wright, Farey Series and a Theorem of Minkowski. Ch. 3 in An Introduction to the Theory of Numbers, 5th ed. Oxford, England: Clarendon Press, (1979), 23-37.

[10] M. Kuroda, H. Michiwaki, S. Saitoh and M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, Int. J. Appl. Math. **27** (2014), no 2, pp. 191-198, DOI: 10.12732/ijam.v27i2.9.

[11] T. Matsuura and S. Saitoh, Matrices and division by zero $z/0 = 0$, Advances in Linear Algebra & Matrix Theory, **6**(2016), 51-58 Published Online June 2016 in SciRes. http://www.scirp.org/journal/alamt http://dx.doi.org/10.4236/alamt.2016.62007.

[12] T. Matsuura, H. Michiwaki and S. Saitoh, $\log 0 = \log \infty = 0$ and applications, Differential and Difference Equations with Applications, Springer Proceedings in Mathematics & Statistics, **230** (2018), 293-305.

[13] H. Michiwaki, S. Saitoh and M.Yamada, Reality of the division by zero $z/0 = 0$, IJAPM International J. of Applied Physics and Math. **6**(2015), 1–8. http://www.ijapm.org/show-63-504-1.html

[14] H. Michiwaki, H. Okumura and S. Saitoh, Division by Zero $z/0 = 0$ in Euclidean Spaces, International Journal of Mathematics and Computation, **2**8(2017); Issue 1, 1-16.

[15] T. Nakanishi, Geometry of complex numbers and circles, Suugaku Tsuusin, **24**(2020), No. 4, 5-15 (in Japanese).

[16] H. Okumura, S. Saitoh and T. Matsuura, Relations of 0 and ∞, Journal of Technology and Social Science (JTSS), **1**(2017), 70-77.

[17] H. Okumura and S. Saitoh, The Descartes circles theorem and division by zero calculus, https://arxiv.org/abs/1711.04961 (2017.11.14).

[18] H. Okumura and S. Saitoh, Remarks for The Twin Circles of Archimedes in a Skewed Arbelos by H. Okumura and M. Watanabe, Forum Geometricorum, **18**(2018), 97-100.

[19] H. Okumura and S. Saitoh, Applications of the division by zero calculus to Wasan geometry, GLOBAL JOURNAL OF ADVANCED RESEARCH ON CLASSICAL AND MODERN GEOMETRIES" (GJARCMG), **7**(2018), 2, 44–49.

[20] H. Okumura and S. Saitoh, Wasan Geometry and Division by Zero Calculus, Sangaku Journal of Mathematics (SJM), **2** (2018), 57–73.

[21] S. Pinelas and S. Saitoh, Division by zero calculus and differential equations. Differential and Difference Equations with Applications. Springer Proceedings in Mathematics & Statistics, **230** (2018), 399-418.

[22] H. G. Romig, Discussions: Early History of Division by Zero, American Mathematical Monthly, **3**1, No. 8. (Oct., 1924), 387-389.

[23] S. Saitoh, Generalized inversions of Hadamard and tensor products for matrices, Advances in Linear Algebra & Matrix Theory, **4** (2014), no. 2, 87–95. http://www.scirp.org/journal/ALAMT/.

[24] S. Saitoh, A reproducing kernel theory with some general applications, Qian,T./Rodino,L.(eds.): Mathematical Analysis, Probability and Applications - Plenary Lectures: Isaac 2015, Macau, China, Springer Proceedings in Mathematics and Statistics, **177**(2016), 151-182.

[25] S. Saitoh, We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense, viXra:1902.0058 submitted on 2019-02-03 22:47:53.

[26] S. Saitoh, What Was Division by Zero?; Division by Zero Calculus and New World, viXra:1904.0408 submitted on 2019-04-22 00:32:30.

[27] S. Saitoh, Fundamental of Mathematics; Division by Zero Calculus and a New Axiom, viXra:1908.0100 submitted on 2019-08-06 20:03:01.

[28] S. Saitoh, Essential Problems on the Origins of Mathematics; Division by Zero Calculus and New World, viXra:1912.0300 submitted on 2019-12-16 18:37:53.

[29] A. Tiwari, Bhartiya New Rule for Fraction (BNRF) www.ankurtiwari.in Legal Documentation © 2011-14 | Protected in 164+ countries of the world by the Berne Convention Treaty. [LINK 1] Copyright Granted by Indian Government. Copyright Registration Number: L-46939/2013 Author and Legal Owner: ANKUR TIWARI, Shubham Vihar, near Sun City, in front of Jaiswal General Store, Mangla Bilaspur, Chhattisgarh - 495001, India.

[30] A. Tiwari, Andhakar - An Autobiography Paperback Hindi By (author) Ankur Tiwari, Product details: Format Paperback, 154 pages, Dimensions 127 x 203 x 9mm, 172g, Publication date 25 Oct 2015, Publisher Educreation Publishing, Language Hindi Illustrations note Illustrations, black and white, ISBN10 8192373517, ISBN13 9788192373515