A Generator for Sums of Powers of Recursive Integer Sequences*

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Abstract

In this paper we will prove a relationship for sums of powers of recursive integer sequences. Also, we will give a possible path to discovery. As corollaries of the main result we will derive relationships for familiar integer sequences like the Fibonacci, Lucas, and Pell numbers. Last, we will discuss some applications and point to further work.

1 Introduction

Let $(U_n)_{n \geqq 1}$ be a recursive integer sequence with initial values of

$$a = U_1 = U_2,$$

where $a > 0$, and, for $n \geqq 2$, a general term of

$$U_{n-1} + p \cdot U_n = U_{n+1},$$

where p is a positive integer. Then we have the following result:

Proposition 1. *Given* $(U_n)_{n \geqq 1}$,

$$\sum_{k=1}^{n} U_k^{m+1} + \sum_{k=1}^{n-1} (U_{k+2} - U_{k+1}) \sum_{l=1}^{k+1} U_l^m = U_{n+1} \sum_{k=1}^{n} U_k^m,$$

where $n \geqq 2$ *and* m *is a positive integer.*

Let $(V_n)_{n \geqq 1}$ also be a recursive integer sequence but with initial values of

$$b = V_1 < V_2 = c,$$

where $b > 0$, and, for $n \geqq 2$, a general term of

$$V_{n-1} + q \cdot V_n = V_{n+1},$$

where q is a positive integer. Then we have the analogous result:

Proposition 2. *Given* $(V_n)_{n \geqq 1}$,

$$\sum_{k=1}^{n} V_k^{m+1} + \sum_{k=1}^{n} (V_{k+1} - V_k) \sum_{l=1}^{k} V_l^m = V_{n+1} \sum_{k=1}^{n} V_k^m,$$

where $n \geqq 1$ *and* m *is a positive integer.*

We will prove these results rigorously. But, before we do that, we will give a possible scenario for how someone might discover them.

2 Discovery

In order to illustrate how someone might discover such results, and to offer a concrete case to be kept in mind for later material, we discuss an example from the Fibonacci numbers.

Let $(F_n)_{n \geqq 1}$ be the sequence of Fibonacci numbers with initial values of

$$1 = F_1 = F_2$$

and, for $n \geqq 2$, a general term of

$$F_{n-1} + F_n = F_{n+1}.$$

There is an argument from antiquity [4, section 2], using little more than a simple diagram, which leads to the fundamental result of

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

We modify it for our present purpose.

Let us start with

$$\begin{aligned} \sum_{k=1}^{4} F_k^5 &= F_1^5 + F_2^5 + F_3^5 + F_4^5 = F_1F_1^4 + F_2F_2^4 + F_3F_3^4 + F_4F_4^4 \\ &= F_1^4 + F_2^4 + 2F_3^4 + 3F_4^4. \end{aligned}$$

We place it in a table as follows:

F_1^4	F_2^4	F_3^4	F_4^4
		F_3^4	F_4^4
			F_4^4

In order to fill in the table we write

F_1^4	F_2^4		
F_1^4	F_2^4	F_3^4	
F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4

This sum is equal to

$$\begin{aligned}&(2-1)\left(F_1^4+F_2^4\right)+(3-2)\left(F_1^4+F_2^4+F_3^4\right)+(5-3)\left(F_1^4+F_2^4+F_3^4+F_4^4\right)\\&=(F_3-F_2)\left(F_1^4+F_2^4\right)+(F_4-F_3)\left(F_1^4+F_2^4+F_3^4\right)+(F_5-F_4)\left(F_1^4+F_2^4+F_3^4+F_4^4\right),\end{aligned}$$

which is

$$\sum_{k=1}^{3}(F_{k+2}-F_{k+1})\sum_{l=1}^{k+1}F_l^4.$$

For the entire table

F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4
F_1^4	F_2^4	F_3^4	F_4^4

we write the sum in a different way:

$$5\sum_{k=1}^{4}F_k^4=F_5\sum_{k=1}^{4}F_k^4.$$

Together we have

$$\sum_{k=1}^{4}F_k^5+\sum_{k=1}^{3}(F_{k+2}-F_{k+1})\sum_{l=1}^{k+1}F_l^4=F_5\sum_{k=1}^{4}F_k^4.$$

This suggests the general case will be

$$\sum_{k=1}^{n}F_k^{m+1}+\sum_{k=1}^{n-1}(F_{k+2}-F_{k+1})\sum_{l=1}^{k+1}F_l^m=F_{n+1}\sum_{k=1}^{n}F_k^m, \tag{1}$$

where $n \geqq 2$ and m is a positive integer.

3 Main Result and Corollaries

Now we prove the main result of the paper. The reason for two cases is the different initial values for the sequences. The proofs will make this apparent.

Proof of Proposition 1

Proof. we proceed by mathematical induction. Again, the relationship we want to establish is

$$\sum_{k=1}^{n} U_k^{m+1} + \sum_{k=1}^{n-1} (U_{k+2} - U_{k+1}) \sum_{l=1}^{k+1} U_l^m = U_{n+1} \sum_{k=1}^{n} U_k^m. \tag{2}$$

For the base case of $n = 2$,

$$\begin{aligned}
\sum_{k=1}^{2} U_k^{m+1} + \sum_{k=1}^{1} (U_{k+2} - U_{k+1}) \sum_{l=1}^{k+1} U_l^m \quad &= \quad U_1^{m+1} + U_2^{m+1} + (U_3 - U_2)(U_1^m + U_2^m) \\
&= U_1^{m+1} + U_2^{m+1} + U_3 U_1^m + U_3 U_2^m - U_2 U_1^m - U_2^{m+1} \\
&= U_3 (U_1^m + U_2^m) + (U_1 - U_2) U_1^m.
\end{aligned}$$

Since $U_1 = U_2$, this is equal to $U_{2+1} \sum_{k=1}^{2} U_k^m$, as desired.

For the inductive step, assume that (2) is true for some $n \geqq 2$. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} U_k^{m+1} + \sum_{k=1}^{n} (U_{k+2} - U_{k+1}) \sum_{l=1}^{k+1} U_l^m \quad &= \quad \sum_{k=1}^{n} U_k^{m+1} + U_{n+1}^{m+1} \\
&+ \sum_{k=1}^{n-1} (U_{k+2} - U_{k+1}) \sum_{l=1}^{k+1} U_l^m + (U_{n+2} - U_{n+1}) \sum_{l=1}^{n+1} U_l^m \\
&= \sum_{k=1}^{n} U_k^{m+1} + \sum_{k=1}^{n-1} (U_{k+2} - U_{k+1}) \sum_{l=1}^{k+1} U_l^m \\
&+ U_{n+1} U_{n+1}^m + (U_{n+2} - U_{n+1}) \sum_{l=1}^{n+1} U_l^m \\
&= U_{n+1} \sum_{k=1}^{n} U_k^m + U_{n+1} U_{n+1}^m + (U_{n+2} - U_{n+1}) \sum_{l=1}^{n+1} U_l^m \\
&= U_{n+1} \sum_{k=1}^{n+1} U_k^m + (U_{n+2} - U_{n+1}) \sum_{l=1}^{n+1} U_l^m.
\end{aligned}$$

Notice that $\sum_{k=1}^{n+1} U_k^m = \sum_{l=1}^{n+1} U_l^m$. The same sum is expressed in two different notations. Therefore

$$(U_{n+1} + U_{n+2} - U_{n+1}) \sum_{k=1}^{n+1} U_k^m = U_{n+2} \sum_{k=1}^{n+1} U_k^m.$$

□

Proof of Proposition 2

Proof. we proceed by mathematical induction. The relationship we want to establish is

$$\sum_{k=1}^{n} V_k^{m+1} + \sum_{k=1}^{n} (V_{k+1} - V_k) \sum_{l=1}^{k} V_l^m = V_{n+1} \sum_{k=1}^{n} V_k^m.$$

The inductive step is analogous to the one just given. Therefore we justify only the base case of $n = 1$:

$$\sum_{k=1}^{1} V_k^{m+1} + \sum_{k=1}^{1} (V_{k+1} - V_k) \sum_{l=1}^{k} V_l^m = V_1^{m+1} + (V_2 - V_1) V_1^m.$$

Since $V_1 < V_2$, we have

$$V_1^{m+1} + V_2 V_1^m - V_1^{m+1} = V_2 V_1^m = V_{1+1} \sum_{k=1}^{1} V_k^m.$$

□

Corollaries

Now we state the main result in terms of more familiar integer sequences like the Fibonacci, Lucas, and Pell numbers. ([1, 2] contain background information on these sequences.) For the Fibonacci numbers this will establish the conjecture of the previous section.

The Fibonacci numbers $(F_n)_{n \geqq 1}$ have equal initial values. Therefore we apply the first case of the main result. Since $F_{k+2} - F_{k+1} = F_k$, we get

Corollary 1. *Given* $(F_n)_{n \geqq 1}$,

$$\sum_{k=1}^{n} F_k^{m+1} + \sum_{k=1}^{n-1} F_k \sum_{l=1}^{k+1} F_l^m = F_{n+1} \sum_{k=1}^{n} F_k^m,$$

where $n \geqq 2$ *and* m *is a positive integer.*

The Lucas numbers $(L_n)_{n \geqq 1}$ are defined identically as the Fibonacci numbers,

$$L_{n-1} + L_n = L_{n+1},$$

where $n \geqq 2$, but with the different initial values of

$$L_1 = 1 \text{ and } L_2 = 3.$$

Also, it is common to set $L_0 = 2$. Since $L_{k+1} - L_k = L_{k-1}$, the second case tells us

Corollary 2. *Given* $(L_n)_{n \geqq 1}$,

$$\sum_{k=1}^{n} L_k^{m+1} + \sum_{k=1}^{n} L_{k-1} \sum_{l=1}^{k} L_l^m = L_{n+1} \sum_{k=1}^{n} L_k^m,$$

where $n \geqq 1$ *and* m *is a positive integer.*

For the Pell numbers $(P_n)_{n \geqq 1}$ we remind ourselves that

$$P_1 = 1 \text{ and } P_2 = 2$$

and, for $n \geqq 2$,

$$P_{n-1} + 2P_n = P_{n+1}.$$

Also, we set $P_0 = 0$. Since $P_{k+1} - P_k = P_{k-1} + P_k$, the second case tells us

Corollary 3. *Given* $(P_n)_{n \geqq 1}$,

$$\sum_{k=1}^{n} P_k^{m+1} + \sum_{k=1}^{n} (P_{k-1} + P_k) \sum_{l=1}^{k} P_l^m = P_{n+1} \sum_{k=1}^{n} P_k^m,$$

where $n \geqq 1$ *and* m *is a positive integer.*

4 Applications and Further Work

Now we discuss some applications of our results and point to further work.

Subsequences

It is quite natural to apply these ideas to subsequences of recursive integer sequences. For example, suppose we look at Fibonacci numbers of even and odd indices, F_{2k} and F_{2k-1}. For even indices, $F_{2(1)} < F_{2(2)}$ tells us

$$\sum_{k=1}^{n} F_{2k}^{m+1} + \sum_{k=1}^{n} (F_{2k+2} - F_{2k}) \sum_{l=1}^{k} F_{2l}^m = F_{2(n+1)} \sum_{k=1}^{n} F_{2k}^m.$$

Since $F_{2k+2} - F_{2k} = F_{2k+1}$, we can write it also as

$$\sum_{k=1}^{n} F_{2k}^{m+1} + \sum_{k=1}^{n} F_{2k+1} \sum_{l=1}^{k} F_{2l}^m = F_{2(n+1)} \sum_{k=1}^{n} F_{2k}^m. \tag{3}$$

For odd indices, $F_{2(1)-1} < F_{2(2)-1}$ tells us

$$\sum_{k=1}^{n} F_{2k-1}^{m+1} + \sum_{k=1}^{n} \left(F_{2(k+1)-1} - F_{2k-1}\right) \sum_{l=1}^{k} F_{2l-1}^m = F_{2(n+1)-1} \sum_{k=1}^{n} F_{2k-1}^m.$$

Since $F_{2k+1} - F_{2k-1} = F_{2k}$, we can write it also as

$$\sum_{k=1}^{n} F_{2k-1}^{m+1} + \sum_{k=1}^{n} F_{2k} \sum_{l=1}^{k} F_{2l-1}^{m} = F_{2(n+1)-1} \sum_{k=1}^{n} F_{2k-1}^{m}. \tag{4}$$

For the Lucas and Pell numbers and other recursive integer sequences we can do the same thing.

Generating Sums of Powers

The title of the paper contains the word "generator." Up until now we have not said anything about that. It should go without saying that the positive integers,

$$1, 2, 3, 4, \ldots,$$

are the prototypical recursive integer sequence. The initial term is 1 and we derive all subsequent terms by adding 1 to the preceding term. Looking at our main result, we see the sequence is of the second type:

Corollary 4. *Given* $\mathbb{Z}$*,*

$$\sum_{k=1}^{n} k^{m+1} + \sum_{k=1}^{n} \sum_{l=1}^{k} l^{m} = (n+1) \sum_{k=1}^{n} k^{m}, \tag{5}$$

where $n \geqq 1$ *and* m *is a positive integer.*

In [4] the author discovered and proved this relationship, and without any consideration of a more general setting. (At a later time he learned al-Haytham might have gotten there 1,000 years earlier ([3, A000537]).) His purpose was the following. Suppose we start with

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n\,(n+1)}{2} = \frac{n^2 + n}{2}.$$

We can "feed" this result into (5) to "generate" an explicit expression for $\sum_{k=1}^{n} k^2$. Then we can use the new result for $\sum_{k=1}^{n} k^2$ to generate an expression for the next case of $\sum_{k=1}^{n} k^3$. We can do this for as many powers as we please.

It is tempting to try to derive sums using the more general expressions of Propositions 1 and 2. Unfortunately, simplifying the intermediate sums would require intricate relationships for $(U_n)_{n \geqq 1}$ and $(V_n)_{n \geqq 1}$. Therefore we will leave this matter for another time.

References

[1] M. Bicknell, A primer on the Pell sequence and related sequences, *The Fibonacci Quarterly* 13 (1975) 345–349.

[2] V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*. The Fibonacci Association, Santa Clara, CA. Available at https://www.fq.math.ca/fibonacci-lucas.html.

[3] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

[4] R. Zielinski, Induction and analogy in a problem of finite sums, https://arxiv.org/abs/1608.04006.