A NEW PROBABILITY DISTRIBUTION AND ITS APPLICATION IN MODERN PHYSICS

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ABSTRACT

In this paper we present a new symetric probability distribution with its properties and we show that is not a uniforme distribution using some standard proofs test like Kolmogorov-Smirnov test and also we may show that is derived from a new another special function by adjusting it using mean and deviation as two parameters , And in the second section we show that PDF present a wave function using rescaled plasma dispersion function such that we define it as a position of massive particle for such charged quantum system .

***K*eywords** Probability distribution · Special relativity · Energy-momentum · quantum mechanics

1 Introduction

In this paper[1] We have studied a new special function behave more like error function [35] which it is defined as :

$$I(a) = \int_0^a \Big(\exp(-x^2 erf(x)) \Big)\, dx$$

such that we showed a little bit its application in probability theory[13] , we are used here the same kind of that function to derive a new probability distribution which it is defined by the following formula :

$$f(h) = h^2 \exp(-h^2 erf(h^2))$$

such that $\int_{-\infty}^{\infty}, f(h) = 0.98953565779600716201258592263911850750446319580 08$,And its Taylor series arround $h = 0$ of order 12 is given by :

$$T = \frac{h^3}{3} - \frac{2h^7}{7\sqrt{\pi}} + \frac{(42 + 14\sqrt{\pi})\, h^{11}}{231\pi} + O\left(h^{13}\right)$$

Now one can look to other analytical properties of that function regarding the new special function montioned [1] , We are ready now to define our PDF by adjusting that function using two parameters: μ, σ

1.1 Definition of new probability distribution:

let $k = \sqrt[4]{\frac{2\pi}{1+\frac{\mu^2}{4}}}; h = k(z - \sigma); 1.01058kh^2 \exp\left(-h^2 erf\left(h^2\right)\right)$, The PDF can be written as :

$$F(z, \mu, \sigma) = 1.0105750026505362 \times \frac{\sqrt[4]{2\pi}}{\sqrt[4]{1 + \mu^2/4}} \frac{(z - \sigma)^2 \sqrt{2\pi}}{\sqrt{1 + \mu^2/4}} \exp\left(-\frac{(z - \sigma)^2 \sqrt{2\pi}}{\sqrt{1 + \mu^2/4}} erf\left(\frac{(z - \sigma)^2 \sqrt{2\pi}}{\sqrt{1 + \mu^2/4}} \right) \right)$$

,The latter PDF is integrand to 1 for σ, and μ are arbitary real numbers and the constant of Normalisation is 1.0105750026505362 ,We could then determine the central moments for a specific value of [27] (say $\mu = 1/20$) such that the odd moments are 0 implies the skweness is also 0 look to Figure 1

$$\left(\int_{-\infty}^{\infty} (z-1)^2 F(z,\mu,\sigma)\, dz\right) = 0.565411$$

,with $\left(z, 1, \frac{1}{20}\right)$ and

$$\left(\int_{-\infty}^{\infty} (z-1)^4 F(z,\mu,\sigma)\, dz\right) = 0.545414$$

,with $\left(z, 1, \frac{1}{20}\right)$ and

$$\left(\int_{-\infty}^{\infty} (z-1)^4 F(z,\mu,\sigma)\, dz\right) = 0.751608$$

,with $\left(z, 1, \frac{1}{20}\right)$, But the i-th central moment for any particular value of μ using Mathematica code will be and given by this mathematica Code :

```
centralMoment[i_, μ_] :=
  If[OddQ[i], 0,
  1.0105750026505362 (2 π / (1 + μ^2 / 4)) ^ (-i / 4) ×
  NIntegrate[h^(2 + i) Exp[-h^2 Erf[h^2]], {h, -∞, ∞}]];
     Evaluate[centralMoment[i_, μ_]]
```

Figure 1: Evaluation of i-th central moment for any particular value of

And Here is the variance (See code in figure 2) since it depend only to μ :

```
FullSimplify[centralMoment[2, μ], μ > 0]
0.282617 Sqrt[4 + μ^2]
```

Figure 2: mathematica code for evaluation of variance

As we said above that PDF is symetric about σ which mean that the Mean Value is zero , here is the plot of that PDF for $\sigma = 0, \mu = 1/20$ see figure 3 :

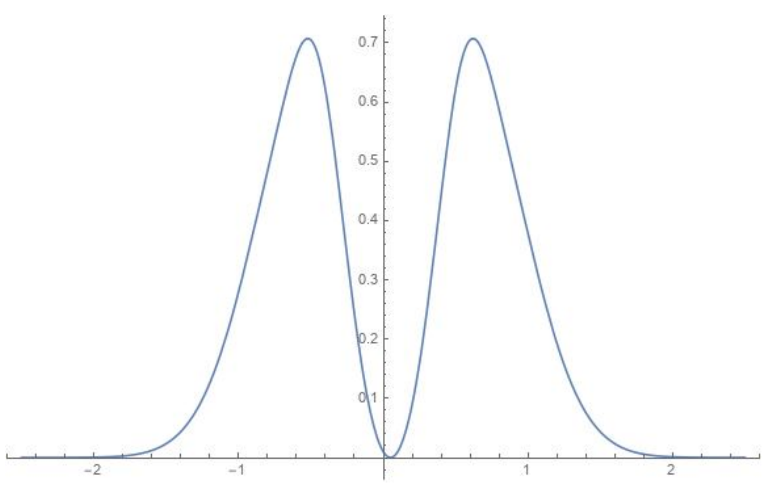

Figure 3: Plot of the new Probability distribution for $\sigma = 0, \mu = 1/20$

We may look to the plot of the new probability distribution for fixed μ and some values of σ , We take as a good example $\mu = 10$, and $\sigma = 0.5, 0.75, 1.5, 2$ as shown below in Figure 4 and we noted that the probability values decay from 0.3 to about 0.22 when we increased the value of μ (Say $\mu = 20$) ,The numerical evidence show us [32] that Probability distribution depend more to the values of μ than σ. See Figures (4, 5)

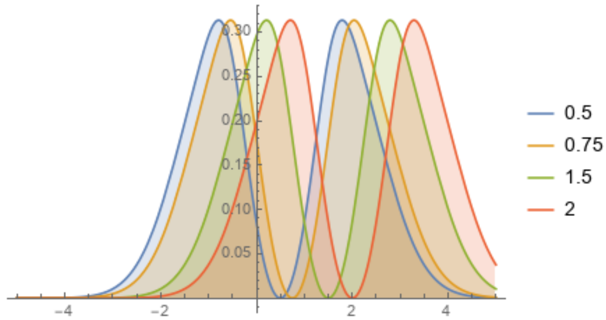

Figure 4: Plot of the new Probability distribution for $\mu = 10, (\sigma = 0.5, 0.75, 1.5, 2)$

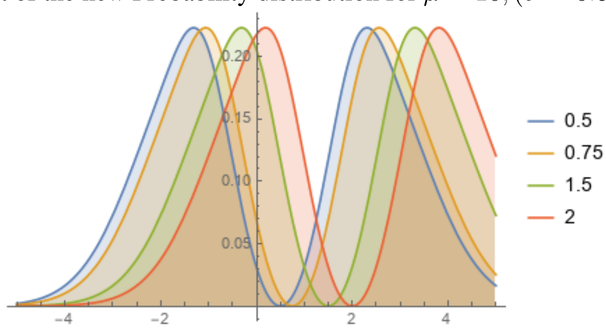

Figure 5: Plot of the new Probability distribution for $\mu = 20, (\sigma = 0.5, 0.75, 1.5, 2)$

We may conclude our first section by Histogram plot and test table for normality of that new probability distribution[18] for $(\sigma = 1, 2), \mu = 10$ and its Histogram for some given data for that Probability distribution, then here is the CDF (Cumulative distribution function) see Figure 6 for $(\sigma = 1, 2), \mu = 10$) .

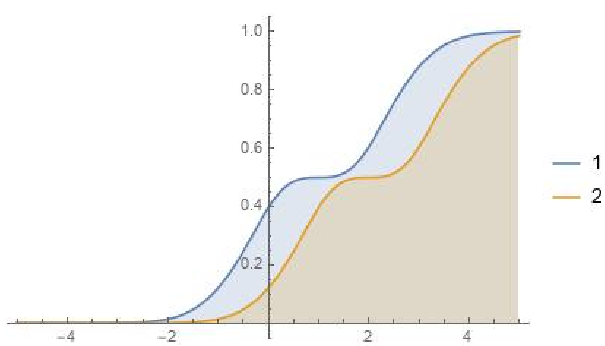

Figure 6: Plot of Cumulative distribution function for a new probability distribution, $(\sigma = 1, 2), \mu = 10$

Here is the mathematica code[33] with plot for Histogram corresponding to the new probability distribution for some given data with $(\sigma = 1, \mu = 10$

Now for normality test [16],[36], We may remember togother the following definition [1] In statistics, normality tests[34] are used to determine if a data set is well-modeled by a normal distribution and to compute how likely it is for a

[1]The Faddeeva function or Kramp function is a scaled complex complementary error function[**?**], It is related to the Fresnel integral, to Dawson's integral, and to the Voigt function. The function arises in various physical problems in describing electromagnetic response in complicated media

```
distr1[μ_, σ_] := ProbabilityDistribution[
 (1.0105750026505362 * (2 * Pi) ^ (1 / 4) * ((z - σ) ^ 2 * Sqrt[2 * Pi])) /
   (E ^ ((((z - σ) ^ 2 * Sqrt[2 * Pi]) * Erf[((z - σ) ^ 2 * Sqrt[2 * Pi]) /
         Sqrt[1 + μ ^ 2 / 4]]) / Sqrt[1 + μ ^ 2 / 4]) *
    ((1 + μ ^ 2 / 4) ^ (1 / 4) * Sqrt[1 + μ ^ 2 / 4])), {z, -Infinity, Infinity}]
data = RandomVariate[distr1[10, 1], 10 ^ 3];

Show[
Histogram[data, 20, "ProbabilityDensity"],
Plot[PDF[distr1[10, 1], z], {z, -9, 9},
PlotStyle → Thick]]
Show[
Histogram[data, Automatic, "ProbabilityDensity"],
DiscretePlot[PDF[distr1[10, 1], z], {z, -10, 10},
PlotStyle → PointSize[Medium]]]
```

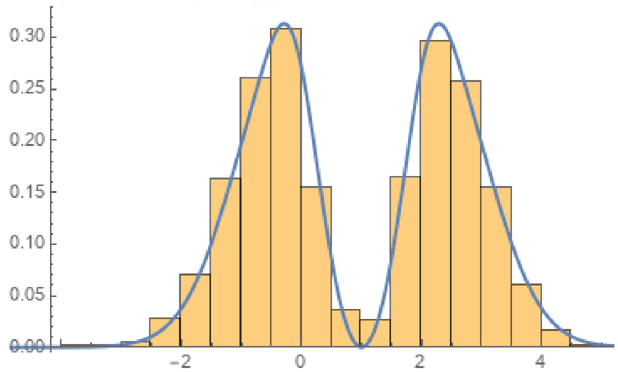

Figure 7: Histogram plot correspending to the new probability distribution for some given data with $(\sigma = 1, \mu = 10$

```
data = RandomVariate[distr[100, 20], 10000];
DistributionFitTest[data]
ℋ =
DistributionFitTest[data, Automatic, "HypothesisTestData"];
ℋ["TestDataTable", All]
```

	Statistic	P-Value
Anderson-Darling	289.862	0.
Baringhaus-Henze	594.105	0.
Cramér-von Mises	56.8307	0.
Jarque-Bera ALM	713.686	0.
Kolmogorov-Smirnov	0.12989	0.
Kuiper	0.254768	0.
Mardia Combined	713.686	0.
Mardia Kurtosis	-26.6754	9.07601×10^{-157}
Mardia Skewness	1.69223	0.193307
Pearson χ^2	4490.37	0.
Watson U^2	56.8139	0.

Figure 8: Table for Normality test using many proofs with $(\sigma = 100, \mu = 20$

random variable underlying the data set to be normally distributed. We say that the random variable is normally[37] is distributed iff The P-value is greater than 0.05

The Importance of Testing for Normality: Many statistical procedures[34] such as estimation and hypothesis testing have the underlying assumption that the sampled data come from a normal distribution. This requires either an effective test of whether the assumption of normality holds or a valid argument showing that non-normality does not invalidate the procedure. Tests of normality are used to formally assess the assumption of the underlying distribution.

Much statistical research has been concerned with evaluating the magnitude of the effect of violations of the normality assumption on the true significance level of a test or the efficiency of a parameter estimate. Geary (1947) showed that for comparing two variances, having a symmetric non-normal underlying distribution can seriously affect the true significance level of the test. For a value of 1.5 for the kurtosis of the alternative distribution, the actual significance level of the test is 0.000089, as compared to the nominal level

and we may show its application to test normality [22] of our probability distribution [16] such that we may show that the random variable of that new probability distribution is not normally distributed using many proofs of test ([3],[17]) as shown in Figure 8 table ([16] ,[15]). For the entropy of that new probability distribution one can compute it using the following simple mathematica Code :

```
[ (* Sample size *)

n = 97

(* Take random sample *)

x= RandomVariate[distr[10,2],n]

(* Calculate entropy *)

Entropy[x]= Log 97 ]
```

Note: for many example we have tried by mathematica always the Entropy[38] of that probability distribution is $Logn$. [2]

2 Application to Modern physics

2.1 A new probability distribution by means of Plasma dispersion relation

In the first we may try to write our PDF in terms of plasma dispersion function or Faddeva function [2] using the fact that : $k = \sqrt[4]{\frac{2\pi}{1+\frac{\mu^2}{4}}}; x = k(z-\sigma); f(z,\mu,\sigma) == 1.0105750026505kx^2 \exp\left(-x^2 erf\left(x^2\right)\right)$, We try to write $erf(x^2)$ in terms of Faddeeva function using the following steps : let $w(-ix)$ be the Faddeva function defined as :

$$w(-ix) = \exp(x^2)(1 + erf(x)) \tag{1}$$

From (1) we can get :

$$erf(x^2) = \exp(-x^4)w(-ix^2) - 1 \tag{2}$$

We use the definition of Fried and Conte for the rescaled function[35] $Z(x) = i\sqrt{\pi}w(x)$ implies that $Z(-ix^2) = i\sqrt{\pi}w(-ix^2)$, Now Multipliying the sides of equation (2) by the factor $-x^2$ using the fact that $Z(-ix^2) = i\sqrt{\pi}w(-ix^2)$, we can get the following equation

$$-x^2 erf(x^2) = i\frac{x^2}{\sqrt{\pi}}(\exp(-x^4)Z(-ix^2) - i\sqrt{\pi}) \tag{3}$$

[2]The wave function is the most fundamental concept of quantum mechanics. It was first introduced into the theory by analogy (Schrödinger 1926); the behavior of microscopic particles likes wave, and thus a wave function is used to describe them. Schrödinger originally regarded the wave function as a description of real physical wave. But this view met serious objections and was soon replaced by Born's probability interpretation (Born 1926), which becomes the standard interpretation of the wave function today. According to this interpretation, the wave function is a probability amplitude, and the square of its absolute value represents the probability density for a particle to be measured in certain locations

Now one can raise the Exp power of the two both sides of (3) and multiplying the obtained equation by the factor kx^2 one can get the final formula which it is the definition of our new PDF in terms of rescaled Faddeeva function

$$kx^2\exp(-x^2erf(x^2)) = kx^2\exp(i\frac{x^2}{\sqrt{\pi}}(\exp(-x^4)Z(-ix^2) - i\sqrt{\pi})) \tag{4}$$

with $k = \sqrt[4]{\frac{2\pi}{1+\frac{\mu^2}{4}}}; x = k(z-\sigma)$ hence we are defined the PDF by means of plasma dispersion relation exactly we have got a new wave function [4]in $3D$ defined In terms of Energy relativistic and position X of a Massive particle at time t and C for a charged quantum system [25] , [5] by:

$$\psi(X,t) = kE_0\exp(i(CX - i\sqrt{\pi}E_0)) \tag{5}$$

such that : $C = \frac{\sqrt{2}E_0}{\exp(E_0^2)}, X(x) = Z(-ix^2)$, Here the position X takes values of rescaled plasma dispersion function , Now we may show how we have got the function defined in (5), Reacall that $x = k(z-\sigma)$, let $z = \sqrt{E} \geq 0$ be the energy of massive particle at time t and let σ be the time t and $\mu = \frac{2p}{m_0c}$ such that c is the speed of light in a vacuum and m_0 is the rest massive and p is the momentum , here K is represented by means of Lorentz factor , using the formula of μ we would get $k = \frac{2\pi^{\frac{1}{4}}}{\sqrt{\gamma}}$, γ is the Lorentz factor [6] which is rarley used although it does appears in Maxwell–Jüttner distribution [7],[30]) and it is defined by this formula : $\gamma = \sqrt{1 + (\frac{p}{m_0c})^2}$, For instance take $t = 0$, we have $x = kzx^2 = k^2z^2$ using the assumption $z = \sqrt{E}$ and $k = \frac{2\pi^{\frac{1}{4}}}{\sqrt{\gamma}}$ gives : $x^2 = \sqrt{2\pi}\frac{E}{\gamma}$ using the fact the law of relativistic mass $mc^2 = m_0\gamma c^2$ then we can deduce directly that $x^2 = \sqrt{2\pi}E_0$ with $E_0 = m_0c^2$,The substitution of x^2 by $\sqrt{2\pi}E_0$ in (4) gives immediately the wave function [28] defined in (5).

The function defined in (5) present a probability distribution for massive particule for charged quantum system [32],[23] such that We get the probability for the obsarvable [24],[29] to be lie in the range $[X, \delta X + X]$ at random time t with initial energy E_0 . We can interpret the histogram plot defined in figure 7 as The number of times that a particle was measured [26]to be in the range $[X, \delta X + X]$

3 Conclusion:

We have showed using our new PDF that for the massive particle with known initial energy[18] at time t_0 and momentum p such that the particule position defined by rescaled plasma function we have always a non -uniform symetric distribution for energy -relativistic .

Data Availability:The data supported this research is The plasma dispersion function and mathematica wolfram language computation and Also Normality test .

There is no conflict of interest for Author of this research

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