IFS$_\alpha$-Open Sets in Intuitionistic Fuzzy Topological Space

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Abstract

The aim of this paper is to introduce the concepts of IFS$_\alpha$-open sets. Also we discussed the relationship between this type of Open set and other existing Open sets in Intuitionistic fuzzy topological spaces. Also we introduce new class of closed sets namely IFS$_\alpha$-closed sets and its properties are studied.

Key words : IF-semi open sets, IFα-closed sets, IFS$_\alpha$-open sets.

1 Introduction

In 1963 Levine initiated semi open set and gave their properties. Mathematicians gave in several papers interesting and different new types of sets. In 1965, O. Njastad introduced α -closed sets and in 2014 A. Alex Francis Xavier introduced Sα-closed sets in topological space.

2 Preliminaries

Throughout this paper (X, τ) (or briefly X) represent Intuitionistic fuzzy topological spaces on which no separation axioms are assumed unless otherwise mentioned.

Definition 2.1. [1] *An Intuitionistic fuzzy topology* (IFT) *on a non empty set X is a family* τ *of IFS in X satisfying the following axioms*
(T_1) $0_\backsim, 1_\backsim \in \tau$
(T_2) $G_1 \cap G_2 \in \tau$, *for any* $G_1, G_2 \in \tau$
(T_3) $\bigcup G_i \in \tau$, *for any arbitrary family* $\{G_i : G_i \in \tau, i \in I\}$
In this case the pari (X, τ) *is called an* ***Intuitionistic Fuzzy Topological Space*** *and any IFS in* τ *is known as* ***Intuitionistic Fuzzy Open Set*** *in X.*

Example 2.2. *Let* $X = \{a, b, c\}$
$A = \langle x, (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.4}) \rangle$
$B = \langle x, (\frac{a}{0.4}, \frac{b}{0.6}, \frac{c}{0.2}), \frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.3}) \rangle$
$C = \langle x, (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.4}), (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3}) \rangle$
$D = \langle x, \frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.2}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4} \rangle)$
Then the family $\tau = \{0_\backsim, 1_\backsim, A, B, C, D\}$ *of IFTs in X is an IFT on X.*

Definition 2.3. *An IFS A of an IFTS X is said to be*

(1) ***IF-α-open***[5] *if* $A \subseteq IFInt(IFCl(IFInt(A)))$.

(2) ***IF-semi-open***[3] *(IFSO) if* $A \subseteq IFCl(IFInt(A))$

(3) ***IF-pre-open***[2] *(IFPO) if* $A \subseteq IFInt(IFCl(A))$.

The complement of an IFα O, IFβ O, IFSO, IFPO is said to be IFα C, IFβ C, IFSC, IFPC.

Definition 2.4. [4] *An IFTS X is said to be* ***IF-locally indiscrete*** *if every IFOS of X is IFCS.*

Definition 2.5. [4] *An IFTS X is said to be* ***IF-hyper-connected space*** *if every non empty IFOS of X is IF-dense in X.*

Definition 2.6. [6] *An IFS A in an IFTS X is said to be* ***IF-dense*** *if there exists no IFCS B in X such that* $A < B < 1_\backsim$.

Definition 2.7. [2] *An IFS A in an IFTS X is said to be* ***IF-regular open*** *(IFRO) if* $A = IFInt(IFCl(A))$.

3 IFS$_\alpha$-Closed Sets

Definition 3.1. *An IFSO A of an IFTS X is said to be* ***IFS$_\alpha$ O*** *if for each x* $\in$ *A, there exists an IFα-closed set F such that* $x \in F \subset A$.

An IFS B of a IFTS X is ***IFS$_\alpha$*** *C, if* $X \setminus B$ *is IFS$_\alpha$ O.*
The family of IFS$_\alpha$ O of X is denoted by IFS$_\alpha$ $O(X)$.

Theorem 3.2. *An IFS A of an IFTS X is IFS$_\alpha$ O if and only if A is IFSO and it is a union of IFα-closed.*

Proof. Let A be an IFS$_\alpha$ O. Then A is IFSO $x \in$ A implies, there exists IFα-closed set F$_x$ such that $x \in$ F$_x \subset$ A. Hence $\cup_{x\in A}$ F$_x \subset$ A. But $x \in$ A, $x \in$ F$_x$ implies A $\subset \bigcup_{x\in A}$ F$_x$. This completes one half of the proof.

Let A be IFSO and A $= \bigcup_{i\in I}$ F$_i$, where each F$_i$ is a IFα-closed. Let $x \in$ A. Then $x \in$ some F$_i \subset$ A. Hence A is IFS$_\alpha$ O. □

The following result shows that any union of IFS $_\alpha$ O is IFS $_\alpha$ O.

Theorem 3.3. *Let* $\{A_\alpha : \alpha \in \Delta\}$ *be a family of IFS*$_\alpha$ *O in an IFTS X. Then* $\bigcup_{\alpha\in\Delta} A_\alpha$ *is an IFS*$_\alpha$ *O.*

Proof. WKT, The union of an arbitrary IFSO is IFSO. Suppose that $x \in \bigcup_{\alpha\in\Delta} A_\alpha$. This implies that there exists $\alpha_0 \in \Delta$ such that $x \in A_{\alpha_0}$ and as A_{α_0} is an IFS α O, there exists a IF α CS F in X such that $x \in F \subset A_{\alpha_0} \subset \bigcup_{\alpha\in\Delta} A_\alpha$. Therefore $\bigcup_{\alpha\in\Delta} A_\alpha$ is a IFS $_\alpha$ O. □

From this theorem, it is clear that any intersection of IFS $_\alpha$ C of a IFTS X is IFS $_\alpha$ C.

Theorem 3.4. *An IFS G of the IFTS X is IFS*$_\alpha$ *O if and only i for each* $x \in G$*, there exists an IFS*$_\alpha$ *O H such that* $x \in H \subset G$*.*

Proof. Let G be an IFS $_\alpha$ O in X. Then for each $x \in$ G, we have G is an IFS $_\alpha$ O such that $x \in G \subset G$.

Conversely, let for each $x \in$ G, there exists an IFS $_\alpha$ O H such that $x \in H \subset G$. Then G is a union of IFS $_\alpha$ O, hence by Theorem 3.3, G is an IFS $_\alpha$ O. □

Theorem 3.5.

1. *IF-Regular Closed set is IFS*$_\alpha$ *O.*
2. *IF-Regular Open set is IFS*$_\alpha$ *C.*

Proof. (1) Let A be an IF-Regular closed in a IFTS X. A = IFCl(IFIntA). A is IFSO. A is IF α-closed. $x \in$ A implies $x \in A \subset A$. Hence A is IFS $_\alpha$ O.

(2) Obivious. □

Theorem 3.6. *If an IFTS X is a IF-*T_1*-space, then IFS*$_\alpha$ $(X) = IFSO(X)$*.*

Proof. Clearly, IFS $_\alpha$ (X) $\subset$ IFSO(X). Let A $\in$ IFSO(X). Let $x \in$ A. Since X is a IF-T $_1$-space, $\{x\}$ is IFCS. Every IFCS in X is a IF α C. Hence $x \in \{x\} \subset$ A $\in$ IFS $_\alpha$ O(X). This completes the proof. □

Theorem 3.7. *If the family of all IFSO of an IFTS is a IFT on X, then the family of IFS*$_\alpha$ *O is also a IFT on X.*

Proof. Obvious. □

Theorem 3.8. *If an IF-space X is IF-hyperconnected, then then only IFS*$_\alpha$ *O of X are* $\emptyset$ *and X.*

Proof. Let A $\subset$ X such that A is IFS $_\alpha$ O in X. If A = X, there is nothing to prove. If A $\neq$ X, we have to prove that A = $\emptyset$. Since A is IFS $_\alpha$ O, for each $x \in$ A, there exists a IF α-closed set F such that $x \in F \subset A$. So X $\setminus$ A $\subset$ X $\setminus$ F. X $\setminus$ A is an IF-semi closed. Therefore, IFInt(IFCl(X $\setminus$ A) $\subset$ X $\setminus$ A. Since S is IF-hyper-connected, then IF-SCl(IFInt(IFCl(X $\setminus$ A))) = X $\subset$ X $\setminus$ A. Hence X $\setminus$ A = X. So A=$\emptyset$. □

Theorem 3.9. *If an IFTS X is IF-locally indiscrete, then every IFSO is IFS*$_\alpha$ *O.*

Proof. Let A be an IFSO in X. Then A $\subset$ IFCl(IFInt A). Since X is IF-locally indiscrete, IFInt A is IFCS. Hence IFInt A = IFCl(IFIntA). So, IFCl(IFInt A) = IFInt A $\subset$ A. So A is IF-Regular closed. By Theorem 2.1.6, A is IFS$_\alpha$O. □

Theorem 3.10. *If an IFTS* (X, τ) *is IF-*T_1 *or IF-locally indiscrete, then* $\tau \subset$ *IFS*$_\alpha$ $O(X)$.

Proof. Let (X, τ) be IF-T$_1$. As every IFOS is IFSO, $\tau \subset$ IFSO(X), IFSO(X) = IFS$_\alpha$O(X). Thus, $\tau \subset$ IFS$_\alpha$O(X).
Let (X, τ) be IF-locally indiscrete, then $\tau \subset$ IFSO(X) $\subset$ IFS$_\alpha$O(X). □

Theorem 3.11. *If B is an IF-clopen subset of a IF-space X and A is IFS*$_\alpha$ *O in X, then* $A \cap B \in$ *IFS*$_\alpha$ $O(X)$.

Proof. Let A be an IFS$_\alpha$O. So A is IFSO. B is IFOS and IFCS in X. Then A$\cap$B is IFSO in X. Let $x \in$ A $\cap$ B. Then $x \in$A and $x \in$B. Since A is IFS$_\alpha$O, there exists a IFα-closed set F such that $x \in$F$\subset$A. B is IFCS and hence IFα-closed. F $\cap$ B is IFα-closed. $x \in$F$\cap$B$\subset$A$\cap$B. So A $\cap$ B is IFS$_\alpha$O. □

Theorem 3.12. *Let X be an IF-locally indiscrete and* $A \subset X$, $B \subset X$. *If* $A \in$ *IFS*$_\alpha$ $O(X)$ *and B is IFOS, and then* $A \cap B$ *is IFS*$_\alpha$ *O in X.*

Proof. Follows from previous theorem. □

Theorem 3.13. *Let X be IF-extremely disconnected and* $A \subset X$,$B \subset X$. *If* $A \in$ *IFS*$_\alpha$ $O(X)$ *and* $B \in$ *IFRO*(X) *then* $A \cap B$ *is IFS*$_\alpha$ *O in X.*

Proof. Let A $\in$ IFS$_\alpha$O(X) and B $\in$ IFRO(X). Then A is IFSO. Hence, A$\cap$B$\in$ IFSO(X). Let $x \in$ A $\cap$ B. This implies $x \in$ A and $x \in$ B. As A is IFS$_\alpha$O, there exists a IFα-closed set F such that $x \in$F$\subset$A. X is IF-extremely disconnected, B is a IF-Regular closed set. This implies F $\cap$ B is IFα-closed. $x \in$ F $\cap$ B $\subset$ A $\cap$ B. So A$\cap$B is IFS$_\alpha$O. □

4 IFS$_\alpha$-Operations

Definition 4.1. *An IFS N of a IFTS X is called* ***IFS***$_\alpha$ ***-neighbourhood*** *of an IFS A of X, if there exists an IFS*$_\alpha$ *O U such that* $A \subset U \subset N$.
When $A = \{x\}$, *we say N is a IFS*$_\alpha$*-neighbourhood of x.*

Definition 4.2. *An IF-point* $x \in X$ *is said to be an* ***IFS***$_\alpha$ ***-interior point of A****, if there exists an IFS*$_\alpha$ *O U containing x such that* $x \in U \subset A$. *The set of all IFS*$_\alpha$*-interior points of A is said to be IFS*$_\alpha$*-interior of A and it is denoted by IFS*$_\alpha$*-Int A.*

Theorem 4.3. *Let A be any IFS of an IFTS X. If x is a IFS*$_\alpha$*-interior point of A, then there exists a IF-semi closed set F of X containing x such that* $F \subset A$.

Proof. Let $x \in$ IFS$_\alpha$-Int A. Then there exists an IFS$_\alpha$O U containing x such that U $\subset$ A. Since U is an IFS$_\alpha$O, there exists a IFα-closed set F of X such that $x \in$ F $\subset$ U $\subset$ A. □

Theorem 4.4. *For any IFS A of an IFTS X, the statements are true.*

1. *The IFS_α-interior of A is the union of all IFS_α O contained in A.*

2. *IFS_α-Int A is the largest IFS_α O contained in A.*

3. *A is IFS_α O if and only if A = IFS_α-Int A.*

Proof. Obvious. □

From 3, are see IFS_α-Int(IFS_α-Int A) = IFS_α-Int A.

Theorem 4.5. *If A and B are any IFS of a IFTS X. Then*

1. *IFS_α-Int $\emptyset = \emptyset$ and IFS_α-Int $X = X$.*

2. *IFS_α-Int $A \subset A$.*

3. *If $A \subset B$, then IFS_α-Int $A \subset IFS_\alpha$-Int B.*

4. *IFS_α-Int A $\cup$ IFS_α-Int B $\subset$ IFS_α-Int $(A \cup B)$.*

5. *IFS_α-Int $(A \cap B)$ $\subset$ IFS_α-Int A $\cap$ IFS_α-Int B.*

6. *IFS_α-Int $(A \setminus B)$ $\subset$ IFS_α-Int A $\setminus IFS_\alpha$-Int B.*

Proof. 1 - 5, Obvious.
(6) Let $x \in$ IFS_α-Int(A $\setminus$ B). There exists IFS_αO U such that $x \in$ U $\subset$ A $\setminus$ B. That is U $\subset$ A. U $\cap$ B = $\emptyset$ and $x \notin$ B. Hence $x \in$ IFS_α-Int A, $x \notin$ IFS_α-Int B. Hence $x \in$ IFS_α-IntA $\setminus$ IFS_α-IntB. This completes the proof. □

Definition 4.6. *Intersection of IFS_α-closed set containing F is called* ***IFS_α-closure of F*** *and is denoted by IFS_α-Cl F.*

Theorem 4.7. *Let A be an IFS of an IFTS X. $x \in X$ is in IFS_α-closed of A if and only if $A \cap U \neq \emptyset$, for every IFS_α O U containing x.*

Proof. To prove the theorem, let us prove contra positive.
$x \notin$ IFS_αCl A $\Leftrightarrow$ There exists an IFS_αO U containing x that does not intersect A. Let $x \notin$ IFS_αCl A. X $\setminus$ IFS_αCl A is an IFS_αO containing x that does not intersect A. Let U be an IFS_αO containing x that does not intersect A. X $\setminus$ U is an IFS_α-closed set containing A.
IFS_αCl A $\subset$ X $\setminus$ U. $x \notin$ X $\setminus$ U $\Rightarrow$ $x \notin$ IFS_αCl A. □

Theorem 4.8. *Let A be any IFS of a IF-space X. $A \cap F \neq \emptyset$, for every IFα-closed set F of X containing x, then the IF-point x is in the IFS_α-closure of A.*

Proof. Let U be any IFS_αO containing x. So, there exists an IFα-closed set F such that $x \in$ F $\subset$ U. A $\cap$ F $\neq \emptyset$ implies A $\cap$ U $\neq \emptyset$, for every IFS_αO U containing x. Hence $x \in$ IFS_αCl A, by previous theorem. □

Theorem 4.9. *For any IFS F of a IFTS X, the following are true.*

1. *IFS_α Cl F is the intersection of all IFS_α-closed set in X containing F.*

2. *IFS_α Cl F is the smallest IFS_α-closed set containing F.*

3. F is IFS_α-closed if and only if $F = IFS_\alpha\, Cl\, F$.

Proof. Obvious. □

Theorem 4.10. *If F and E are any IFS of a IFTS X, then*

1. $IFS_\alpha\, Cl\, \emptyset = \emptyset$ and $IFS_\alpha\, Cl\, X = X$.
2. For any IFS F of X, $F \subset IFS_\alpha\, Cl\, F$.
3. If $F \subset E$, then $IFS_\alpha\, Cl\, F \subset IFS_\alpha\, Cl\, E$.
4. $IFS_\alpha\, Cl\, F \cup IFS_\alpha\, Cl\, E \subset IFS_\alpha\, Cl\, (F \cup E)$.
5. $IFS_\alpha\, Cl\, (F \cap E) \subset IFS_\alpha\, Cl\, F \cap IFS_\alpha\, Cl\, E$.

Proof. Obvious. □

Theorem 4.11. *For any IFS A of an IFTS X. the following are true.*

1. $X \setminus IFS_\alpha\, Cl\, A = IFS_\alpha\text{-}Int\, (X \setminus A)$.
2. $X \setminus IFS_\alpha\text{-}Int\, A = IFS_\alpha\, Cl\, A$.
3. $IFS_\alpha\, Cl\, A = X \setminus IFS_\alpha\, Cl\, A$.

Proof. (1) $\mathrm{X} \setminus \mathrm{IFS}_\alpha\, \mathrm{Cl\, A}$ is an $\mathrm{IFS}_\alpha\mathrm{O}$ contained in $\mathrm{X} \setminus \mathrm{A}$. Hence, $\mathrm{X} \setminus \mathrm{IFS}_\alpha\, \mathrm{Cl\, A} \subset \mathrm{IFS}_\alpha\text{-}\mathrm{Int\, X} \setminus \mathrm{A}$. If $\mathrm{X} \setminus \mathrm{IFS}_\alpha\, \mathrm{Cl\, A} \neq \mathrm{IFS}_\alpha\text{-}\mathrm{Int\, X} \setminus \mathrm{A}$ is a IFS_α-closed set properly contained in $\mathrm{IFS}_\alpha\, \mathrm{Cl}$, a contradiction. Hence, $\mathrm{X} \setminus \mathrm{IFS}_\alpha\, \mathrm{Cl\, A} = \mathrm{IFS}_\alpha\text{-}\mathrm{Int\, X} \setminus \mathrm{A}$.

(2) and (3) follows from (1). □

References

[1] K. T. Atanassov, Intuitionistic Fuzzy Sets Past, Present and Future, CLBME - Bul- garian Academy of Sciences.

[2] J. Dontchev, Survey on pre open sets, The proceedings of Yatsushiro topological conference (1998) 1-18.

[3] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Mat. Monthly 70 (1963), 36-41.

[4] N. Levine, Strong continuity in topological spaces, Amer. Math. Monthly, 67 (1960),269.

[5] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.

[6] Thangaraj, G., Balasubramanian, G. (2003). On somewhat fuzzy continuous functions. Journal of Fuzzy Mathematics, 11(2), 725-736.