NEW CLUES ON ARBITRARY-PRECISION CALCULATION OF THE RIEMANN ZETA FUNCTION ON THE CRITICAL LINE

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ABSTRACT

The Riemann Hypothesis, is considered by many mathematicians to be the most important unsolved problem, consist in the assertion that all of zeta's nontrivial zeros line up at the so called critical line, $\zeta(1/2 + it)$.

This paper presents an algorithm, based on a closed-form system of equations, that computes directly at n^{th} decimal digit each non-trivial zeros of the Riemann Zeta Function.

Keywords Riemann Hypothesis · Riemann Zeta Function · Non-trivial Zeros

1 Introduction

The non-trivial zeros of Riemann Zeta Function has been focus of intense investigation, actually considered the most important unsolved problem in pure mathematics.

The Riemann-Siegel formula is an approximation algorithm that permits very fast evaluation of the zeta function, and the accuracy of the approximations of $\zeta(1/2 + it)$ improves with increasing t [1].

Very recent formulas found by Guilherme França and LeClair André [2] and Simon Plouffe [3] are also allowing very fast calculations of non-trivial zeros.

2 On the Transcendental Equations Satisfying Zeta Function

From an explicit expression given by Guilherme França and LeClair André [2], we can get approximated values of imaginary part for every non-trivial zero of zeta function:

$$t_n = \frac{2\pi\left(n - \frac{11}{8}\right)}{\mathrm{W}\left(\frac{n - \frac{11}{8}}{e}\right)} \tag{1}$$

Related from this formula we propose a system of equations which provides an unprecedentedly accurate estimation of the zeros on the critical line.

$$
\begin{cases}
t_{m_1} = \dfrac{2\pi\left(m - \dfrac{11}{8}\right)}{\mathrm{W}\left(\dfrac{m - \dfrac{11}{8}}{e}\right)} & (2) \\[2em]
t_{m_2} = \dfrac{t_m \cdot \mathrm{W}\left(\dfrac{8m-11}{8e}\right)}{\mathrm{W}\left(\dfrac{t_m \cdot \mathrm{W}\left(\dfrac{8m-11}{8e}\right)}{2e\pi}\right)} & (3)
\end{cases}
$$

Based on this system of equations, we can compute every zeta non-trivial roots and gram points.

3 Algorithm and Experimental Results

3.0.1 Algorithm description

As is known the non-trivial zeros of zeta are denoted by $\rho_n = 1/2 + i\gamma n$ for $n \neq 0$, we describe the follow algorithm to compute γn at desired decimal digit of accuracy.

By simple trial and error method, we can find a value (m) that satisfy the proposed system of equations.

In each iteration we got 2 values for the equation system (t_{m_1} and t_{m_2}) which correspond to equation (2) and (3) respectively.

3.0.2 Algorithm pseudocode

Algorithm 1: Computation of γn at n^{th} digit

```
Result: t_m ≃ γn
Decimal digits accuracy = d;
compute t_n for n;
m = n;
for i ← 1 to d do
    for j ← 0 to 9 do
        compute t_m1 and t_m2;
        if d digit of t_m1 and t_m2 = d digit of t_m then
            break;
        else
            d digit of t_m = j;
        end
        for k ← 0 to 9 do
            compute t_m1 and t_m2;
            if d digit of t_m1 and t_m2 = d digit of t_m then
                break;
            else
                d digit of m = k;
            end
        end
    end
end
```

3.1 Examples

Using the algorithm with twenty one decimal digits of accuracy
$m = 0.9492778727433179471299999999999999999999999999999999$
gives
$t_{m_1} =$ **14.134725141734693790481**42839749565535146511422969 3
$t_{m_2} =$ **14.13472514173469379048**70603144109744995286678323 87

Using the algorithm for some other examples of zeros at ten decimal digits of accuracy

Table 1: First four zeros of $\zeta(1/2 + it)$: t_n, t_m and γn for $n \in \{1, 2, 3, 4\}$

n	t_n	m	t_m	γn
1	**14**.5213469531	0.9492778727	**14.13472514**22	14.1347251417
2	**2**0.6557403558	2.0698956896	**21.02203963**91	21.0220396388
3	**25**.4926754322	2.8933326859	**25.01085758**31	25.0108575801
4	29.7394116323	4.1708464753	**30.42487612**75	30.4248761259

4 Conclusions

We only present a new way to compute Riemann Zeta Function, based on a system of equations which uses the Lambert W function that could finally lead to a truly fundamental formula.

There are no proofs of all this, just empirical results. Further investigations must be done to confirm/reject or even improve those findings.

References

[1] Carl Ludwig Siegel, Komaravolu Chandrasekharan, and Heinrich Maass. Über riemanns nachlaß zur analytischen zahlentheorie. 1966.

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[3] Simon Plouffe. Le Calcul de P(n) et Pi(n). *viXra preprint viXra:2002.0463 [math.NT]*, 2020.