The Math Underlying the Schrodinger Equation

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1 Properties of light

The observed speed of light (c) is 300,000km/second.

$$c = 300,000km/second \tag{1}$$

If we think of light as coming in waves, then the frequency is the number of waves that pass by each second.

$$f = waves/second \tag{2}$$

The wavelength (λ) can be calculated by taking the distance light travels in one second (300,000km), and dividing that by the number of waves.

$$\lambda = 300,000km/waves \tag{3}$$

How they relate is, the frequency times the wavelength is equal to the speed of light.

$$f \times \lambda = c \tag{4}$$

$$\frac{waves}{second} \times \frac{300,000km}{waves} = \frac{300,000km}{second} \tag{5}$$

If we work in units of radians (multiply the left side of the equation by $2\pi/2\pi$ which is 1), then

$$\frac{radians}{second} \times \frac{300,000km}{radians} = \frac{300,000km}{second} \tag{6}$$

And if we multiply both sides by radians/300,000km (the inverse of the radian-length) then we get an interesting equation.

$$\frac{radians}{second} = \frac{300,000km}{second} \times \frac{radians}{300,000km} \tag{7}$$

This equation tells us that the change in time is equal to the speed of light times the change in space.

$$\Delta time = c \times \Delta space \tag{8}$$

The variables ω and k are used to represent the Δtime and Δspace respectively.

$$\omega = ck \tag{9}$$

2 The energy of light

Early in the 20th century, Albert Einstein learned through experiments, that the number of radians/second (ω) is directly proportional to the energy. More specifically, the energy (E) is equal to the constant $\hbar$ (pronounced h-bar) times ω.

$$E = \hbar\omega \tag{10}$$

And similarly, the momentum (p) is equal to $\hbar$ times k.

$$p = \hbar k \tag{11}$$

3 Modeling the behavior of light

Below is a fundamental representation of the behavior of light.

$$\Psi = e^{i(kx - \omega t)} \tag{12}$$

[Note the "ωt" term above is subtracted to stay consistent with accepted math, and since technically this term can be added or subtracted as long as the measurement direction is assigned appropriately.]

This is a somewhat mysterious equation, but we know some things about it. We can see that it uses the change in time and space (k and ω) information. And since $E = \hbar\omega$ (formula 10), then

$$\omega = E/\hbar \tag{13}$$

And since $p = \hbar k$ (formula 11)

$$k = p/\hbar \tag{14}$$

So our fundamental equation can be rewritten in a form that is easier to work with.

$$\Psi = e^{i(\frac{p}{\hbar}x - \frac{E}{\hbar}t)} \tag{15}$$

4 Deriving a complex energy equation

We also know something about the derivative of Ψ (with respect to time). The first derivative of Ψ ($d\Psi/dt$) is defined to be how Ψ changes in time - which we know is measured in radians/second.

$$\frac{d\Psi}{dt} = \frac{\Delta\Psi}{\Delta t} = radians/second \tag{16}$$

So we can take the first derivative of Ψ

$$\frac{d\Psi}{dt} = -i\frac{E}{\hbar}\Psi \tag{17}$$

And then multiply both sides by $i\hbar$ to get an energy formula for Ψ ($E\Psi$) based on the number of radians/second.

$$\begin{aligned} i\hbar\frac{d\Psi}{dt} &= E\Psi & (18)\\ E\Psi &= i\hbar\frac{d\Psi}{dt} & (19)\\ E\Psi &= i\hbar \times \text{radians/second} & (20) \end{aligned}$$

The energy equation by Einstein ($E = \hbar\omega$) also says that the amount of energy can be calculated by multiplying $\hbar$ times the number of radians/second. Only the $E\Psi$ formula carries polarity information (i).

5 Schrodinger's kinetic energy formula

Schrodinger knew the formulas for the energy ($E\Psi$) and momentum ($p\Psi$). He also knew that the momentum (mv) was the derivative of the kinetic energy $\left(\frac{1}{2}mv^2\right)$.

$$\frac{d}{dv}\left[\frac{1}{2}mv^2\right] = mv \tag{21}$$

So Schrodinger may have reasoned, since the first derivative of Ψ yielded the momentum formula, one more derivative ($d^2\Psi/dx^2$) should yield the kinetic energy of Ψ (KEΨ).

It almost worked, however the second derivative of Ψ yields.

$$\frac{d^2\Psi}{dx^2} = \frac{c^2m^2v^2\Psi}{\hbar^2} = \frac{-m^2v^2\Psi}{\hbar^2} \tag{22}$$

and it is close, but not equal to the kinetic energy.

$$\frac{-m^2v^2\Psi}{\hbar^2} \neq \frac{1}{2}mv^2\Psi \tag{23}$$

The problem was, the momentum is the derivative of the kinetic energy with respect to the velocity (v). The derivative of Ψ is with respect to $imv/\hbar$ - and the two are not equal.

$$\frac{imv}{\hbar} \neq v \tag{24}$$

So if you use Ψ to find the kinetic energy, then you need a "patch" backing out two unneeded factors of i and $\hbar$, one too many factors of m, and 1/2.

$$\text{patch} = \frac{\hbar^2}{i^2m2} = -\frac{\hbar^2}{2m} \tag{25}$$

But if you multiply the second derivative of Ψ $(d^2\Psi/dx^2)$ by the patch, you get the kinetic energy.

$$\frac{d^2\Psi}{dx^2} \times \text{patch} = KE\Psi \tag{26}$$

$$\frac{-m^2v^2}{\hbar^2}\Psi \times -\frac{\hbar^2}{2m} = \frac{1}{2}mv^2\Psi \tag{27}$$

$$\frac{d^2\Psi}{dx^2} \times -\frac{\hbar^2}{2m} = KE\Psi \tag{28}$$

$$KE\Psi = -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} \tag{29}$$

6 The Schrodinger equation

The kinetic energy of Ψ can be added to any potential energy of Ψ ($PE\Psi$) to get the total energy.

$$\text{Total energy of } \Psi = KE\Psi + PE\Psi \tag{30}$$

The Schrodinger equation sets $E\Psi$ (the energy equation based on the radians/second) equal to his kinetic energy formula plus any potential energy (written as $\upsilon\Psi$).

$$E\Psi = KE\Psi + PE\Psi \tag{31}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}\Psi + \upsilon\Psi \tag{32}$$