On some mathematical connections concerning the relation between three-dimensional gravity related to Chern-Simons gauge theory, p-adic Hartle-Hawking wave function, Rumanujan's modular functions and some equations describing the Riemann zeta-function.

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Abstract

In this paper we've described some mathematical connections that we've obtained between three-dimensional gravity related to Chern-Simons gauge theory and p-adic Hartle-Hawking wave function, Ramanujan's modular functions and some equations describing the Riemann zeta-function.

In the **Chapter 1**, we have described the very recent paper "Three-Dimensional Gravity Reconsidered", where Witten consider the problem of identifying the CFT's that may be dual to pure gravity in three dimensions with negative cosmological constant.

In the section 2 of Witten's paper, "**Gauge Theory And The Value Of** c", the goal is to determine what values of the cosmological constant, or equivalently of the central charge c of the boundary CFT, are suggested by the relation between three-dimensional gravity and Chern-Simons gauge theory.

In this section, Witten says that the Hartle-Hawking wavefunction Ψ is computed by integrating over three-manifolds W with a give boundary C.

The Hartle-Hawking wavefunction is a functional of metrics on C. For every metric h on C, Witten define $\Psi(h)$ as the result of performing a path integral over three-manifolds W whose boundary is C and whose metric g coincides with h on the boundary. Formally, one can try to argue that $\Psi(h)$ obeys the Wheeler-de Witt equation and thus is a vector in a Hilbert space H_C of solutions of this equations. Moreover, one can formally match the Wheeler-de Witt equations of gravity with the conditions for a physical state in Chern-Simons gauge theory. Though many steps in these arguments work nicely, one runs into trouble because a Riemann surface can be immersed, rather than embedded, in a three-manifold, and hence it is possible for W to degenerate without C degenerating. As a result, the Hartle-Hawking wavefunction does not obey the Wheeler-de Witt equation and is not a vector in H_C. In the case of negative cosmological constant, the boundary CFT gives a sort of cure for the problem with the Hartle-Hawking wavefunction. Instead of thinking of C as an ordinary boundary of W, Witten think of it as a conformal boundary at infinity. The partition function $\hat{\Psi}(h)$ of the boundary CFT is defined by performing the path integral over all choices of W with C as conformal boundary. This is well-behaved, because, with C at conformal

infinity, it is definitely embedded rather than immersed. Moreover, $\hat{\Psi}(h)$ is a sort of limiting value of the Hartle-Hawking wavefunction.

We have observed that, from this section, it is possible to obtain some interesting mathematical connections with some equations concerning p-adic models in the Hartle-Hawking proposal.

In the section 3 of Witten's paper, "**Partition Functions**", Witten will determine what he propose to be the exact spectrum of physical states of three-dimensional gravity or supergravity with negative cosmological constant, in a spacetime asymptotic at infinity to Anti de Sitter space. Equivalently, Witten will determine the genus one partition function of the dual CFT.

Hence, further the possible connections with the Hartle-Hawking wave function, there are various equations describing the partition functions, in this section, that, we have observed, can be related to the Ramanujan's modular functions and also some equations that can be related with some equations describing the Riemann-zeta function.

In the **section 1**, we have described some parts of the three-dimensional pure quantum gravity and relation to gauge theory, of the Witten's paper above mentioned. In the **section 2**, we have described some equations of the 2+1 dimensional gravity as an exactly soluble system. In the **section 3**, we have described some equations concerning the three dimensional charged black string solution. In the **section 4**, we have described some equations concerning a tachyon condensate phase that replaces the spacelike singularity in certain cosmological and black hole spacetimes in string theory.

In conclusion, in the **section 5**, we have described some possible mathematical connections between p-adic Hartle-Hawking wave function and the arguments above mentioned.

1. Three-dimensional pure quantum gravity and relation to gauge theory. [1]

Three-dimensional pure quantum gravity, with the Einstein-Hilbert action

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right), \quad (1.1)$$

has been studied from many points of view. Classically, 2+1-dimensional pure gravity can be expressed in terms of gauge theory. The spin connection ω is an $SO(2,1)$ gauge field (or an $SO(3)$ gauge field in the case of Euclidean signature). It can be combined with the "vierbein" e to make a gauge field of the group $SO(2,2)$ if the cosmological constant is negative. We simply combine ω and e to a 4×4 matrix A of one-forms:

$$A = \begin{pmatrix} \omega & e/\ell \\ -e/\ell & 0 \end{pmatrix}. \quad (1.2)$$

What is special in $d = 3$ is that it is also possible to write the action in a gauge-invariant form. Indeed the usual Einstein-Hilbert action (1.1) is equivalent to a Chern-Simons Lagrangian for the gauge field A:

$$I = \frac{k}{4\pi}\int tr^*\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right). \quad (1.3)$$

We remember that tr^* denotes an invariant quadratic form on the Lie algebra of $SO(2,2)$, defined by $tr^* ab = tr a * b$, where tr is the trace in the four-dimensional representation and * is the Hodge star, $(*b)_{ij} = \frac{1}{2}\varepsilon_{ijkl}b^{kl}$.

It is very important that we note that one of the precursors of the AdS/CFT correspondence was the discovery by Brown and Henneaux of an asymptotic Virasoro algebra in three-dimensional gravity. They considered three-dimensional gravity with negative cosmological constant possibly coupled to additional fields. The action is

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2} + \ldots\right), \quad (1.4)$$

where the ellipses reflect the contributions of other fields. Their main result is that the physical Hilbert space obtained in quantizing this theory (is an asymptotically Anti de Sitter or AdS spacetime) has an action of left- and right-moving Virasoro algebras with $c_L = c_R = 3\ell / 2G$. In our modern understanding, this is part of a much richer structure – the boundary conformal field theory.

Now we describe what values of the cosmological constant, or equivalently of the central charge c of the boundary CFT, are suggested by the relation between three-dimensional gravity and Chern-Simons gauge theory.

As long as the three-dimensional spacetime is oriented, three-dimensional gravity can be generalized to include an additional interaction, the Chern-Simons functional of the spin connection ω:

$$\Delta_0 I = \frac{k'}{4\pi}\int_W tr\left(\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right). \quad (1.5)$$

Here we think of ω as an $SO(2,1)$ gauge field (or an $SO(3)$ gauge field in the case of Euclidean signature). Also, tr is the trace in the three-dimensional representation of $SO(2,1)$, and k' is quantized for topological reasons. Equivalently, instead of ω, we could use the $SO(2,2)$ gauge field A introduced in eq. (1.2), and add to the action a term of the form

$$\Delta I = \frac{k'}{4\pi}\int_M tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right), \quad (1.6)$$

where now tr is the trace in the four-dimensional representation of $SO(2,2)$. If one adds to ω a multiple of e, the Einstein action (1.1) transforms in a way that cancels the e-dependent part of (1.6), reducing it to (1.5). The $SO(2,2)$-invariant form (1.6) is more useful. This way of writing the Chern-Simons functional places it precisely in parallel with the Einstein-Hilbert action, which has in (1.3) can similarly be expressed as a Chern-Simons interaction, defined with a different quadratic form. We start with the fact that the group $SO(2,2)$ is locally equivalent to $SO(2,1)\times SO(2,1)$. Moreover, we will in performing the computation assume to start with that $SO(2,1)\times SO(2,1)$ is the right global form of the gauge group.

Thus, by taking suitable linear combinations of ω and e, we will obtain a pair of $SO(2,1)$ gauge fields A_L and A_R. These have Chern-Simons interactions

$$I = \frac{k_L}{4\pi}\int tr\left(A_L\wedge dA_L + \frac{2}{3}A_L\wedge A_L\wedge A_L\right) - \frac{k_R}{4\pi}\int tr\left(A_R\wedge dA_R + \frac{2}{3}A_R\wedge A_R\wedge A_R\right). \quad (1.7)$$

Both k_L and k_R are integers for topological reasons, and this will lead to a quantization of the ratio G/ℓ that appears in the Einstein-Hilbert action, as well as the gravitational Chern-Simons coupling (1.6).

Now we describing the quantization of the Chern-Simons coupling in gauge theory. The basic case to consider is that the gauge group is $U(1)$. The gauge field A is a connection on a complex line bundle L over a three-manifold W, which for simplicity we will assume to have no boundary. The Chern-Simons action is

$$I = \frac{k}{2\pi}\int_W A \wedge dA \quad (1.8)$$

with some coefficient k. If the line bundle L is trivial, then we can interpret A as a one-form, and I is well-defined as a real-valued functional.

Now we pick a four-manifold M of boundary W and such that L extends over M. Such an M always exists. Then we pick an extension of L and A over M, and replace the definition (1.8) with

$$I_M = \frac{k}{2\pi}\int_M F \wedge F, \quad (1.9)$$

where $F = dA$ is the curvature. Now there is no Dirac string singularity, and the definition of I_M makes sense. But I_M does depend on M. To quantify the dependence on M, we consider two different four-manifolds M and M' with boundary W and chosen extensions of L. We can build a four-manifold X with no-boundary by gluing together M and M' along W, with opposite orientation for M' so that they fit smoothly along their common boundary. Then we get

$$I_M - I_{M'} = \frac{k}{2\pi}\int_X F \wedge F. \quad (1.10)$$

Now, on the closed four-manifold X, the quantity $\int_X F \wedge F/(2\pi)^2$ represents $\int_X c_1(L)^2$ and so is an integer. In quantum mechanics, the action function I should be defined modulo 2π. Requiring $I_M - I_{M'}$ to be an integer multiple of 2π, we learn that k must be an integer.

Now let us move on to the case of gauge group $SO(2,1)$. The group $SO(2,1)$ is contractible onto its maximal compact subgroup $SO(2)$, which is isomorphic to $U(1)$. So quantization of the Chern-Simons coupling for an $SO(2,1)$ gauge field and define the Chern-Simons coupling

$$I = \frac{k}{4\pi}\int_W tr\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right), \quad (1.11)$$

where tr is the "trace" in the three-dimensional representation of $SO(2,1)$. Then, in order for I to be part of the action of a quantum theory, k must be an integer.

We consider a $U(1)\times U(1)$ gauge theory with gauge fields A, B and a Chern-Simons action

$$I = \frac{k_L}{2\pi}\int_W A \wedge dA - \frac{k_R}{2\pi}\int_W B \wedge dB. \quad (1.12)$$

To define I in the topologically non-trivial case, we pick a four-manifold M over which everything extends and define

$$I_M = \int_M \left(\frac{k_L}{2\pi} F_A \wedge F_A - \frac{k_R}{2\pi} F_B \wedge F_B \right), \quad (1.13)$$

where F_A and F_B are the two curvatures. This is well-defined $\bmod 2\pi$ if

$$I_X = \int_X \left(\frac{k_L}{2\pi} F_A \wedge F_A - \frac{k_R}{2\pi} F_B \wedge F_B \right) \quad (1.14)$$

is a multiple of 2π for any $U(1)\times U(1)$ gauge field over a closed four-manifold X.
We have understood the appropriate gauge theory normalizations for the Chern-Simons action

$$I = k_L I_L + k_R I_R = \frac{k_L}{4\pi}\int tr\left(A_L \wedge dA_L + \frac{2}{3} A_L \wedge A_L \wedge A_L \right) - \frac{k_R}{4\pi}\int tr\left(A_R \wedge dA_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right) \quad (1.15)$$

Now we want to express A_L and A_R, which are gauge fields of $SO(2,1)\times SO(2,1)$, in terms of gravitational variables, and thereby determine the constraints on the gravitational couplings. We have

$$I = \frac{k_L + k_R}{2}\left(I_L - I_R\right) + \left(k_L - k_R\right)\frac{\left(I_L + I_R\right)}{2}. \quad (1.16)$$

The term in (1.16) proportional to $I_L - I_R$ will gave the Einstein-Hilbert action (1.1), while the term proportional to $\left(I_L + I_R\right)/2$ is equivalent to the gravitational Chern-Simons coupling (1.6) with coefficient $k' = k_L - k_R$. The spin connection $\omega^{ab} = \sum_i dx^i \omega_i^{ab}$ is a one-form with values in antisymmetric 3×3 matrices. The vierbein is conventionally a one-form valued in Lorentz vectors, $e^a = \sum_i dx^i e_i^a$. The metric is expressed in terms of e in the usual way, $g_{ij} dx^i \otimes dx^j = \sum_{ab} \eta_{ab} e^a \otimes e^b$, where $\eta = diag(-1,1,1)$ is the Lorentz metric; and the Riemannian volume form is $d^3x\sqrt{g} = \frac{1}{6}\varepsilon_{abc} e^a \wedge e^b \wedge e^c$, where ε_{abc} is the antisymmetric tensor with, say, $\varepsilon_{012} = 1$. It is convenient to introduce ${}^*e_{ab} = \varepsilon_{abc} e^c$, which is a one-form valued in antisymmetric matrices, just like ω. We raise and lower local Lorentz indices with the Lorentz metric η, so $\frac{1}{2}\varepsilon^{abc}\varepsilon_{bcd} = -\delta_d^a$, and $e^c = -\frac{1}{2}\varepsilon^{abc\,*}e_{bc}$. We can combine ω and ${}^*e$ and set $A_L = \omega - {}^*e/\ell$, $A_R = \omega + {}^*e/\ell$. We obtain

$$I_L - I_R = -\frac{1}{\pi\ell}\int tr\,{}^*e\left(d\omega + \omega\wedge\omega\right) - \frac{1}{3\pi\ell^3}\int tr\left({}^*e \wedge {}^*e \wedge {}^*e\right). \quad (1.17)$$

In terms of the matrix-valued curvature two-form $R^{ab} = \left(d\omega + \omega\wedge\omega\right)^{ab} = \frac{1}{2}\sum_{ij} dx^i \wedge dx_j R_{ij}^{ab}$, where R_{ij}^{ab} is the Riemann tensor, and the metric tensor g, this is equivalent to

$$I_L - I_R = \frac{1}{\pi\ell}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right). \quad (1.18)$$

Remembering the factor of $(k_L + k_R)/2$ in (1.16), we see that this agrees with the Einstein-Hilbert action (1.1) precisely if

$$k_L + k_R = \frac{\ell}{8G}. \quad (1.19)$$

The central charge of the boundary conformal field theory was originally computed by Brown and Henneaux for the case that the gravitational Chern-Simons coupling $k' = k_L - k_R$ vanishes. In this case, we set $k = k_L = k_R = \ell/16G$. The formula for the central charge is $c = 3\ell/2G$, and this leads to $c = 24k$. For the case $k' = 0$, the boundary CFT is left-right symmetric, with $c_L = c_R$, so in fact $c_L = c_R = 24k$. In general, the boundary CFT has left- and right-moving Virasoro algebras that can be interpreted as boundary excitations associated with A_L and A_R respectively. Hence, we obtain

$$(c_L, c_R) = (24k_L, 24k_R). \quad (1.20)$$

Our discussion of the quantization of the gravitational Chern-Simons coupling (1.1)

$$\Delta_0 I = \frac{k'}{4\pi}\int_W tr\left(\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right) \quad (1.21)$$

has been based entirely on gauge theory. Now, let ω a connection on an $SO(2,1)$ or (in Euclidean signature) $SO(3)$ bundle over a three-manifold W. To define $\Delta_0 I$ more precisely, we pick an oriented four-manifold M of boundary W with an extension of ω over W. Then we define

$$I_M = \frac{k'}{4\pi}\int_M trF\wedge F. \quad (1.22)$$

If M is replaced by some other four-manifold M', and $X = M - M'$ is a four-manifold without boundary obtained by gluing together M and M', then

$$I_M - I_{M'} = \frac{k'}{4\pi}\int_X trF\wedge F = 2\pi k'\int_X p_1(F), \quad (1.23)$$

where $p_1(F) = (1/8\pi^2)trF\wedge F$ is the first Pontryagin form. In general, $\int_X p_1(F)$ can be any integer, so the condition that the indeterminacy in I_M is an integer multiple of 2π means simply that k' is an integer. Then, we can rewrite the eq. (1.23) as follow:

$$I_M - I_{M'} = \frac{k'}{4\pi}\int_X trF\wedge F = 2\pi k'\int_X\left(\frac{1}{8\pi^2}\right)trF\wedge F. \quad (1.23b)$$

Now replace TW by $TW\oplus\varepsilon$, where ε is a trivial real line bundle. Then ω can be regarded as a connection on $TW\oplus\varepsilon$ in an obvious way, and $TW\oplus\varepsilon$ extends over M as the tangent bundle of M. With this choice, (1.22) becomes

$$I_M = \frac{k'}{4\pi}\int_M trR \wedge R\,, \quad (1.24)$$

where R is the curvature form of M, and (1.23) becomes

$$I_M - I_{M'} = 2\pi k'\int_X p_1(R). \quad (1.25)$$

The effects of this is that instead of the first Pontryagin number of a general bundle over X as in (1.23), we have here the first Pontryagin number $p_1(TX)$ of the tangent bundle of X. This number is divisible by 3, because of the signature theorem, which says that for a four-manifold X, $p_1(TX)/3$ is an integer, the signature of X. Hence, in the gravitational interpretation, the condition on k' is

$$k' \in \frac{1}{3}Z. \quad (1.26)$$

But the signature of a four-dimensional spin manifold is divisible by 16. So in this situation $p_1(TX)$ is a multiple of 48, and the result for k' under these assumptions is

$$k' \in \frac{1}{48}Z. \quad (1.27)$$

In three-dimensional Chern-Simons gauge theory, one can fix a Riemann surface C and construct a Hilbert space H_C of physical states obtained by quantizing the given theory on C. This Hilbert space depends on the Chern-Simons couplings and we will call it $H_C(k_L, k_R)$.

We may start by asking what we mean by the physical Hilbert space H_C obtained in quantum gravity by quantizing on a closed manifold C. Quantization on, for example, an asymptotically flat spacetime leads to a Hilbert space that can be interpreted in a relatively straightforward way, but the physical meaning of a Hilbert space obtained by quantizing on a compact spatial manifold is not clear. ***One line of thought is to consider the Hartle-Hawking wavefunction and claim that is a vector in*** H_C**.**

1.1 ***The Hartle-Hawking wave function***.

According to the no boundary proposal, the quantum state of the universe is defined by path integrals over Euclidean metrics $g_{\mu\nu}$ on compact manifolds M. From this it follows that the probability of finding a three-metric h_{ij} on a spacelike surface Σ is given by a path integral over all $g_{\mu\nu}$ on M that agree with h_{ij} on Σ. If the spacetime is simply connected, the surface Σ will divide M into two parts, M_+ and M_-. One can then factorise the probability of finding h_{ij} into a product of two wave functions, Ψ_+ and Ψ_-. Ψ_+ (Ψ_-) is given by a path integral over all metrics $g_{\mu\nu}$ on the half-manifold M_+ (M_-) which agree with h_{ij} on the boundary Σ. In most situations Ψ_+ equals Ψ_-. ***We refer to*** Ψ ***as the wave function of the universe***. Under inclusion of matter fields, one arrives at the following prescription:

$$\Psi[h_{ij}, \Phi_\Sigma] = \int D(g_{\mu\nu}, \Phi)\exp[-I(g_{\mu\nu}, \Phi)], \quad (1.28)$$

where (h_{ij}, Φ_Σ) are the 3-metric and matter fields on a spacelike boundary Σ and the path integral is taken over all compact Euclidean four geometries $g_{\mu\nu}$ that have Σ as their only boundary and matter field configurations Φ that are regular on them; $I(g_{\mu\nu}, \Phi)$ is their action. The gravitational part of the action is given by

$$I_E = -\frac{1}{16\pi}\int_{M_+} d^4x g^{1/2}(R-2\Lambda) - \frac{1}{8\pi}\int_\Sigma d^3x h^{1/2} K, \quad (1.29)$$

where R is the Ricci-scalar, Λ is the cosmological constant, and K is the trace of K_{ij}, the second fundamental form of the boundary Σ in the metric g.

One is interested in two types of inflationary universes: one with a pair of black holes, and one without. One then calculate the Euclidean actions I of the two types of saddle-point solutions. Semiclassically, it follows from eq. (1.28) that the wave function is given by

$$\Psi = \exp(-I), \quad (1.30)$$

neglecting a prefactor. One can thus assign a probability measure to each type of universe:

$$P = |\Psi|^2 = \exp(-2I^{\mathrm{Re}}), \quad (1.31)$$

where the superscript "Re" denotes the real part.

Now we consider the Hartle-Hawking wavefunction and claim that it is a vector in H_C. **The Hartle-Hawking wavefunction is a functional of metrics on C.** For every metric h on C, we define $\Psi(h)$ as the result of performing a path integral over three-manifolds W whose boundary is C and whose metric g coincides with h on the boundary. Formally, one can try to argue that $\Psi(h)$ obeys the Wheeler–de Witt equation and thus is a vector in a Hilbert space H_C of solutions of this equation. Moreover, one can formally match the Wheeler–de Witt equations of gravity with the conditions for a physical state in Chern-Simons gauge theory. Though many steps in these arguments work nicely, one runs into trouble because a Riemann surface can be immersed, rather than embedded, in a three-manifold, and hence it is possible for W to degenerate without C degenerating. As a result, the Hartle-Hawking wavefunction does not obey the Wheeler–de Witt equation and is not a vector in H_C. ***In the case of negative cosmological constant, the boundary CFT gives a sort of cure for the problem with the Hartle-Hawking wavefunction. Instead of thinking of C as an ordinary boundary of W, we think of it as a conformal boundary at infinity. The partition function $\hat{\Psi}(h)$ of the boundary CFT is defined by performing the path integral over all choices of W with C as conformal boundary. This is well-behaved, because, with C at conformal infinity, it is definitely embedded rather than immersed. Moreover, $\hat{\Psi}(h)$ is a sort of limiting value of the Hartle-Hawking wavefunction. Indeed, let ϕ be a positive function on C. Then $\hat{\Psi}(h)$ is essentially the limiting value of $\Psi(e^\phi h)$ as $\phi \to \infty$. This suggests that we should be able to think of $\hat{\Psi}(h)$ as a vector in the Hilbert space H_C associated with three-dimensional gravity and a two-manifold C.***

The phase space of $SO(2,1)\times SO(2,1)$ Chern-Simons theory on C is the space of $SO(2,1)\times SO(2,1)$ flat connections on C. The space of $SO(2,1)$ flat connections on C has several topological components. One of these components, the only one that can be simply interpreted in terms of classical gravity with negative cosmological constant, is isomorphic to Teichmuller space T. Thus, this component of the classical phase space M is a product of two copies of T, parametrized by a pair of points τ, $\tau' \in \mathrm{T}$. One can quantize M naively by using the standard holomorphic structure of T. If we do this, ***the wavefunction of a physical state is a "function" of τ and τ' that is holomorphic in τ and antiholomorphic in τ'. (Antiholomorphy in one variable reflects the relative minus sign in the Chern-Simons action (1.15); we assume that k_L and k_R are positive***).

A physical state wavefunction $\Psi(\tau,\bar{\tau}')$ is a form of weights determined by k_L and k_R. So it takes values in a Hilbert space $\mathrm{H}_C(k_L,k_R)$ that depends on the Chern-Simons couplings. Such a wavefunction $\Psi(\tau,\bar{\tau}')$ is determined by its restriction to the diagonal subspace $\tau=\tau'$. Moreover, if we want to make a relation to gravity, it is natural to require that Ψ should be invariant under the diagonal action of the mapping class group on τ and τ'; this condition is compatible with restricting to $\tau=\tau'$.

Similarly, **the partition function of a CFT on the Riemann surface C is a not necessarily holomorphic "function" $\Psi(\tau,\bar{\tau})$. Being real analytic, Ψ can be analytically continued to a function $\Psi(\tau,\bar{\tau})$ with τ' al least slightly away from τ**. It does not seem to be a standard fact that Ψ analytically continues to a holomorphic function on $\mathrm{T}\times\mathrm{T}$. However, this is true in genus 1, since the partition function can be defined as $Trq^{L_0}\bar{q}'^{\bar{L}_0}$, where we can take q and q' to be independent complex variables of modulus less than 1. It seems very plausible that the statement is actually true for all values of the genus, since one can move on Teichmuller space by "cutting" on a circle and inserting $q^{L_0}\bar{q}'^{\bar{L}_0}$. If so, the partition function of the CFT can always be interpreted as a vector in the Chern-Simons Hilbert space $\mathrm{H}_C(k_L,k_R)$. ***If we are given a theory of three-dimensional gravity, possibly coupled to other fields, the partition function of the dual CFT is a wavefunction $\Psi(\tau,\bar{\tau}')$ which is a vector in $\mathrm{H}_C(k_L,k_R)$.***

From this point of view, it seems that we should not claim (see the following chapter 2) that $\mathrm{H}_C(k_L,k_R)$ is a space of physical states that are physically meaningful in pure three-dimensional gravity. Thus, $\mathrm{H}_C(k_L,k_R)$ is in a sense a universal target for gravitational theories (with arbitrary matter fields) of given central charges.

We have formulated this for a particular Riemann surface C, but in either the gravitational theory or the dual CFT, C can vary and there is a nice behaviour when C degenerates. So it is more natural to think of this as a structure that is defined for all Riemann surfaces.

Now, we consider a two-dimensional CFT with $(0,1)$ supersymmetry, that is, with $\mathrm{N}=1$ supersymmetry for right-movers and none for left-movers. Then left-movers have an ordinary Virasoro symmetry and right-movers have an $\mathrm{N}=1$ super-Virasoro symmetry. Such a theory can be dual to a three-dimensional supergravity theory, which classically can be described by a Chern-Simons gauge theory in which the gauge supergroup is $SO(2,1)\times OSp(1|2)$. We assume that the gauge group is precisely $SO(2,1)\times OSp(1|2)$. The action is the obvious analogue of (1.15):

$$I = k_L I_L + k_R I_R = \frac{k_L}{4\pi}\int tr\left(A_L \wedge dA_L + \frac{2}{3}A_L \wedge A_L \wedge A_L\right) - \frac{k_R}{4\pi}\int str\left(A_R \wedge dA_R + \frac{2}{3}A_R \wedge A_R \wedge A_R\right). \quad (1.32)$$

Here A_L is an $SO(2,1)$ gauge field, A_R is an $OSp(1|2)$ gauge field, and str is the supertrace in the adjoint representation of $OSp(1|2)$. A_L is simply an $SO(2,1)$ gauge field, so k_L must be an integer. As for A_R, we can for topological purposes replace the supergroup $OSp(1|2)$ by its bosonic reduction $SL(2,R)$, since the fermionic directions are infinitesimal and carry no topology. So, we obtain the following result:

$$k_L \in Z \qquad k_R \in \frac{1}{4}Z\,. \quad (1.33)$$

We have $(c_L, c_R) = (24k_L, 24k_R)$, since the Brown-Henneaux computation of the central charge depends only on the bosonic part of the action. So c_L must be a multiple of 24 and c_R a multiple of 6.

Now, we want to define

$$I_R = \frac{1}{4\pi}\int_W str\left(A_R \wedge dA_R + \frac{2}{3}A_R \wedge A_R \wedge A_R\right) \quad (1.34)$$

in the presence of a Ramond world-line L on W. Let W' be a new three-manifold obtained by taking a double cover of W branched over the line L. We let $I_R(W)$ be the action (1.34) and $I_R(W')$ be the corresponding action for the gauge field A_R pulled back to W'. When pulled back to W', the singularity of A_R along the Ramond line disappears, so $I_R(W')$ is defined modulo $4\cdot 2\pi$. There is no better way to define $I_R(W)$ in the presence of a Ramond line than to say that $I_R(W) = I_R(W')/2$. So $I_R(W)$ is defined modulo $2\cdot 2\pi$. This means that k_R should be a multiple of 1/2 , not 1/4. So in other words, if including Ramond lines is the right think to do, we get

$$k_L \in Z \qquad k_R \in \frac{1}{2}Z\,, \quad (1.35)$$

and hence c_L and c_R are multiples of 24 and 12, respectively.

A good reason to focus on the case that c_R is a multiple of 12 is that there are interesting candidate superconformal field theories (SCFT's) in that case. In the supersymmetric case it is convenient to express the Chern-Simons coupling k as $k = k^*/2$, were we will focus on the case that k^* is an integer. In terms of k^*, the central charge is $c = 12k^*$.

1.2 **Partition Functions**.

Witten in the paper "Three-Dimensional Gravity Reconsidered", has determined what he proposed to be the exact spectrum of physical states of three-dimensional gravity or supergravity with negative cosmological constant, in a spacetime asymptotic at infinity to Anti de Sitter space. Equivalently, he has determined the genus one partition function of the dual CFT.

In a conformal field theory with central charge $c = 24k$, the ground state energy is $L_0 = -c/24 = -k$. The contribution of the ground state $|\Omega\rangle$ to the partition function $Z(q) = Trq^{L_0}$ is therefore q^{-k}. The Virasoro generators L_n, $n \geq -1$ annihilate $|\Omega\rangle$, but by acting with $L_{-2}, L_{-3}, \ldots$, we can make new states of the general form $\prod_{n=2}^{\infty} L_{-n}^{s_n}|\Omega\rangle$, with energy $-k + \sum_n ns_n$. If these are the only states to consider, then the partition function would be

$$Z_0(q) = q^{-k}\prod_{n=2}^{\infty}\frac{1}{1-q^n}\,. \quad (1.36)$$

There must be additional states such that $Z_0(q)$ is completed to a modular-invariant function.
Additional states are expected, because the theory also has BTZ black holes. The classical BTZ black hole is characterized by its mass M and angular momentum J. In terms of the Virasoro generators,

$$M = \frac{1}{\ell}\left(L_0 + \overline{L}_0\right) \quad J = \left(L_0 - \overline{L}_0\right), \quad (1.37)$$

so $L_0 = (\ell M + J)/2$, $\overline{L}_0 = (\ell M - J)/2$. The classical BTZ black hole obeys $M\ell \geq |J|$, or L_0, $\overline{L}_0 \geq 0$. The BTZ black hole is usually studied in the absence of the gravitational Chern-Simons coupling, that is for $k_L = k_R = k$.
Its entropy is $S = \pi(\ell/2G)^{1/2}\left(\sqrt{M\ell - J} + \sqrt{M\ell + J}\right)$. With $\ell/G = 16k$ as in (1.19), this is equivalent to $S = 4\pi\sqrt{k}\left(\sqrt{L_0} + \sqrt{\overline{L}_0}\right)$. For the holomorphic sector, the entropy is therefore

$$S_L = 4\pi\sqrt{k_L L_0}\,, \quad (1.38)$$

and similarly for the antiholomorphic sector. The quantum states corresponding to black holes exist only if $L_0 > 0$, that is $L_0 \geq 1$. This means that the exact partition function $Z(q)$ should differ from the function $Z_0(q)$ in (1.36) by terms of order q:

$$Z(q) = q^{-k}\prod_{n=2}^{\infty}\frac{1}{1-q^n} + \mathrm{O}(q). \quad (1.39)$$

This result follows from the fact that the moduli space M_1 of Riemann surfaces of genus 1 is itself a Riemann surface of genus 0, in fact parametrized by the j-function. If E_4 and E_6 are the usual Eisenstein series of weights 4 and 6, then $j = 1728E_4^3/\left(E_4^3 - E_6^2\right)$. Its expansion in powers of q is

$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \ldots \quad (1.40)$$

Actually, it is more convenient to use the function

$$J(q) = j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \ldots, \quad (1.41)$$

which likewise parametrizes the moduli space.
The J-function has a pole at $q = 0$ and no other poles. The statement that J parametrizes the moduli space means precisely that any modular-invariant function can be written as a function of J. The partition function $Z(q)$ has a pole at $q = 0$, that is at $J = \infty$.
Hence, as the pole in $Z(q)$ at $q = 0$ is of order k, Z must be a polynomial in J of degree k. Thus

$$Z(q) = \sum_{r=0}^{k} f_r J^r\,, \quad (1.42)$$

with some coefficients f_r. The terms in $Z(q)$ of order $q^{-n}, n = 0,\ldots,k$, coincide with the function $Z_0(q)$. We get a function that we will call $Z_k(q)$, $k = 1,2,3,\ldots$. This function is our candidate for the generating function that counts the quantum states of three-dimensional gravity in a spacetime asymptotic to AdS_3. For example, for $k = 1$ we have simply $Z_1(q) = J(q)$, and the next few examples are

$$Z_2(q) = J(q)^2 - 393767 = q^{-2} + 1 + 42987520q + 40491909396q^2 + ...$$
$$Z_3(q) = J(q)^3 - 590651J(q) - 64481279 = q^{-3} + q^{-1} + 1 + 2593096794q + 12756091394048q^2 + ...$$
$$Z_4(q) = J(q)^4 - 787535J(q)^2 - 8597555039J(q) - 644481279 =$$
$$= q^{-4} + q^{-2} + q^{-1} + 2 + 81026609428q + 1604671292452452276q^2 + ... \quad (1.43)$$

Frenkel, Lepowsky and Meurman constructed an extremal CFT with $k = 1$, that is, a holomorphic CFT with $c = 24$ and partition function $J(q) = Z_1(q)$. (the FLM construction)
The main point of the FLM construction was that their theory has as a group of symmetries the Fischer-Griess monster group M, the largest of the sporadic finite groups.
The FLM interpretation is that 196884 is the number of operator of dimension 2 in their theory. One of these operators is the stress tensor, while the other 196883 are primary fields transforming in the smallest non-trivial representation of M. In Witten's interpretation, the 196883 primaries are operators that create black holes. It is illuminating to compare the number 196883 to the Bekenstein-Hawking formula. An exact quantum degeneracy of 196883 corresponds to an entropy of $\ln 196883 \cong 12.19$. By contrast, the Bekenstein-Hawking entropy at $k = 1$ and $L_0 = 1$ is $4\pi \cong 12.57$. We should not expect perfect agreement, because the Bekenstein-Hawking formula is derived in a semiclassical approximation which is valid for large k.
Agreement improves rapidly if one increases k. For example, at $k = 4$, and again taking $L_0 = 1$, the exact quantum degeneracy of primary states is 81026609426, according to eq. (1.43). This corresponds to an entropy $\ln 81026609426 \cong 25.12$, compared to the Bekenstein-Hawking entropy $8\pi = 25.13$.
With regard the numbers 12,19 and 25,12 **F. Di Noto** has obtained the following numerical results. Considering 12,19 = 12 + 0,19 and 25,12 = 25 + 0,12 we obtain with some mean:

12 + 19 = 31, 12 + 25 = 37, (31 + 37) / 2 = 68 / 2 = **34**
12 + 12 = 24, 19 + 25 = 44, (24 + 44) / 2 = 68 / 2 = **34**

(12 + 12 + 19 + 25) / 4 = 68 / 4 =17 = **34** / 2.

Furthermore:

$12 \cong c^{31}$, $12 \cong c^{31}$, $19 \cong c^{37}$, $25 \cong (c^{41} + c^{40})/2$. (Here c represent the Legendre's number, $c = 1{,}08366$)

Now, we take the mean of the exponents 31, 31, 37 and 41, hence:

(31 + 31 + 37 + 41) / 4 = 140 / 4 = 35 = **34** +1.

Furthermore, we take the following mean, subtracting one unit at the four values:

(30 + 30 + 36 + 40) / 4 = 136 / 4 = **34**; and $34 \pm 3 = 31$ and 37 (prime numbers), that are the sum of 12 + 19 and 12 + 25 and the exponents of $12 \cong c^{31}$ and $19 \cong c^{37}$.
While, 41 of c^{41} is equal to 25 + 19 – 3 = 44 – 3, with $44 \cong 44{,}5 = (34 + 55)/2$, that is the mean of the two Fibonacci's numbers **34** and **55**.
With regard the ratio between the three number 12, 19 and 25, he has:

$19/12 = 1{,}58 \cong 1{,}618 = \Phi \cong c^6$; $12/19 = 0{,}631 \cong 0{,}618 = \varphi$; $25/12 = 2{,}08 \cong c^9$;
$12/25 = 0{,}48$ with $\varphi/0{,}48 = 1{,}28 \cong c^3 = 1{,}27\ldots$.

Thus, he obtain the exponents of c, i.e., 3, 6 and 9 related in the ratio between the three numbers. Thence, these numbers can be explained in the following way:

$12 = 2\times 6 = 9+3$; $19 = 3\times 6 + 1 = 9+6+3+1$; $25 = 3\times 9 - 2 = (6\times 9)/2 - 2 = 3+6+6+9+1$.

With regard the natural prime numbers, he has

12 = **11** + 1 = **13** – 1; Fibonacci's coefficient **2**, because $6\times 2 \pm 1 =$ **11** and **13**;
19 = 6×3+1; Fibonacci's coefficient 3;
25 = **23** + 2 = 6×**4** – 1+2, coefficient 4 = **2** + **2** = **3** + **1.**

11, 13, 19 and 23 are the natural prime numbers very near to 12, 19 and 25, while 2 and 3 are Fibonacci's numbers related to the exponents of c 3, 6 and 9, above mentioned, with

3 = **3**; **6** = **2**×**3** = **3**+**3**; **9** = **2**+**3**+(**2**+**2**) = **3**×**3**.

Furthermore, always with Fibonacci, he has:

12 = **13** – 1; **19** = **21** – 2; **25** = **21** + 4 = **34** – 9, with **13** and **21** Fibonacci's numbers.

In conclusion, an "interlacing" of natural prime numbers, Fibonacci's numbers, specially the number 34, tenth number of the series and that can be connected to the 10 dimensions of the superstring and with one dimension associated at one Fibonacci's number, and powers of $c = 1{,}08366$ with possible involving of φ and Φ, look to connect the Witten's numbers 12, 19 and 25.
Now we consider the analog for supergravity. We can explicitly describe a function K that parametrizes $\mathcal{H}/\Gamma_\theta$ and has a pole only at the NS cusp. This can be done in several ways. One formula is

$$K(\tau) = \frac{\Delta(\tau)^2}{\Delta(2\tau)\Delta(\tau/2)} - 24, \quad (1.44)$$

where $\Delta = q\prod_{n=1}^{24}(1-q^n)^{24}$ is the discriminant, a modular form of weight 24. In (1.44), we have subtracted the constant 24 so that the expansion of $K(\tau)$ in powers of $q^{1/2}$ has no constant term:

$$K(\tau) = q^{-1/2} + 276q^{1/2} + 2048q + 11202q^{3/2} + 49152q^2 + 184024q^{5/2} + 614400q^3 + \\ + 1881471q^{7/2} + 5373952q^4 + 14478180q^{9/2} + \ldots \quad (1.45)$$

This is analogous to the definition of the J-function without a constant term. Another formula for K is

$$K = \frac{q^{-1/2}}{2}\left(\prod_{n=1}^{\infty}\left(1+q^{n-1/2}\right)^{24} + \prod_{n=1}^{\infty}\left(1-q^{n-1/2}\right)^{24}\right) + 2048q\prod_{n=1}^{\infty}\left(1+q^n\right)^{24}. \quad (1.46)$$

The product formula in (1.44), since it converges for all $|q|<1$, shows that K is non-singular as a function on $\mathcal{H}$. As for the behaviour at the cusps, either formula shows that K has a simple pole at the NS cusp, that is, it behaves for $q \to 0$ as $q^{-1/2}$. K is regular at the Ramond cusp; in fact

$$K(\tau=1)=-24 . \quad (1.47)$$

This statement is equivalent to the statement that $K+24=\frac{\Delta(\tau)^2}{\Delta(2\tau)\Delta(\tau/2)}$ vanishes at $\tau=1$. Indeed, that function has a pole at the NS cusp, so it must have a zero somewhere. Its representation as a convergent infinite product shows that it is nonzero for $0<|q|<1$, so the zero is at the Ramond cusp. Now let us consider the Neveu-Schwarz partition function F of a holomorphic SCFT. Any Γ_θ-invariant function F on $\mathcal{H}$ can be written as a function of K.

The function F arising in a holomorphic SCFT is actually polynomial in K. Indeed, since the definition of F as Trq^{L_0} is convergent for $0<|q|<1$, F is regular as a function on $\mathcal{H}$.

So the only pole of F is at the Neveu-Schwarz cusp, that is at $K=\infty$. Consequently, in any holomorphic SCFT, the Neveu-Schwarz partition function F is a polynomial in K. The degree of this polynomial is precisely k^*, since $F \approx q^{-k^*/2}$ for $q \to 0$. So

$$F=\sum_{r=0}^{k^*} f_r K^r . \quad (1.48)$$

Thus, F depends on k^*+1 coefficients. Either statement would mean that up to terms of order $q^{1/2}$, F would coincide with the naive function

$$F_0(k^*)=q^{-k^*/2}\prod_{n=2}^{\infty}\frac{1+q^{n-1/2}}{1-q^n} \quad (1.49)$$

that counts superconformal descendants of the identity. For each positive integer k^*, there is a uniquely determined function F_{k^*} that is a polynomial in K and coincides with $F_0(k^*)$ up to order $q^{1/2}$. This is a natural analog of the partition function Z_k that we defined for an extremal CFT without supersymmetry. The number of Ramond primaries of $L_0=0$ in a theory with Neveu-Schwarz partition function F_{k^*} is, $h_0=(-1)^{k^*}F_{k^*}(1)$. Let us call this number β_{k^*} and this is uniquely determined for each k^*. A practical way to determine it, using (1.47), is to write

$$\beta_{k^*}=(-1)^{k^*}F_{k^*}(1)=(-1)^{k^*}\sum_{r=0}^{k^*} f_r K(1)^r=(-1)^{k^*}\sum_{r=0}^{k^*} f_r(-24)^r . \quad (1.50)$$

The first ten values of β_{k^*} are given in the following Table:

k^*	β_{k^*}
1	24
2	24
3	95
4	1
5	143
6	1
7	262
8	-213
9	453
10	-261

Witten has interpreted the numbers in this table as quantum corrections. For example, at $k^* = 9$, where β takes the relatively large value 453, the multiplicity of the lowest mass classically allowed black hole, namely $L_0 = 1/2$ in the NS sector, turns out to be 135149371 if the partition function F_9 can be trusted.
With regard these numbers, **F. Di Noto**, has obtained the constant number 48, hence a multiple of the number 24 that are the physical vibrations of a bosonic string. Indeed, we have:

24 + 24 = 48 = **48**×1 = **24**×2,

95 + 1 = 96 = **48**×2 = **24**×4,

143 + 1 = 144 = **48**×3 = **24**×6,

262 – 213 = 49 = **48**+1 = **24**×2+1,

453 – 261 = 192 = **48**×4 = **24**×8.

With regard the connections with the Fibonacci's numbers and the natural prime numbers, he has obtained:

Witten's numbers $a(n)$	Natural prime numbers near $6f \pm 1$	Coefficient f	Fibonacci's numbers
24	$23 = 6\times 4 - 1$	4	4 = **1+3** = **2+2**
95	$97 = 6\times 16 + 1$	16	16 = **13+3** = **3+5+8**
1	$1 = 6\times 0 + 1 = 1$	0	
143	$139 = 6\times 23 + 1$	23	23 = **21+2** = **2+8+13**
262	$262 = 6\times 44 - 1$	44	44 = **34+8+2** = **2+8+13+21**
-213	$211 = 6\times 35 + 1$	35	35 = **34+1** = **1+13+21**

453	$449 = 6\times 75 - 1$	75	75 = **2+5+13+55**
-261	$263 = 6\times 44 - 1$	44	44 = **34+8+2** = **2+8+13+21**

Now we need to understand what are the Neveu-Schwarz and Ramond vertex operators. A Ramond vertex operator O has a square root singularity in the presence of the supercurrent W:

$$W(x)\mathrm{O}(x') \approx \frac{\mathrm{O}'}{(x-x')^{n-1/2}} \quad (1.51)$$

for some integer n and some operators O'. With W understood as a spin operator with respect to the original fermions λ, those fermions have precisely this property. So they are Ramond fields. This enables us to analyze all of the states in the original NS_0 sector. The operators in the NS_0 sector that have no branch cut with W are those that are products of an even number of λ's and their derivatives. The partition function that counts the corresponding states is

$$\frac{q^{-1/2}}{2}\left(\prod_{n=1}^{\infty}\left(1+q^{n-1/2}\right)^{24}+\prod_{n=1}^{\infty}\left(1-q^{n-1/2}\right)^{24}\right). \quad (1.52)$$

We recognize this as part of the formula (1.46) for the function K. In the R_0 sector, of the 4096 ground states, half have one chirality or fermion number and half have the other. So for each L_0 eigenvalue in the R_0 sector, precisely half the states contribute to the NS sector and half to the R sector. The contribution of R_0 states to the NS sector is therefore

$$2048q\prod_{n=1}^{\infty}\left(1+q^{n}\right)^{24}. \quad (1.53)$$

Adding up (1.52) and (1.53), we see that the total partition function F_1 of the NS sector in this model is precisely what we have called K:

$$F_1 = K = \frac{q^{-1/2}}{2}\left(\prod_{n=1}^{\infty}\left(1+q^{n-1/2}\right)^{24}+\prod_{n=1}^{\infty}\left(1-q^{n-1/2}\right)^{24}\right)+2048q\prod_{n=1}^{\infty}\left(1+q^{n}\right)^{24}. \quad (1.54)$$

We can similarly compute the Ramond partition function H_1 of this model. The contribution of the NS_0 sector is obtained from (1.52) by changing a sign so as to project onto states of odd fermion number, rather than even fermion number. And the contribution of the R_0 sector is the same as (1.53). So

$$H_1 = \frac{q^{-1/2}}{2}\left(\prod_{n=1}^{\infty}\left(1+q^{n-1/2}\right)^{24}-\prod_{n=1}^{\infty}\left(1-q^{n-1/2}\right)^{24}\right)+2048q\prod_{n=1}^{\infty}\left(1+q^{n}\right)^{24} =$$
$$= 24+4096q+98304q^2+1228800q^3+10747904q^4+\dots \quad (1.55)$$

Except for states of $L_0 = 0$, the global supercharge $\mathcal{G}_0$ of the Ramond sector exchanges the part of the Ramond sector coming from NS_0 with the part coming from R_0. This implies that we can alternatively write

$$H_1 = 24 + 4096q\prod_{n=1}^{\infty}\left(1+q^n\right)^{24}, \quad (1.56)$$

where we have removed the NS_0 contribution except for the ground states, and doubled the R_0 contribution.

A hyperelliptic Riemann surface C is a double cover of the complex plane, for example a double cover of the complex x-plane, which we will call C_0, described by an equation

$$y^2 = \prod_{i=1}^{2g+2}\left(x - e_i\right). \quad (1.57)$$

The 2 : 1 cover $C \to C_0$ is branched at the points $e_1,...,e_{2g+2}$. C is smooth if the e_i are distinct, and has genus g if the number of branch points is precisely $2g+2$.
The partition function of a conformal field theory $\mathcal{W}$ on the hyperelliptic Riemann surface C can be determined by computing, in a doubled theory, the correlation function of $2g+2$ copies of a "twist field" ε, inserted at the points $e_1,...,e_{2g-2}$ in C_0. Consider any CFT $\mathcal{W}$ of central charge c. Away from branch points, the theory $\mathcal{W}$ on the double cover C looks locally like the theory $\mathcal{W}\times\mathcal{W}$ on C_0. Here we have one copy of $\mathcal{W}$ for each of the two branches of $C \to C_0$.
Now, we will calculate the details of the $\varepsilon\cdot\varepsilon$ operator product. For this, we start with a double cover C of the x-plane branched at e and $e' = -e$, and so described by an equation $y^2 = x^2 - e^2$. If $u = x + y$, $v = x - y$, then the equation is $uv = e^2$. The two branches C_+ and C_- correspond respectively to $u \to \infty$, $v \to 0$ and $u \to 0$, $v \to \infty$. **The path integral over C gives a quantum state Ψ in the theory $\mathcal{W}\times\mathcal{W}$, that is, one copy of $\mathcal{W}$ for each branch**. This state is invariant under exchange of the two branches by the symmetry $y \to -y$, so it is really a state in the symmetric product theory $\mathrm{Sym}^2\mathcal{W}$. We are really only interested in the part of Ψ proportional to the vacuum state and its descendants. This part suffices to describe the desired chiral algebra if $k \le 2$. **We will determine Ψ by using the fact that certain elements in the product $\mathcal{V}\times\mathcal{V}$ of two Virasoro algebras annihilate Ψ**. This is so because there are globally-defined holomorphic vector fields on C, of the form

$$V_n = 2^{-n}u^{n+1}d/du = -2^{-n}\left(e^2/v\right)^n v\,d/dv.$$

Let S be a contour on the surface C that wraps once around the "hole". If T is the stress tensor, the contour integral $\int_{\mathrm{S}} V_n T$ can be regarded, for any n, as an operator acting on the state Ψ. This operator is invariant under deformation of the contour. It can be deformed to a contour S_+ in the upper branch C_+ or a contour S_- in the lower branch C_-. So we have for all n

$$\left(\int_{\mathrm{S}_+} V_n T - \int_{\mathrm{S}_-} V_n T\right)\Psi = 0. \quad (1.58)$$

We want to express the two contour integrals that appear here in terms of Virasoro generators on the two branches. To do this, we simply express V_n as a vector field on the branch C_+ or C_-, either of which we identify with the x-plane. On the branch C_+, we write explicitly

$$y = \sqrt{1 - \frac{e^2}{x^2}} = 1 - \frac{e^2}{2x^2} - \frac{e^4}{8x^4} + \mathrm{O}\left(e^6\right).$$

We have carried the expansion far enough to determine (for $k \leq 2$) all singular terms in the product $\varepsilon(e) \cdot \varepsilon(-e)$. So

$$V_n = x^{n+1}\left[1 - (n+2)\frac{e^2}{4x^2} + \frac{e^4}{x^4}\left(\frac{n^2+n-4}{32}\right) + \mathrm{O}\left(e^6\right)\right]\frac{d}{dx}. \quad (1.59)$$

This means, if we ignore the conformal anomaly for the moment, that $\int_{S_+} V_n T$ corresponds, on the branch C_+, to the operator

$$Q_n^+ = L_n^+ - \frac{n+2}{4}e^2 L_{n-2}^+ + \left(\frac{n^2+n-4}{32}\right)e^4 L_{n-4}^+ + \mathrm{O}\left(e^6\right). \quad (1.60)$$

Similarly, $\int_{S_-} V_n T$ corresponds on the branch C_-, to the operator

$$Q_n^- = \left(\frac{e^2}{4}\right)^n\left[L_{-n}^- - e^2\left(\frac{-n+2}{4}\right)L_{-n-2}^- + e^4\left(\frac{n^2-n-4}{32}\right)L_{-n-4}^- + \mathrm{O}\left(e^6\right)\right]. \quad (1.61)$$

The state Ψ is determined for each value of e by the condition that $\hat{Q}_n \Psi = 0$, where $\hat{Q}_n = Q_n^+ - Q_n^-$. Hence, we have:

$$L_n^+ - \frac{n+2}{4}e^2 L_{n-2}^+ + \left(\frac{n^2+n-4}{32}\right)e^4 L_{n-4}^+ + \mathrm{O}\left(e^6\right) -$$
$$-\left(\frac{e^2}{4}\right)^n\left[L_{-n}^- - e^2\left(\frac{-n+2}{4}\right)L_{-n-2}^- + e^4\left(\frac{n^2-n-4}{32}\right)L_{-n-4}^- + \mathrm{O}\left(e^6\right)\right]\Psi = 0. \quad (1.61b)$$

However, because of the Virasoro anomaly some c-number terms must be added to the above formulas, reflecting the conformal anomaly in the mapping from u to x. The c-numbers in $\hat{Q}_n$ for other n can be conveniently determined by requiring that $\left[\hat{Q}_n, \hat{Q}_m\right] = (n-m)\hat{Q}_{n+m}$. For our purposes, the only formulas we need are

$$\hat{Q}_0 = \left(L_0^+ - \frac{e^2}{2}L_{-2}^+ - \frac{e^4}{8}L_{-4}^-\right) - \left(L_0^+ - \frac{e^2}{2}L_{-2}^- - \frac{e^4}{8}L_{-4}^-\right) + \ldots$$

$$\hat{Q}_1 = \left(L_1^+ - \frac{3e^2}{4} L_{-1}^+ - \frac{e^4}{16} L_{-3}^+ \right) - \frac{e^2}{4} \left(L_{-1}^- - \frac{e^2}{4} L_{-3}^- \right) + \dots$$

$$\hat{Q}_2 = L_2^+ - e^2 L_0^+ - 3ke^2 + \frac{e^4}{16} L_{-2}^+ - \frac{e^4}{16} L_{-2}^- + \dots. \quad (1.62)$$

The constant in $\hat{Q}_2$ was obtained from $\left[\hat{Q}_0, \hat{Q}_2\right] = -2\hat{Q}_2$. By requiring that $\hat{Q}_m \Psi = 0$ for $m = 0,1,2$, and that Ψ converges to the Fock vacuum $|\Omega\rangle$ for $e \to 0$, we now find Ψ to be

$$\Psi(e) = \left[1 + \frac{e^2}{4}\left(L_{-2}^+ + L_{-2}^-\right) + \frac{e^4}{32}\left(L_{-4}^+ + L_{-4}^-\right) + \frac{e^4}{32}\left(L_{-2}^+ + L_{-2}^-\right)^2 + \frac{e^4}{192k} L_{-2}^+ L_{-2}^- + \dots \right] |\Omega\rangle. \quad (1.63)$$

2. 2+1 Dimensional Gravity as an exactly soluble system. [2]

Let X be 2+1 dimensional Minkowski space, with coordinates t, x and y and metric $(ds)^2 = -(dt)^2 + (dx)^2 + (dy)^2$. Let X^+ be the interior of the future light cone, that is, the points of $t > 0$ and $t^2 - x^2 - y^2 > 0$. Let X^- be the interior of the past light cone, consisting of points of $t < 0$ and $t^2 - x^2 - y^2 > 0$. The 2+1 dimensional Lorentz group is $SO(2,1)$; the 2+1 dimensional Poincaré group is $ISO(2,1)$ (the "I" means that we are including the translations).
For a space-time manifold M of dimension three, the Einstein-Hilbert action would be

$$I = \frac{1}{2} \int_M \varepsilon^{ijk} \varepsilon_{abc} \left(e_i^a \left(\partial_j \omega_k^{bc} - \partial_k \omega_j^{bc} + \left[\omega_j, \omega_k\right]^{bc} \right) \right). \quad (2.1)$$

If we interpret the e's and ω's as gauge fields, this is of the general form $AdA + A^3$, and might conceivably be interpreted as a Chern-Simons three form.
Witten has described that three dimensional general relativity, without a cosmological constant, is equivalent to a gauge theory with gauge group $ISO(2,1)$ and a pure Chern-Simons action.
For a compact gauge group G, the Chern-Simons interaction may be written

$$I_{CS} = \frac{1}{2} \int_M Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (2.2)$$

Here we are regarding the gauge field A as a Lie algebra valued one form, and "Tr" really represents a non-degenerate invariant bilinear form on the Lie algebra.
Thus, if we choose a basis of the Lie algebra, and write $A = A^a T_a$, then the quadratic part of (2.2) becomes

$$Tr(T_a T_b) \cdot \int_M \left(A^a \wedge dA^b \right). \quad (2.3)$$

Here $d_{ab} = Tr(T_a T_b)$ plays the role of a metric on the Lie algebra, and this should be non-degenerate so that (2.2) or (2.3) contains a kinetic energy for all components of the gauge field.

Now we consider the general case of $ISO(d-1,1)$. The Lorentz generators are J^{ab}, and the translations are P^a, with $a,b=1...d$. A Lorentz invariant bilinear expression in the generators would have to be of the general form

$$W = xJ_{ab}J^{ab} + yP_aP^a,$$

with some constants x and y. However, in requiring that W should commute with the P^b, we learn that we must set $x=0$. For $d=3$ we can set $W=\varepsilon_{abc}P^aJ^{bc}$. This is easily seen to be $ISO(2,1)$ invariant as well as non-degenerate. For $d=3$ it is convenient to replace J^{ab} with $J^a = \frac{1}{2}\varepsilon^{abc}J_{bc}$. The invariant quadratic form of interest is then

$$< J_a, P_b >= \delta_{ab}, < J_a, J_b >=< P_a, P_b >= 0. \quad (2.4)$$

The commutation relations of $ISO(2,1)$ then take the form

$$[J_a, J_b] = \varepsilon_{abc}J^c \qquad [J_a, P_b] = \varepsilon_{abc}P^c \qquad [P_a, P_b] = 0. \quad (2.5)$$

Let use these formulas and construct gauge theory for the group $ISO(2,1)$. The gauge field is a Lie algebra valued one form,

$$A_i = e_i^a P_a + \omega_i^a J_a. \quad (2.6)$$

An infinitesimal gauge parameter would be $u=\rho^a P_a + \tau^a J_a$, with ρ^a and τ^a being infinitesimal parameters. The variation of A_i under a gauge transformation should be

$$\delta A_i = -D_i u, \quad (2.7)$$

where by definition

$$D_i u = \partial_i u + [A_i, u]. \quad (2.8)$$

Upon evaluating (2.6), we arrive at the transformation laws

$$\delta e_i^a = -\partial_i \rho^a - \varepsilon^{abc} e_{ib}\tau_c - \varepsilon^{abc}\omega_{ib}\rho_c, \qquad \delta\omega_i^a = -\partial_i\tau^a - \varepsilon^{abc}\omega_{ib}\tau_c. \quad (2.9)$$

Now we calculate the curvature tensor,

$$F_{ij} = [D_i, D_j] = P_a\left[\partial_i e_j^a - \partial_j e_i^a + \varepsilon^{abc}\left(\omega_{ib}e_{jc} - e_{ib}\omega_{jc}\right)\right] + J_a\left(\partial_i\omega_j^a - \partial_j\omega_i^a + \varepsilon^{abc}\omega_{ib}\omega_{jc}\right). \quad (2.10)$$

If now we were studying $ISO(2,1)$ gauge theory on a manifold without boundary of dimension four, we would form a topological invariant of the form $\int F^a \wedge F^b d_{ab}$ where d_{ab} is an invariant quadratic form on the Lie algebra. Using the quadratic form (2.4), we get for a four manifold Y the invariant

$$\frac{1}{2}\int_Y \varepsilon^{ijkl}\left[\partial_i e_j^a - \partial_j e_i^a + \varepsilon^{abc}\left(\omega_{ib}e_{jc} - e_{ib}\omega_{jc}\right)\right]\cdot\left(\partial_k\omega_{la} - \partial_l\omega_{ka} + \varepsilon_{ade}\omega_k^d\omega_l^e\right). \quad (2.11)$$

Denoting the integrand in (2.11) as U, a straightforward computation shows that U is a total derivative, $U = dV$. Therefore, if the four manifold Y has for its boundary a three manifold M, (2.11) reduces to an integral on M. This integral is by definition the Chern-Simons action, and one easily finds it to be

$$I_{CS} = \int_M \varepsilon^{ijk} \left[e_{ia} \left(\partial_j \omega_k^a - \partial_k \omega_j^a + \varepsilon_{abc} \omega_j^b \omega_k^c \right) \right]. \quad (2.12)$$

By this construction, (2.12) is automatically invariant under the gauge transformations (2.9).
Now we include a cosmological constant in three dimensional gravity. We note, from (2.12), that the generalized Lagrangian is

$$I = \int_M \varepsilon^{ijk} \left[e_{ia} \left(\partial_j \omega_k^a - \partial_k \omega_j^a \right) + \varepsilon_{abc} e_i^a \omega_j^b \omega_k^c + \frac{\lambda}{3} \varepsilon_{abc} e_i^a e_j^b e_k^c \right]. \quad (2.13)$$

The equations of motion now say that spacetime is locally homogeneous, with curvature proportional to λ. The simply connected covering space of such a spacetime is a portion of de Sitter or anti de Sitter space, depending on the sign of λ.
We generalize (2.5) to

$$[J_a, J_b] = \varepsilon_{abc} J^c \qquad [J_a, P_b] = \varepsilon_{abc} P^c \qquad [P_a, P_b] = \lambda \varepsilon_{abc} J^c . \quad (2.14)$$

Introducing the gauge field and covariant derivatives as in (2.6), (2.8), we find that the transformation laws (2.9) generalize to

$$\delta e_i^a = -\partial_i \rho^a - \varepsilon^{abc} e_{ib} \tau_c - \varepsilon^{abc} \omega_{ib} \rho_c \qquad \delta \omega_i^a = -\partial_i \tau^a - \varepsilon^{abc} \omega_{ib} \tau_c - \lambda \varepsilon^{abc} e_{ib} \rho_c . \quad (2.15)$$

And formula (2.10) for the curvature is replaced by

$$F_{ij} = P_a \left[\partial_i e_j^a - \partial_j e_i^a + \varepsilon_{abc} \left(\omega_i^b e_j^c - e_i^b \omega_j^c \right) \right] + J_a \left[\partial_i \omega_j^a - \partial_j \omega_i^a + \varepsilon^{abc} \left(\omega_{ib} \omega_{jc} + \lambda e_{ib} e_{jc} \right) \right]. \quad (2.16)$$

The formula (2.4) gives an invariant quadratic form on the generalized Lie algebra (2.14). Using it, we find that the Chern-Simons three form comes out to be precisely the Einstein Lagrangian (2.13) with cosmological constant included. The equations of motion derived from this Lagrangian are precisely the vanishing of the field strength (2.16). Vanishing of the coefficient of P_a in (2.16) is the assertion that ω is the Levi-Civita connection; and vanishing of the coefficient of J_a is then the Einstein equation with a cosmological constant.
Now, with regard the three dimensional gravity, in addition to the invariant quadratic form (2.4), there is a second invariant quadratic form on the Lie algebra (2.14), namely

$$< J_a, J_b > = \delta_{ab} \qquad < J_a, P_b > = 0 \qquad < P_a, P_b > = \lambda \delta_{ab} . \quad (2.17)$$

From the quadratic form (2.17), one arrives at the new fundamental Chern-Simons Lagrangian

$$\tilde{I} = \int d^3x \varepsilon^{jkl} \left[\omega_j^a \left(\partial_k \omega_l^a - \partial_l \omega_k^a + \frac{2}{3} \varepsilon_{abc} \omega_k^b \omega_l^c \right) + \lambda e_j^a \left(\partial_k e_l^a - \partial_l e_k^a \right) + 2\lambda \varepsilon_{abc} \omega_j^a e_k^b e_l^c \right]. \quad (2.18)$$

Therefore, (2.18) is invariant under (2.15) and it makes sense to add it, with an arbitrary coefficient, to the original Einstein Lagrangian (2.13). For generic values of this coefficient, the classical equations are unchanged, i.e., they still assert the vanishing of the field strength (2.16).
The bulk of this chapter is devoted to a systematic field theoretic analysis of quantum gravity on $M = \Sigma \times R^1$, with M be a flat space-time, the key insight being that the constraints of the canonical formalism can be neatly untangled by making an equivalence of 2+1 dimensional gravity with a suitable gauge theory. (For the foundations of the canonical formalism of general relativity, see also the Hartle-Hawking's paper "The wave Function of The Universe" – Physical Review D28 (1983) 2960).
We now turn to construct a canonical formalism, with a view toward quantization. Thus, we consider the Lagrangian (2.13) on a three manifold $M = \Sigma \times R^1$, with Σ being a Riemann surface that plays the role of an initial value surface. The first step in constructing a canonical formalism is to introduce new variables, if necessary, to get a Lagrangian that is linear in time derivatives.
The variables whose time derivatives appear in (2.13) are the "spatial" components of the vierbein and connection, namely e_i^a and ω_i^a, for $i = 1,2$. The variables whose time derivatives are absent in (2.13) are the "time" components e_0^a and ω_0^a. This convenient, global separation between variables whose time derivatives appear in the Lagrangian and variables whose time derivatives do not appear, and the fact that the Lagrangian is linear in the latter, make the construction of a canonical formalism relatively straightforward.
Then, the eq. (2.13) may be rewritten

$$I = -2\int dt \int_\Sigma \varepsilon^{ij} e_{ia} \frac{d}{dt}\omega_j^a + \int dt \int_\Sigma \left[e_0^a \cdot \varepsilon^{ij}\left(\partial_i \omega_j^a - \partial_j \omega_i^a + \varepsilon^{abc}\omega_{ib}\omega_{jc} + \lambda\varepsilon^{abc} e_{ib} e_{jc}\right) + \omega_0^a \cdot \varepsilon^{ij}\left(\partial_i e_j^a - \partial_j e_i^a + \varepsilon^{abc}\left(\omega_{ib} e_{jc} - e_{ib}\omega_{jc}\right)\right)\right]. \quad (2.19)$$

The Poisson brackets can be read off from the terms in (2.19) that contain time derivatives. They are

$$\left\{\omega_i^a(x), e_j^b(y)\right\} = \frac{1}{2}\cdot \varepsilon_{ij}\eta^{ab}\delta^2(x-y), \quad \left\{e_{ia}(x), e_{jb}(y)\right\} = \left\{\omega_{ia}(x), \omega_{jb}(y)\right\} = 0 . \quad (2.20)$$

In addition, we must impose the constraint equations. They are simply the equations $\delta I / \delta e_0^a = \delta I / \delta\omega_0^a = 0$, or

$$\varepsilon^{ij}\left[\partial_i e_j^a - \partial_j e_i^a + \varepsilon^{abc}\left(\omega_{ib} e_{jc} - e_{ib}\omega_{jc}\right)\right] = 0, \quad \varepsilon^{ij}\left[\partial_i \omega_j^a - \partial_j \omega_i^a + \varepsilon^{abc}\left(\omega_{ib}\omega_{jc} + \lambda e_{ib} e_{jc}\right)\right] = 0 . \quad (2.21)$$

Let G be the group $ISO(2,1)$ if $\lambda = 0$, and its generalization $SO(3,1)$ or $SO(2,2)$ if λ is not zero. It is natural to regard e_i^a and ω_i^a, for $i = 1,2$, as a gauge field on the Riemann surface Σ. The space of all such gauge fields is a phase space on which we have defined Poisson brackets (2.20).
The canonical variables e_i^a and ω_i^a fit together into a G gauge field on Σ. The constraint equations (2.21) assert that this gauge field is a "flat connection", that is, the field strength vanishes. As for the group of transformations generated by the constraints, these are just gauge transformations. One may easily check, using the Poisson brackets (2.20), that the constraint operators that appear on the left of (2.21) are the generators of the very gauge transformations that we have discussed in (2.15):

$$\delta e_i^a = -\partial_i \rho^a - \varepsilon^{abc} e_{ib}\tau_c - \varepsilon^{abc}\omega_{ib}\rho_c, \qquad \delta\omega_i^a = -\partial_i \tau^a - \varepsilon^{abc}\omega_{ib}\tau_c - \lambda\varepsilon^{abc} e_{ib}\rho_c . \quad (2.22)$$

In gauge theories, $\delta L/\delta A_0$ is always the generator of gauge transformations. Thus, to construct the classical phase space which should be quantized, one simply takes the space of solutions of the constraints (the space of flat connections) and divides by the group generated by the constraints (the group of gauge transformations). Consequently, the phase space M of 2+1 dimensional gravity is the same as the moduli space of flat G connections modulo gauge transformations.
Now, we consider the geometrical applications of quantum gravity to the 2+1 dimensional gravity.
The most important case for geometrical applications, is likely to be the case of Euclidean signature with negative cosmological constant – the relevant gauge group is then $SO(3,1)$. In this case, the Lagrangian is

$$\hat{I} = \frac{1}{\hbar} I + \frac{ik}{8\pi} I' \ , \quad (2.23)$$

where I is the standard Einstein action (2.13) with cosmological constant, and I' is the exotic action (2.18). Hence, we have the following interesting equation:

$$\hat{I} = \frac{1}{\hbar}\int_M \varepsilon^{ijk}\left[e_{ia}\left(\partial_j \omega_k^a - \partial_k \omega_j^a\right) + \varepsilon_{abc} e_i^a \omega_j^b \omega_k^c + \frac{\lambda}{3}\varepsilon_{abc} e_i^a e_j^b e_k^c \right] +$$
$$+ \frac{ik}{8\pi}\int d^3x \varepsilon^{jkl}\left[\omega_j^a \left(\partial_k \omega_l^a - \partial_l \omega_k^a + \frac{2}{3}\varepsilon_{abc}\omega_k^b \omega_l^c \right) + \lambda e_j^a \left(\partial_k e_l^a - \partial_l e_k^a\right) + 2\lambda \varepsilon_{abc} \omega_j^a e_k^b e_l^c \right]. \quad (2.23b)$$

The standard action I appears with a real coefficient, which we have written as $1/\hbar$; here $\hbar$ is Planck's constant. But I' has a quantized coefficient, with k an integer. Once $\hbar$ is explicitly introduced in this way, one may as well set $\lambda = 1$ in (2.13). Now one wishes to study the Feynman integral over all choices of field variables on an arbitrary three manifold M, to get the partition function defined by

$$Z(M) = \int De D\omega e^{-\hat{I}} \ . \quad (2.24)$$

Understanding quantum gravity on a general three manifold would mean understanding how to compute $Z(M)$ as a function of the variables $\hbar$ and k that appear in the Lagrangian.
The connection with classical geometry should be particularly striking in the limit of small $\hbar$. According to the standard conjectures about three manifolds, almost all interesting (irreducible) three manifolds are "hyperbolic", and the action (2.23b) would have a unique non-trivial critical point up to gauge transformation. The action for this critical point is $-(V/\hbar + 2\pi ikC)$, where V and C are known as the volume and Chern-Simons invariant of the hyperbolic three manifold. The small $\hbar$ limit of the partition function would be $Z \approx \exp(V/\hbar + 2\pi ikC)$ (up to a power of $\hbar$), so that the classical invariants V and C could be extracted from the asymptotic behaviour of Z, if indeed it is possible to define the partition function Z as an invariant of three manifolds.

3. On the three dimensional charged black string solution. [3]

We take the following form of anti-de Sitter space:

$$ds^2 = \left(1 - \frac{\hat{r}^2}{l^2}\right) d\hat{t}^2 + \left(\frac{\hat{r}^2}{l^2} - 1\right)^{-1} d\hat{r}^2 + \hat{r}^2 d\hat{\varphi}^2 \ . \quad (3.1)$$

Since $\hat{t}$ and $\hat{\varphi}$ are both parameters along a boost, they can take any real value. If we identify $\hat{\varphi} = \hat{\varphi} + 2\pi$, (3.1) describes a black hole.
Now, we choose two constants r_+, r_- and introduce new coordinates $\hat{t} = (r_+ t / l) - r_- \varphi$, $\hat{\varphi} = (r_+ \varphi / l) - (r_- t / l^2)$, $\hat{r}^2 = l^2 (r^2 - r_-^2)/(r_+^2 - r_-^2)$. Then the metric (3.1) becomes

$$ds^2 = \left(M - \frac{r^2}{l^2} \right) dt^2 - J dt d\varphi + r^2 d\varphi^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right)^{-1} dr^2 \quad (3.2)$$

where the constants M and J are related to $r_\pm$ by

$$M = \frac{r_+^2 + r_-^2}{l^2} \quad J = \frac{2 r_+ r_-}{l}. \quad (3.3)$$

Identifying φ with $\varphi + 2\pi$, yields a two parameter family of black holes.
We now turn to string theory. We consider the black holes in the context of the low energy approximation, and then consider the exact conformal field theory. In three dimensions, the low energy string action is

$$S = \int d^3 x \sqrt{-g} e^{-2\phi} \left[\frac{4}{k} + R + 4(\nabla \phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right]. \quad (3.4)$$

We note that it is possible to obtain the following mathematical connection with eq. (1.18). Indeed, we have:

$$\frac{1}{\pi \ell} \int d^3 x \sqrt{g} \left(R + \frac{2}{\ell^2} \right) = \int d^3 x \sqrt{-g} e^{-2\phi} \left[\frac{4}{k} + R + 4(\nabla \phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right]. \quad (3.4b)$$

The equations of motion which follows from this action are

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\mu\lambda\sigma} H_\nu^{\lambda\sigma} = 0, \quad (3.5a) \quad \nabla^\mu \left(e^{-2\phi} H_{\mu\nu\rho} \right) = 0, \quad (3.5b)$$

$$4\nabla^2 \phi - 4(\nabla \phi)^2 + \frac{4}{k} + R - \frac{1}{12} H^2 = 0. \quad (3.5c)$$

If we assume $\phi = 0$, then (3.5b) yields $H_{\mu\nu\rho} = (2/l)\varepsilon_{\mu\nu\rho}$ where l is a constant with dimensions of length. Substituting this form of H into (3.5a) yields

$$R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu}, \quad (3.6)$$

which is exactly Einstein's equation with a negative cosmological constant. The dilaton equation (3.5c) will also be satisfied provided $k = l^2$. Thus every solution to three dimensional general relativity with negative cosmological constant, is a solution to low energy string theory with $\phi = 0$, $H_{\mu\nu\rho} = (2/l)\varepsilon_{\mu\nu\rho}$ and $k = l^2$. In particular, the two parameter family of black holes (3.2) is a solution with

$$B_{\varphi t} = \frac{r^2}{l}, \quad \phi = 0 \quad (3.7)$$

where $H = dB$.

We now consider the dual of this solution. Given a solution $(g_{\mu\nu}, B_{\mu\nu}, \phi)$ that is independent of one coordinate, say x, then $(\tilde{g}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{\phi})$ is also a solution where

$$\tilde{g}_{xx} = 1/g_{xx}, \quad \tilde{g}_{x\alpha} = B_{x\alpha}/g_{xx}, \quad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - (g_{x\alpha}g_{x\beta} - B_{x\alpha}B_{x\beta})/g_{xx},$$
$$\tilde{B}_{x\alpha} = g_{x\alpha}/g_{xx}, \quad \tilde{B}_{\alpha\beta} = B_{\alpha\beta} - 2g_{x[\alpha}B_{\beta]x}/g_{xx}, \quad \tilde{\phi} = \phi - \frac{1}{2}\ln g_{xx}, \quad (3.8)$$

and α, β run over all directions except x. Applying this transformation to the φ translational symmetry of the black hole solution (3.2) (3.7) yields

$$\tilde{d}s^2 = \left(M - \frac{J^2}{4r^2}\right)dt^2 + \frac{2}{l}dtd\varphi + \frac{1}{r^2}d\varphi^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1} dr^2,$$
$$\tilde{B}_{\varphi t} = -\frac{J}{2r^2}, \quad \phi = -\ln r. \quad (3.9)$$

To better understand this solution, we diagonalize the metric. Let

$$t = \frac{l(\hat{x} - \hat{t})}{(r_+^2 - r_-^2)^{1/2}}, \quad \varphi = \frac{r_+^2\hat{t} - r_-^2\hat{x}}{(r_+^2 - r_-^2)^{1/2}}, \quad r^2 = l\hat{r}. \quad (3.10)$$

Then the solution becomes

$$\tilde{d}s^2 = -\left(1 - \frac{\mathrm{M}}{\hat{r}}\right)d\hat{t}^2 + \left(1 - \frac{Q^2}{\mathrm{M}\hat{r}}\right)d\hat{x}^2 + \left(1 - \frac{\mathrm{M}}{\hat{r}}\right)^{-1}\left(1 - \frac{Q^2}{\mathrm{M}\hat{r}}\right)^{-1}\frac{l^2 d\hat{r}^2}{4\hat{r}^2},$$
$$\phi = -\frac{1}{2}\ln \hat{r}l, \quad B_{\hat{x}\hat{t}} = \frac{Q}{\hat{r}}, \quad (3.11)$$

where $\mathrm{M} = r_+^2/l$ and $Q = J/2$. This is precisely the three dimensional charged black string solution.

4. On a tachyon condensate phase that replaces the spacelike singularity in certain cosmological and black hole spacetimes in string theory. [4]

We consider a general relativistic solution approaching a curvature singularity in the past or future. The metric is of the form

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = -(dx^0)^2 + R_i(x^0)^2 d\Omega_i^2 + ds_\perp^2 \quad (4.1)$$

with $R_i\left(x^0\right)\to 0$ for some i at some finite time. Here Ω_i describe spatial coordinates whose scale factor is varying in time and $ds_\perp^2$ describes some transverse directions not directly participating in the time dependent physics. In the large radius regime where general relativity applies, the background (4.1) is described by a worldsheet sigma model with action in conformal gauge

$$S_0 \equiv \frac{1}{4\pi\alpha'}\int d^2\sigma G_{\mu\nu}(X)\partial_a X^\mu \partial^a X^\nu + \text{fermions} + \text{ghosts}. \quad (4.2)$$

Here we are considering a type II or heterotic string with worldsheet supersymmetry in order to avoid bulk tachyons.
Consider the Milne spacetime described by the metric

$$ds^2 = -\left(dx^0\right)^2 + v^2\left(x^0\right)^2 d\Omega^2 + d\vec{x}^2. \quad (4.3)$$

For $x^0 > 0$, this solution describes a growing S^1 along the Ω direction. At $x^0 = 0$ there is a spacelike big bang singularity, and general relativity breaks down. The evolution from $x^0 = -\infty$ to $x^0 = 0$ similarly describes an evolution toward a big crunch singularity. This geometry appears inside 2+1 dimensional black holes, BTZ black holes in AdS_3.
In the heterotic theory we have target space coordinates given by $(0,1)$ scalar supefields $\chi^\mu = X^\mu + \theta^+\psi_+^\mu$ and left moving fermion superfields $\Psi_-^a = \psi_-^a + \theta^+ F^a$ containing auxiliary fields F^a. In terms of these fields we have a Lorentzian signature path integral

$$G(\{V_n\}) \equiv \int [d\chi][d\Psi_-][d(g)]d(m)e^{iS}\prod_n \left(i\int d\sigma d\tau V_n[\chi]\right) \quad (4.4)$$

where (g) and (m) are ghosts and moduli and where the semiclassical action is

$$iS = i\int d\sigma d\tau d\theta^+\left(D_{\theta^+}\chi^\mu\partial_-\chi^\nu G_{\mu\nu}(\chi) - \mu\Psi_- : e^{-\kappa\chi^0}\cos\left(\omega\tilde{\Omega}\right): + \Psi_-^a D_{\theta^+}\Psi_-^a + (d)\right) + iS_g\,, \quad (4.5)$$

where (d) is the dilaton, g of S_g is the ghost and $V_n[\chi]$ are vertex operator insertions.
Here $\tilde{\Omega}$ is the T-dual of the coordinate Ω on the smallest circle in the space; $\cos\omega\tilde{\Omega}$ is the winding operator for strings wrapped around the Ω direction. The fluctuations of the worldsheet fields in (4.4) generates corrections to the action (4.5). Because the bulk region of the geometry (4.3) is approximately flat space, we may identify the V_n with operators of the form

$$V_{\vec{k},n} \to e^{i\vec{k}\cdot\vec{\chi}}e^{i\omega(\vec{k},n)\chi^0}\hat{V}_n \quad \text{as} \quad X^0 \to \infty\,, \quad (4.6)$$

where we have pulled out the oscillator and ghost contributions into $\hat{V}$. At the semiclassical level the dilaton is also known: it goes to a constant

$$\Phi \to \Phi_0 \quad \text{as} \quad X^0 \to +\infty. \quad (4.7)$$

In particular, the tachyon vertex operator in (4.5) is semiclassically marginal without an additional dilaton contribution and the metric terms solve Einstein's equations. The path integral over fluctuations of the fields will generate corrections to these semiclassical statements (4.5) (4.6) (4.7).

Let us Wick rotate the worldsheet time coordinate τ, the spatial target space coordinates $\vec{\chi}(\sigma,\tau)$ (including $\tilde{\Omega}$), and the parameters μ and $\vec{k}$ by

$$\tau \equiv e^{i\gamma}\tau_\gamma \quad \vec{\chi} \equiv e^{i\gamma}\vec{\chi}_\gamma \quad \mu = e^{-i\gamma}\mu_\gamma \quad \vec{k} = e^{-i\gamma}\vec{k}_\gamma, \quad (4.8)$$

where γ is a phase which we will rotate from 0 to $\pi/2$. This produces a Euclidean path integral for the worldsheet theory (where we label the quantities rotated to $\gamma = \pi/2$ by a subscript E)

$$G(\{V_n\}) \equiv \int [d\vec{\chi}_E][d\chi^0][d\psi_-][d(g)]d(m)e^{-S_E}\prod_n \int (-1)d\sigma d\tau_E V_{n,-i\vec{k}_E}[\chi^0, i\vec{\chi}_E], \quad (4.9)$$

(where g and m are the ghosts and the moduli) with Euclidean action

$$S_E = \int d\sigma d\tau_E d\theta^+ \Big(D_{\theta^+}\chi^0\partial_-\chi^0 + v^2(\chi^0)^2 D_{\theta^+}\tilde{\Omega}_E\partial_-\tilde{\Omega}_E + G_{ij}D_{\theta^+}\chi^i_{\perp,E}\partial_-\chi^j_{\perp,E}$$
$$- i\mu_E e^{-\kappa\chi^0}\cosh(\omega\tilde{\Omega}_E) + \Psi^a_- D_{\theta^+}\Psi^a_- + (d)\Big) + S_E(g), \quad (4.10)$$

where (d) is the dilaton and (g) is the ghost.
Here $\vec{\chi}_E \equiv (\Omega_E, \vec{\chi}_{\perp E})$ refers to the worldsheet superfields corresponding to the spatial target space coordinates, and we have plugged in the spacetime metric (4.3).
In the type II theory, we have (1,1) scalar superfields $\chi^\mu = X^\mu + \theta^+\psi^\mu_+ + \theta^-\psi^\mu_- + \theta^+\theta^- F^\mu$. In terms of these, we have a Lorentzian signature path integral

$$G(\{V_n\}) \equiv \int [d\chi][d(ghosts)]d(\mathrm{mod}\,uli)e^{iS}\prod_n \Big(i\int d\sigma d\tau V_n[\chi]\Big), \quad (4.11)$$

where the semiclassical action is

$$iS = i\int d\sigma d\tau d\theta^+ d\theta^- \Big(D_{\theta^+}\chi^\mu D_{\theta^-}\chi^\nu G_{\mu\nu}(\chi) - \mu : e^{-\kappa\chi^0}\cos(\omega\tilde{\Omega}) : + (d)\Big) + iS_g, \quad (4.12)$$

where (d) is the dilaton and g of S_g is the ghost and $V_n[\chi]$ are vertex operator insertions. As in the heterotic case, the form of the vertex operators is known in the flat space region to be of the form (4.6). The dilaton is (4.7). Let us Wick rotate the worldsheet time coordinate τ, the spatial target space coordinates $\vec{X}(\sigma,\tau)$ (including $\tilde{\Omega}$), and the parameters μ and $\vec{k}$ by

$$\tau \equiv e^{i\gamma}\tau_\gamma \quad \vec{\chi} \equiv e^{i\gamma}\vec{\chi}_\gamma \quad \mu = e^{-i\gamma}\mu_\gamma \quad \vec{k} = e^{-i\gamma}\vec{k}_\gamma, \quad (4.13)$$

where γ is a phase which we will rotate from 0 to $\pi/2$. This produces a Euclidean path integral for the worldsheet theory (where we label the quantities rotated to $\gamma = \pi/2$ by a subscript E)

$$G(\{V_n\}) \equiv \int [d\vec{\chi}_E][d\chi^0][d(ghosts)]d\,(\text{moduli})\,e^{-S_E}\prod_n \int (-)d\sigma d\tau_E V_{n,-i\vec{k}_E}[\chi^0, i\vec{\chi}_E], \quad (4.14)$$

with Euclidean action

$$S_E = \int d\sigma d\tau_E d\theta^+ d\theta^- \Big(D_{\theta^+}\chi^0 D_{\theta^-}\chi^0 + v^2 (\chi^0)^2 D_{\theta^+}\tilde{\Omega} D_{\theta^-}\tilde{\Omega} + G_{ij} D_{\theta^+}\chi^i_{\perp,E} D_{\theta^-}\chi^j_{\perp,E}$$
$$- i\mu_E e^{-\kappa\chi^0} \cosh(\omega\tilde{\Omega}_E) + (d)\Big) + S_E(g), \quad (4.15)$$

where (d) is the dilaton and (g) is the ghost.

Now we compute the quantity $\partial Z_1/\partial\mu$ and perform the path integral by doing the integral over the X^0 zero modes first. That is, decompose

$$X^0 \equiv X^0_0 + \hat{X}^0(\sigma, \tau_E), \quad (4.16)$$

where $\hat{X}^0$ contains the nonzero mode dependence on the worldsheet coordinates σ, τ_E. The path integral measure then decomposes as $[dX^0] = dX^0_0 [d\hat{X}^0]$. We obtain for heterotic and type II respectively

$$\frac{\partial Z_1^{(Het)}}{\partial \mu_E} = \int [d\vec{\chi}_E][d\Psi_-][d(gh)]d(mo)[d\hat{\chi}^0] dX^0_0 \left(-\int d\sigma d\tau_E d\theta^+ \Psi_- e^{-\kappa\chi^0} i \cosh(w\tilde{\Omega}_E) \right) e^{-S_E}, \quad (4.17)$$

$$\frac{\partial Z_1^{(II)}}{\partial \mu_E} = \int [d\vec{\chi}_E][d(gh)]d(mo)[d\hat{\chi}^0] dX^0_0 \left(-\int d\sigma d\tau_E d\theta^+ d\theta^- e^{-\kappa\chi^0} i \cosh(w\tilde{\Omega}_E) \right) e^{-S_E}. \quad (4.18)$$

Decomposing $e^{-\kappa\chi^0} = e^{-\kappa X^0_0} e^{-\kappa\hat{\chi}^0}$, we can change variables in the zero mode integral to $y \equiv e^{-\kappa X^0_0}$ and integrate from $y = 0$ to $y = \infty$ as X^0_0 ranges from ∞ to $-\infty$. For each point in worldsheet field space, the zero mode integral is of the form

$$\int_0^\infty dy e^{-Cy} = \frac{1}{C}, \quad (4.19)$$

where the coefficient C is the nonzeromode part of the tachyon vertex operator in S_E, integrated over worldsheet superspace.

The analytic continuation (4.13) included a rotation $\mu = e^{-i\pi/2}\mu_E$. This means that as a function of our original parameter μ, we have an imaginary part in the partition function:

$$Z = \left(-\frac{1}{\kappa} \ln \frac{\mu}{\mu_*} + i \frac{\pi}{2\kappa} \right) \hat{Z}. \quad (4.20)$$

A thermal system is described in a real-time formalism by shifting time by i times half the inverse temperature: $t \to t + i\beta_T/2$. The result (4.20) arises from the bulk vacuum result via such a shift, with $\beta_T = \pi/\kappa$ corresponding to a temperature $T = \kappa/\pi$.

Now, we analyze the derivative of the correlation function (4.14) with respect to μ_E by doing the integral over X^0's mode X^0_0 first. From that we can determine its dependence on μ_E, and finally use (4.13) to determine its dependence on μ. This is similar to the computation of the partition function, except now the integral over $y = e^{-\kappa X^0_0}$ (which gave (4.19) in the case of the partition function) is of the form

$$\frac{\partial G(\{V_{n,-i\vec{k}_E}\})}{\partial \mu_E} = \int [d\hat{\chi}^0][d\vec{\chi}][d(g)] \int dy y^{\sum_n i\frac{\omega_n(\vec{k}_n)}{\kappa}} e^{-Cy} e^{-\hat{S}_E}. \quad (4.21)$$

This yields a result for $G(\{V_{n,-i\vec{k}_E}\})$ proportional to $\mu_E^{-i\sum_n \omega_n/\kappa}$ (4.22)
times a complicated path integral over nonzero modes, which would be difficult to evaluate directly. In the case of the 2-point function, we can use a simple aspect of the analytic continuation we used to define the path integral to determine the magnitude of the result. The two-point function of two negative frequency modes in the bulk is

$$G(\vec{k}, n; \vec{k}', n') = \delta_{nn'} \delta(\vec{k} - \vec{k}') \frac{\beta_{\vec{k},n}}{\alpha_{\vec{k},n}}, \quad (4.23)$$

where $\alpha_{\vec{k},n}$ and $\beta_{\vec{k},n}$ are the Bogoliubov coefficients describing the mixing of positive and negative frequency modes. This is the timelike Liouville analogue of the reflection coefficients describing the mixing of positive and negative momentum for modes bouncing off a spacelike Liouville wall.
After performing the Euclidean path integral defined via the rotations (4.13), we must continue back to $\mu = -i\mu_E$ in order to obtain the amplitude. The regions where the worldsheet potential is positive translate in the Euclidean path integral to a positive Liouville wall. For these regions, the Euclidean 2-point function is a reflection coefficient of magnitude 1. The physical two point function is given by continuing back in μ to the physical value. The continuation above (4.8) (4.13) in μ,

$$\mu \to e^{-i\frac{\pi}{2}} \mu \quad (4.24)$$

therefore yields a 2-point function of magnitude

$$\left| \frac{\beta_{\vec{k},n}}{\alpha_{\vec{k},n}} \right| = e^{-\omega(\vec{k},n)\pi/\kappa}. \quad (4.25)$$

Using the relations $|\alpha_\omega|^2 \mp |\beta_\omega|^2 = 1$ for bosonic and fermionic spacetime fields, and the fact that the number of particles produced $N_{\vec{k},n}$ is given by $|\beta_{\vec{k},n}|^2$, this result translates into a distribution of pairs of particles of a thermal form

$$N_{\vec{k},n} = \frac{1}{e^{2\pi\omega(\vec{k},n)/\kappa} \mp 1}. \quad (4.26)$$

This corresponds to a Boltzmann suppression of the distribution of pairs of particles by a temperature $T = \kappa/\pi$. This temperature is the same as that deduced from the imaginary part of the 1-loop partition function (4.20).
The state in the bulk $X^0 \to \infty$ region has a thermal distribution of pairs of particles (4.26), with temperature κ/π. These pairs are created during the phase where the tachyon condensate is order one, and hence the calculation is self-consistent if we tune the bare dilaton to weak coupling.
This choice of state is analogous to the Hartle-Hawking, or Euclidean, State in the theory of quantum fields on curved space, but it arises here in a perturbative string system via crucially stringy effects. In quantum field theory on curved space, the Euclidean vacuum is obtained by calculating Greens functions in the Euclidean continuation of the spacetime background

(when it exists) and continuing them back to Lorentzian signature. In our case, a similar continuation has been made, but here the Euclidean system is a spacelike Liouville field theory.

Now, we generalize our techniques to strings in geometries of the form (4.1) where the Ω are coordinates on higher dimensional spheres. The worldsheet theory will be described by an $O(N)$ model at an energy scale related to X^0.

Hence, we compute the mass gap of the $O(N)$ model. Consider N two-dimensional scalar fields arranged into an $O(N)$ vector $\vec{n}$. The partition function of the $O(N)$ model is

$$Z = \int [dn] e^{-\int d^2 z R^2 (\partial_\mu \vec{n})^2} \prod_z \delta(n^2(z) - 1). \quad (4.27)$$

A way to see the mass term appear is to use a Lagrange multiplier to enforce the delta function localizing the path integral onto a sphere, and large N to simplify the resulting dynamics:

$$Z = \int [dn] \int [d\lambda] e^{-\int d^2 z [R^2 \vec{n}(-\partial^2 + i\lambda)\vec{n} + i\lambda]}, \quad (4.28)$$

where λ is the Lagrange multiplier field introduced to represent the delta function. Now integrate out n:

$$Z = \int [d\lambda] e^{-N/2 tr \ln(-\partial^2 + \lambda) + R^2 \int d^2 z \lambda}. \quad (4.29)$$

At large N, the λ integral has a well-peaked saddle at

$$\lambda(x) = -im^2, \quad (4.30)$$

where the mass m satisfies

$$R^2 = N \int^\Lambda \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 + m^2} = \frac{N}{2\pi} \ln \frac{\Lambda}{m}. \quad (4.31)$$

Renormalize by defining the running coupling at the scale M by

$$R^2(M) = R_0^2 + \frac{N}{2\pi} \ln \Lambda / M. \quad (4.32)$$

Plugging back into the action for n, we have a mass for the n-field which runs like

$$m = M e^{-\frac{2\pi R^2}{N}}. \quad (4.33)$$

5. Mathematical connections

A. *Mathematical connections with Ramanujan's modular equations*. [5]

Now we consider the following Ramanujan's tau-function

$$\sum_{n=1}^{\infty} \tau(n) q^n = q(q;q)_{\infty}^{24} . \quad (5.1)$$

With regard the modulus 2 of $\tau(n)$, it is easy to see that the coefficients of q^n in the expansion of $q(q;q)_{\infty}^{24}$ and $q(q^8;q^8)_{\infty}^{3}$ are both odd or both even, where here and in the sequel

$$(a;q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n), \quad (5.2)$$

where $|q| < 1$. But

$$q(q^8;q^8)_{\infty}^{3} = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(2n+1)^2} .$$

It follows that $\tau(n)$ is odd or even according as n is an odd square or not. Thus we see that the number of values of n not exceeding n for which $\tau(n)$ is odd is only

$$\left[\frac{1+\sqrt{n}}{2} \right].$$

Recall that the Ramanujan function $\tau(n)$ is defined by the Fourier expansion of $\Delta(\tau)$, the unique normalized cusp form of weight 12 on $SL_2(Z)$. In particular, we have

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n . \quad (5.3)$$

Furthermore, Ramanujan have shown that the definite integral

$$\phi_w(t) = \int_0^{\infty} \frac{\cos \pi t x}{\cosh \pi x} e^{-\pi w x^2} dx ,$$

can be evaluated in finite terms if w is any rational multiple of i. Furthermore, this integral can be evaluated not only for these values but also for many other values of t and w. Now we have

$$\phi_w(t) = 2 \int_0^{\infty} \int_0^{\infty} \frac{\cos 2\pi x z}{\cosh \pi z} \cos \pi t x e^{-\pi w x^2} dx dz = \frac{e^{-\frac{\pi t^2 w'}{4}}}{\sqrt{w}} \int_0^{\infty} \frac{\cosh \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx , \quad (5.4a)$$

here w' stands for $1/w$. It follows that

$$\phi_w(t) = \frac{1}{\sqrt{w}} e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw'). \quad (5.4b)$$

Now, it is possible to obtain the π value utilizing the following expression

$$\phi_w(t) = \frac{e^{-\frac{\pi t^2 w'}{4}}}{\sqrt{w}} \int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx = \frac{1}{\sqrt{w}} e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw'), \quad (5.5)$$

$$e^{-\frac{\pi t^2 w'}{4}} \frac{1}{\sqrt{w}} \int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx = \phi_w(t),$$

$$e^{-\frac{\pi t^2 w'}{4}} = \frac{\phi_w(t)}{\frac{1}{\sqrt{w}} \int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx} = \frac{\frac{1}{\sqrt{w}} e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}{\frac{1}{\sqrt{w}} \int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx},$$

$$e^{-\frac{\pi t^2 w'}{4}} = \frac{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}; \quad \log \frac{\pi t^2 w'}{4} = \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')};$$

$$\pi = 4 \left[anti \log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{1}{t^2 w'}. \quad (5.6)$$

With regard the number 24, from the following Ramanujan's modular equation

$$\pi = \frac{24}{\sqrt{142}} \log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right],$$

for the eq. (5.6), we have that

$$\frac{24}{\sqrt{142}} \log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right] = 4 \left[anti \log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{1}{t^2 w'};$$

$$24 = \frac{4 \left[anti \log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]}. \quad (5.7)$$

With regard the number 12, from the following Ramanujan's modular equation

$$\pi = \frac{12}{\sqrt{130}} \log\left[\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right],$$ hence $$12 = \frac{\pi\sqrt{130}}{\log\left[\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right]},$$ from the eq. (5.6), we have that

$$12 = \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{130}}{t^2 w'}}{\log\left[\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right]}. \quad (5.8)$$

For the number 8, from the eq. (5.7), we obtain

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (5.9)$$

But, with regard the number 8, we have that, from the following Ramanujan's modular equation:

$$\pi = \frac{4}{\sqrt{522}} \log\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)^2 (5\sqrt{29}+11\sqrt{6}) \times \left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)} + \sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}^{7.5}\right],$$

we obtain:

$$8 = 2 \cdot \frac{\pi\sqrt{522}}{\log\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)(5\sqrt{29}+11\sqrt{6}) \times \left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)} + \sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}^{7.5}\right]},$$

hence, for the eq. (5.6), we have that:

$$8 = 2 \cdot \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'} \phi_{w'}(itw')}\right] \frac{\sqrt{522}}{t^2 w'}}{\log\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)(5\sqrt{29}+11\sqrt{6}) \times \left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)} + \sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}^{7.5}\right]}. \quad (5.10)$$

When a string moves in space-time by splitting and recombining, a large number of mathematical identities must be satisfied. These are the identities of Ramanujan's modular function (Ramanujan' modular equations).
The Ramanujan function, has 24 "modes" that correspond to the physical vibrations of a bosonic string. When the Ramanujan function is generalized, 24 is replaced by 8 (8 + 2 = 10), hence, has 8 "modes" that correspond to the physical vibrations of a superstring.
Now, we consider various equations and describe the possible mathematical connections with the Ramanujan's modular equations.
We take the eq. (1.20). It is possible the following connection with the eq. (5.7):

$$(c_L, c_R) = (24k_L, 24k_R) \Rightarrow 24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (5.11)$$

For the eq. (1.44), we have the following interesting connections with the eqs. (5.7) and (5.3):

$$24 = \frac{\Delta(\tau)^2}{\Delta(2\tau)\Delta(\tau/2)} - K(\tau) \Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \quad (5.12)$$

and

$$\Delta = q\prod_{n=1}^{24}(1-q^n)^{24} \Rightarrow \Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n. \quad (5.13)$$

Also the eqs. (1.55) and (1.56) can be related with the eqs. (5.7) and (5.3). For the eq. (1.55), we have that:

$$H_1 = \frac{q^{-1/2}}{2}\left(\prod_{n=1}^{\infty}(1+q^{n-1/2})^{24} - \prod_{n=1}^{\infty}(1-q^{n-1/2})^{24}\right) + 2048q\prod_{n=1}^{\infty}(1+q^n)^{24} =$$

$$= 24 + 4096q + 98304q^2 + 1228800q^3 + 10747904q^4 + \ldots =$$

$$= \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} + 4096q + 98304q^2 + 1228800q^3 + 10747904q^4 + \ldots \Rightarrow$$

$$\Rightarrow \Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n. \quad (5.14)$$

Furthermore, we have that $2048 = 8^3 \cdot 2^2$; $4096 = 8^3 \cdot 2^3$ and $98304 = 4096 \cdot 24$. We note that 2 and 8 are Fibonacci's number and that 8 and 24 are the "modes" corresponding to the physical vibrations of strings.

For the eq. (1.56), we have that:

$$24 = H_1 - 4096q\prod_{n=1}^{\infty}\left(1+q^n\right)^{24} \Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow \Delta(\tau) = q\prod_{n=1}^{\infty}\left(1-q^n\right)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n . \quad (5.15)$$

With regard the eq. (1.63):

$$\Psi(e) = \left[1+\frac{e^2}{4}\left(L_{-2}^+ + L_{-2}^-\right)+\frac{e^4}{32}\left(L_{-4}^+ + L_{-4}^-\right)+\frac{e^4}{32}\left(L_{-2}^+ + L_{-2}^-\right)^2+\frac{e^4}{192k}L_{-2}^+ L_{-2}^- + ...\right]\left|\Omega\right\rangle ,$$

we note that the number $32 = 4 \cdot 8 = 2^5$, while the number $192 = 24 \times 8 = 2^6 \cdot 3$, and that the numbers 8 and 24 are the "modes" corresponding to the physical vibrations of strings. Hence, we can write the following connections with the eqs. (5.7) and (5.10):

$$\Psi(e) = \left[1+\frac{e^2}{4}\left(L_{-2}^+ + L_{-2}^-\right)+\frac{e^4}{32}\left(L_{-4}^+ + L_{-4}^-\right)+\frac{e^4}{32}\left(L_{-2}^+ + L_{-2}^-\right)^2+\frac{e^4}{192k}L_{-2}^+ L_{-2}^- + ...\right]\left|\Omega\right\rangle \Rightarrow$$

$$\Rightarrow 2\cdot\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\frac{\sqrt{522}}{t^2 w'}}{\log\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)\left(5\sqrt{29}+11\sqrt{6}\right)\times\left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}^{7.5}\right]} \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (5.16)$$

In conclusion, with regard the expression $\ln 196883 \cong 12.19$, we have the connection with eq. (5.8):

$$\ln 196883 \cong 12.19 \cong \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{130}}{t^2 w'}}{\log\left[\frac{\left(2+\sqrt{5}\right)\left(3+\sqrt{13}\right)}{\sqrt{2}}\right]}, \quad (5.17)$$

furthermore, for the following Ramanujan identity

$$(c)^{31} - \left\{\frac{1}{2\times 5}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\} \approx 12,$$

we have also the following connection:

$$\ln 196883 \cong 12.19 \cong (c)^{31} - \left\{\frac{1}{2\times 5}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}. \quad (5.18)$$

With regard the p-adic Hartle-Hawking wave function concerning the de Sitter minisuperspace model in D = 4 space-time dimensions, we have the following equations:

$$\Psi_p(q) = \int_{|T|_p\le 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2 T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right), \quad (5.18a)$$

$$\psi_p(q) = \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left[-\frac{\lambda^2 T^3}{24}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right], \quad (5.18b)$$

Hence, the following connections with the equation (5.7):

$$\int_{|T|_p\le 1} dT \frac{\lambda_p(-8T)}{|4T|_p^{1/2}}\chi_p\left(-\frac{\lambda^2 T^3}{24}+(\lambda q-2)\frac{T}{4}+\frac{q^2}{8T}\right)\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left[-\frac{\lambda^2T^3}{24}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right]\Rightarrow$$

$$\Rightarrow \int_{Q_p} dx\chi_p(qx)\int DT\chi_p\left\{-\lambda^2T^3\frac{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}+\left(\frac{\lambda q}{4}-\frac{1}{2}-2x^2\right)T\right\}. \quad (5.18c)$$

B. *Mathematical connections with Lemma 3 of Goldston-Montgomery Theorem and some equations of the Riemann zeta function.* [6] [7]

Now we describe the possible mathematical connections with some equations concerning the Goldston-Montgomery theorem and the Riemann zeta function.
With regard the Goldston-Montgomery theorem, we take the equations concerning the Lemma 3.
Let $f(t)\geq 0$ a continuous function defined on $[0,+\infty)$ such that $f(t)<<\log^2(t+2)$. If

$$I(k)=\int_0^\infty\left(\frac{\sin ku}{u}\right)^2 f(u)du=\left(\frac{\pi}{2}+\varepsilon'(k)\right)k\log\frac{1}{k}, \quad (5.19)$$

then

$$J(T)=\int_0^T f(t)dt=(1+\varepsilon')T\log T, \quad (5.20)$$

with $|\varepsilon'|$ small if $|\varepsilon(k)|\leq\varepsilon$ uniformly for $\frac{1}{T\log T}\leq k\leq\frac{1}{T}\log^2 T$.

With regard the Riemann zeta function, we take some equations concerning the study of the behaviour of the argument of the Riemann function $\zeta(s)$ with the condition that s lies on the critical line $s=\frac{1}{2}+it$, where t is real.
We have:

$$\int_T^{T+H}\left|S(t)+\frac{1}{\pi}\sum_{p<x^3}\frac{\sin(t\log p)}{\sqrt{p}}\right|^{2k}dt=O\left(\sum_{j=1}^4 K_j\right), \quad (5.21)$$

where

$$K_1=\int_T^{T+H}\left|\sum_{p<x^3}\frac{\Lambda(p)-\Lambda_x(p)}{\sqrt{p}\log p}p^{-it}\right|^{2k}dt, \quad K_2=\int_T^{T+H}\left|\sum_{p<x^{1.5}}\frac{\Lambda_x(p^2)}{p\log p}p^{-i2t}\right|^{2k}dt,$$

$$K_3 = (\log T)^{2k} \int_T^{T+H} \left(\sigma_{x,t} - \frac{1}{2} \right)^{2k} dt ,$$

$$K_4 = \int_T^{T+H} \left(\sigma_{x,t} - \frac{1}{2} \right)^{2k} x^{2k\left(\sigma_{x,t}-\frac{1}{2}\right)} \times \left\{ \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right| du \right\}^{2k} dt . \quad (5.22)$$

Applying Cauchy's inequality to K_4, we obtain

$$K_4 << \left\{ \int_T^{T+H} \left(\sigma_{x,t} - \frac{1}{2} \right)^{4k} x^{4k\left(\sigma_{x,t}-\frac{1}{2}\right)} dt \right\}^{1/2} \times \left\{ \int_T^{T+H} \left(\int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right| du \right)^{4k} dt \right\}^{1/2} . \quad (5.23)$$

The second integral in (5.23) is estimated by

$$I_2 << (\log x)^{-4k+1} \int_T^{T+H} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right|^{4k} dtdu =$$

$$= (\log x)^{4k+1} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left(\int_T^{T+H} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it} \log^2 x} \right|^{4k} dt \right) du << H (\log x)^{4k} . \quad (5.24)$$

We take the eqs. (3.4) and (3.11), we obtain the following connections:

$$S = \int d^3x \sqrt{-g} e^{-2\phi} \left[\frac{4}{k} + R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] \Rightarrow$$

$$\phi = -\frac{1}{2} \ln \hat{r} l \Rightarrow (1+\varepsilon') T \log T = \int_0^T f(t) dt \Rightarrow \left\{ \int_T^{T+H} \left(\int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right| du \right)^{4k} dt \right\}^{1/2} <<$$

$$<< (\log x)^{-4k+1} \int_T^{T+H} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right|^{4k} dtdu =$$

$$= (\log x)^{4k+1} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left(\int_T^{T+H} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it} \log^2 x} \right|^{4k} dt \right) du << H (\log x)^{4k} . \quad (5.25)$$

From eq. (4.20), we have that:

$$-\frac{1}{\kappa} \ln \frac{\mu}{\mu_*} \hat{Z} = Z - i \frac{\pi}{2\kappa} \hat{Z} ,$$

hence, the following connections:

$$Z - i \frac{\pi}{2\kappa} \hat{Z} = -\frac{1}{\kappa} \ln \frac{\mu}{\mu_*} \hat{Z} \Rightarrow (1+\varepsilon') T \log T = \int_0^T f(t) dt \Rightarrow$$

$$\Rightarrow \left\{ \int_T^{T+H} \left(\int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right| du \right)^{4k} dt \right\}^{1/2} <<$$

$$<< (\log x)^{-4k+1} \int_T^{T+H} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right|^{4k} dt du =$$

$$= (\log x)^{4k+1} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left(\int_T^{T+H} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it} \log^2 x} \right|^{4k} dt \right) du << H(\log x)^{4k} \ . \quad (5.26)$$

Furthermore, we take the eq. (4.31) and we obtain the following connections:

$$R^2 = N \int^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{k^2+m^2} = \frac{N}{2\pi} \ln \frac{\Lambda}{m} \Rightarrow (1+\varepsilon')T \log T = \int_0^T f(t) dt \Rightarrow$$

$$\left\{ \int_T^{T+H} \left(\int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right| du \right)^{4k} dt \right\}^{1/2} <<$$

$$<< (\log x)^{-4k+1} \int_T^{T+H} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it}} \right|^{4k} dt du =$$

$$= (\log x)^{4k+1} \int_{0.5}^{\infty} x^{\frac{1}{2}-u} \left(\int_T^{T+H} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log(xp)}{p^{u+it} \log^2 x} \right|^{4k} dt \right) du << H(\log x)^{4k} \ . \quad (5.27)$$

C. *Mathematical connections between p-adic Hartle-Hawking wave function, three-dimensional gravity and Euclidean State in the theory of quantum fields on curved space arising in a perturbative string system where the Euclidean system is a spacelike Liouville field theory*. [8]

In the de Sitter minisuperspace model in D = 3 dimensions, the p-adic Hartle-Hawking wave function is:

$$\Psi_p(a) = \int_{|N|_p \le 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda} \coth(N\sqrt{\lambda})}{2} a^2 \right). \quad (5.28)$$

Now, we take the eqs. (1.11), (1.15), (1.16), (1.18) and (1.19). We have the following possible connections:

$$I = \frac{k}{4\pi} \int_W tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \Rightarrow$$

$$\Rightarrow k_L I_L + k_R I_R = \frac{k_L}{4\pi} \int tr \left(A_L \wedge dA_L + \frac{2}{3} A_L \wedge A_L \wedge A_L \right) - \frac{k_R}{4\pi} \int tr \left(A_R \wedge dA_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right) \Rightarrow$$

$$\Rightarrow \frac{\ell}{16G} \left[\frac{1}{\pi \ell} \int d^3x \sqrt{g} \left(R + \frac{2}{\ell^2} \right) \right] + (k_L - k_R) \frac{(I_L + I_R)}{2} \Rightarrow$$

$$\Rightarrow \int_{|N|_p \le 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda} \coth(N\sqrt{\lambda})}{2} a^2 \right). \quad (5.29)$$

Furthermore, for the equation (5.6), we can also write the following connections:

$$I = \frac{k}{4\pi}\int_W tr\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \Rightarrow$$

$$\Rightarrow k_L I_L + k_R I_R = \frac{k_L}{4\pi}\int tr\left(A_L \wedge dA_L + \frac{2}{3}A_L \wedge A_L \wedge A_L\right) - \frac{k_R}{4\pi}\int tr\left(A_R \wedge dA_R + \frac{2}{3}A_R \wedge A_R \wedge A_R\right) \Rightarrow$$

$$\Rightarrow \frac{\ell}{16G}\left\{\frac{1}{\ell 4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi x w'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{1}{t^2 w'}}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right)\right\} + (k_L - k_R)\frac{(I_L + I_R)}{2} \Rightarrow$$

$$\Rightarrow \int_{|N|_p \le 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}}\chi_p\left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth(N\sqrt{\lambda})}{2}a^2\right). \quad (5.30)$$

Now, we take the eq. (1.58). We have the following connection with the eq. (5.28):

$$\left(\int_{S_+} V_n T - \int_{S_-} V_n T\right)\Psi = \left(\int_{S_+} V_n T - \int_{S_-} V_n T\right)\int_{|N|_p \le 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}}\chi_p\left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth(N\sqrt{\lambda})}{2}a^2\right) = 0. \quad (5.31)$$

With regard the eq. (1.63), we obtain the following connections with the eq. (5.28):

$$\left[1 + \frac{e^2}{4}\left(L_{-2}^+ + L_{-2}^-\right) + \frac{e^4}{32}\left(L_{-4}^+ + L_{-4}^-\right) + \frac{e^4}{32}\left(L_{-2}^+ + L_{-2}^-\right)^2 + \frac{e^4}{192k}L_{-2}^+ L_{-2}^- + \dots\right]|\Omega\rangle =$$

$$= \int_{|N|_p \le 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}}\chi_p\left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth(N\sqrt{\lambda})}{2}a^2\right), \quad (5.32)$$

$$\left[1 + \frac{e^2}{4}\left(L_{-2}^+ + L_{-2}^-\right) + \frac{e^4}{32}\left(L_{-4}^+ + L_{-4}^-\right) + \frac{e^4}{32}\left(L_{-2}^+ + L_{-2}^-\right)^2 + \frac{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi x w'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}\cdot\right.$$

$$\left.\cdot\frac{e^4}{8k}L_{-2}^+ L_{-2}^- + \dots\right]|\Omega\rangle = \int_{|N|_p \le 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}}\chi_p\left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth(N\sqrt{\lambda})}{2}a^2\right). \quad (5.33)$$

Now, we take the eq. (2.23b). It is possible to obtain the following connection with the eq. (5.28):

$$\hat{I} = \frac{1}{\hbar}\int_M \varepsilon^{ijk}\left[e_{ia}\left(\partial_j \omega_k^a - \partial_k \omega_j^a\right) + \varepsilon_{abc} e_i^a \omega_j^b \omega_k^c + \frac{\lambda}{3}\varepsilon_{abc} e_i^a e_j^b e_k^c \right] +$$
$$+ \frac{ik}{8\pi}\int d^3 x \varepsilon^{jkl}\left[\omega_j^a \left(\partial_k \omega_l^a - \partial_l \omega_k^a + \frac{2}{3}\varepsilon_{abc}\omega_k^b \omega_l^c \right) + \lambda e_j^a \left(\partial_k e_l^a - \partial_l e_k^a\right) + 2\lambda \varepsilon_{abc}\omega_j^a e_k^b e_l^c \right] \Rightarrow$$
$$\Rightarrow \int_{|N|_p \le 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth\left(N\sqrt{\lambda}\right)}{2} a^2 \right). \quad (5.34)$$

With regard the eq. (4.10), also in this case it is possible to obtain the following connection with the eq. (5.28):

$$S_E = \int d\sigma d\tau_E d\theta^+ \Big(D_{\theta^+}\chi^0 \partial_- \chi^0 + v^2 \left(\chi^0\right)^2 D_{\theta^+}\tilde{\Omega}_E \partial_- \tilde{\Omega}_E + G_{ij} D_{\theta^+}\chi^i_{\perp,E}\partial_- \chi^j_{\perp,E}$$
$$- i\mu_E e^{-\kappa\chi^0}\cosh\left(w\tilde{\Omega}_E\right) + \Psi_-^a D_{\theta^+}\int_{|N|_p \le 1} dN \frac{\lambda_p(-2N)}{|N|_p^{1/2}} \chi_p \left(-\frac{N}{2} + \frac{\sqrt{\lambda}\coth\left(N\sqrt{\lambda}\right)}{2} a^2 \right) + (d) \Big) + S_E(g).$$

(5.35)

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