Bypassing the Axion Paradigm through Fractional Field Theory

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Abstract

A long-standing challenge of the Standard Model (SM) is to convincingly explain the physical mechanism behind the $U(1)$ axial anomaly of Quantum Chromodynamics (QCD). While instantons are routinely invoked as solution to this anomaly, they create new puzzles – the strong CP problem and the postulated existence of axions. Here we suggest that fractional field theory offers an alternative resolution to the axial anomaly, shedding new light on the strong CP problem and bypassing the axion hypothesis.

Key words: $U(1)$ axial anomaly, instantons, strong CP problem, axions, fractional field theory, fractional differential operators.

1. The axial anomaly and the axion hypothesis

The $U(1)$ symmetry of QCD has two components. The vectorial component $U(1)_V$ represents the symmetry generated by the global quark transformation [1-3]

$$U(1)_V : q_i \to \exp(i\alpha)\, q_i \tag{1}$$

in which α is an arbitrary real number. Considering only the first two quark flavors and by the Noether theorem, (1) gives rise to the current

$$J_\mu^B = \bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d \tag{2}$$

where γ_μ are Dirac's gamma matrices. (2) is called the baryon-number current and the $U(1)_V$ symmetry reflects the observed baryon-number conservation

$$\partial^\mu J_\mu^B = 0 \tag{3}$$

The second component of the $U(1)$ symmetry is generated by the axial transformation

$$U(1)_A : q_i \rightarrow \exp(i\beta\gamma_5)\, q_i \tag{4}$$

where β is an arbitrary real number and γ_5 is the fifth gamma matrix (otherwise called the chirality operator). The axial current generated by (4) is given by

$$J_\mu^5 = \bar{u}\,\gamma_\mu\gamma_5 u + \bar{d}\,\gamma_\mu\gamma_5 d \tag{5}$$

Conservation of (5) requires vanishing of the its divergence, that is,

$$\partial^\mu J_\mu^5 = 0 \tag{6}$$

However, there is no observed symmetry in the hadron spectra demanded by (6). Specifically, if (4) were a real symmetry of Nature, the number of baryon states would double (effect known as *parity doubling*). For example, the π meson would have a scalar partner, which is missing from experimental observations. Likewise, there is no evidence for additional Goldstone bosons that would arise if the axial symmetry were spontaneously broken. The absence of either parity doubling or of Goldstone bosons related to (6) represents the $U(1)$ *axial anomaly* [1-3].

A deceivingly simple way to avoid the axial anomaly is to account for the coupling of gluons and quarks, which naturally leads to the violation of (6). Yet this scenario turns out to be wrong. To see why this is the case, consider the limit of nearly vanishing quark masses ($m_u, m_d \to 0$), where the divergence of the axial current assumes the form [1-3]

$$\partial^{\mu} J^{5}_{\mu} = \frac{g_s^2}{4\pi^2} tr(G_{\mu\nu}\overline{G}^{\mu\nu}) + 2im_u \overline{u}\gamma_5 u + 2im_d \overline{d}\gamma_5 d \tag{7}$$

in which g_s is the strong coupling, $G_{\mu\nu}$ the gluon tensor matrix and $\overline{G}^{\mu\nu}$ its dual. However, the trace of $G\overline{G}$ may be cast as a divergence of a current K_μ

$$\partial^{\mu} K_{\mu} = \frac{g_s^2}{4\pi^2} tr(G_{\mu\nu}\overline{G}^{\mu\nu}) \tag{8}$$

If $A_\mu = \frac{A_\mu^a \lambda^a}{2}$ stands for the gluon field and λ^a, $a = (1, 2, ..., 8)$ for the generators of the $SU(3)$ group, the current K_μ is explicitly given by

$$K_{\mu} = \frac{g_s^2}{4\pi^2} \varepsilon_{\mu\nu\lambda\rho} tr(G^{\nu\lambda} A^{\rho}) \tag{9}$$

Hence, a new axial vector current can be introduced as

$$\overline{J}^{5}_{\mu} = J^{5}_{\mu} - K_{\mu} \tag{10}$$

whose conservation is guaranteed in the limit $m_u, m_d \to 0$, e. g.,

$$\partial^{\mu}\overline{J}_{\mu}^{5} = 2im_u\overline{u}\gamma_5 u + 2im_d\overline{d}\gamma_5 d \to 0 \tag{11}$$

Moreover, there is a conserved charge arising from (11), namely,

$$\frac{d\overline{Q}_5}{dt} = \int d^3x \frac{\partial \overline{J}_0^5}{\partial t} \to 0 \tag{12}$$

Both (11) and (12) show that the axial anomaly persists even in the presence of quark-gluon coupling.

Historically, it was 't Hooft who first realized that the nonperturbative dynamics of gauge theory has the potential of violating the condition (12) [1-3, 12]. Specifically, topological solutions of gauge theory called *instantons* bring a nonvanishing contribution to the charge $\overline{Q}_5$ and prevent its conservation. In turn, violation of charge conservation driven by instantons explains the absence of any Goldstone bosons associated with the axial anomaly.

While instantons offer an attractive way of circumventing the axial anomaly, they create two other challenges for the QCD theory, namely the *strong charge-parity (CP) problem* and its proposed resolution in terms of *axions*.

In particular, instantons bring an extra parameter to the QCD sector (the so-called θ parameter) which quantifies the magnitude of CP violation in strong interactions. *Axions* are hypothetical scalar particles that cancel the contribution of the θ parameter and restore the CP symmetry of QCD [1-3]. Despite a vast array of thorough searches for their signature, as of today, axions are still eluding observations.

2. Solving the axial anomaly through fractional field theory

The goal of this section is to explore the axial anomaly from the vantage point of *fractional field theory*, introduced and developed in our previous publications [4-6].

We start by recalling the chiral representation of gamma matrices (γ's) [7]

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{13}$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{14}$$

where I is the identity matrix and σ_i, $(i = 1,2,3)$ the three Pauli matrices. Since (γ's) contain constant entries, they cannot describe a regime that is *out of equilibrium*, where all dynamic variables and parameters of the theory evolve with time. Working in the spirit of fractional field theory, it makes sense to extend the mathematics of gamma matrices to the *minimal fractal manifold* (MFM), a spacetime continuum characterized by arbitrarily small and scale-dependent deviations from four dimensions ($\varepsilon = 4 - D << 1$). As shown in [4], MFM reflects an evolving setting that starts outside equilibrium and asymptotically approaches the equilibrium conditions mandated by QFT in the limit of four-dimensional spacetime ($\varepsilon = 0$).

Drawing on fractional Dirac equation [see e.g., 8], as well as on topics related to non-equilibrium dynamics [see e.g., 13], we conjecture below that the gamma matrices are explicitly dependent on both time and energy scale. Specifically, we assume that:

1) Each entry in the gamma matrices represents the density of a well-behaved distribution function $f(t)$. The most straightforward hypothesis is that $f(t)=t$, a linear scaling function reflecting the transition to critical behavior in the strong coupling regime of QCD [10, 11].

2) This density is characterized by long-range (memory) properties, which means that it is best described by using *fractional derivatives*.

Based on these two premises, each entry "1" and each entry "i" in the composition of gamma matrices is replaced with, respectively,

$$1 \Rightarrow D_C^{1\pm\varepsilon} f(t) \approx t^{\mp\varepsilon}$$

(15)

$$i \Rightarrow \sqrt{-D_C^{1\pm\varepsilon} f(t)} \approx \sqrt{-t^{\mp\varepsilon}}$$

Here, $D_C^{1\pm\varepsilon}(t)$ is the Caputo differential operator [9] and t the dimensionless time variable defined as

$$t \Rightarrow \frac{t}{t_{UV}} = O\left(\frac{\mu^{-1}}{M_{Pl}^{-1}}\right) = O\left(\frac{M_{Pl}}{\mu}\right) \tag{16}$$

where μ is the running energy scale, t_{UV} a high-energy normalization "time" and M_{Pl} the Planck scale. The time corresponding to the infrared regime of QCD takes the form

$$t_{QCD} = O\left(\frac{M_{Pl}}{\Lambda_{QCD}}\right) >> 1 \tag{17}$$

in which $\mu = \Lambda_{QCD}$ is the strong interaction scale ($\approx 250\,MeV$).

The far ultraviolet (UV) limit $\mu = O(M_{Pl})$ and $t = O(1) << t_{QCD}$ defines the "*asymptotic freedom*" sector of QCD in which (15) recovers the standard numerical entries of the gamma matrices. Consider now the scenario where the change rate of ε is far slower than unity, that is,

$$\left|\frac{d\varepsilon}{dt}\right| << \left|\frac{dt}{dt}\right| = 1 \tag{18}$$

In this case, the infrared (IR) limit of large timescales expressed by (17) $t = O(t_{QCD}) \to \infty,\ \varepsilon \to 0$, defines the transition to the strong coupling regime of QCD. It yields an undetermined expression (15) with two diametrically opposite outcomes, namely $t^{+\varepsilon} \to \infty$ and $t^{-\varepsilon} \to 0$. This transition can be conveniently parameterized using a generic power law relationship of the form

$$t = t_{QCD}^{\pm\varepsilon} \sim \varepsilon^{\mp\lambda},\ \ \varepsilon \to 0\,,\ \lambda > 0 \tag{19}$$

What remains to be explained is which one of the two "modes" of (19) is the dominant one. The $t_{QCD}^{+\varepsilon} = \varepsilon^{-\lambda} >> 1$ mode is bound to produce a large amplification in the numbers entering the gamma matrices, an effect likely to carry over to the laboratory timescale due to the long-range attributes of fractional derivatives. By the same token, the same numbers collapse to zero in the opposite mode ($t_{QCD}^{-\varepsilon} = \varepsilon^{+\lambda} << 1$), which renders the gamma matrices unobservable at the laboratory timescale. It is for this reason that this mode fails to survive.

An important observation is now in order. There is a key difference between the baryon number current (2) and the axial current (5) in that the latter includes the product of γ_μ matrices with the γ_5 matrix. By (14), either one of the two terms entering (5) contains the product

$$\gamma_\mu \gamma_5 = \gamma_\mu (i\gamma_0 \gamma_1 \gamma_2 \gamma_3), \quad \mu = 0,1,2,3 \tag{20}$$

On account of (20), since each entry of either γ_μ or γ_5 vary with ε according to (19), there is a significant *amplification* effect in (5) that scales as $\varepsilon^{\mp 5\lambda}$ versus the same effect in (2), which only scales as $\varepsilon^{\mp\lambda}$. As a result, a cursory comparison of (3) to (6) reveals that the axial current violation is *far more likely* than the baryon number violation, in agreement with observations.

In conclusion, our analysis points out that (19) leads to the violation of (6), (11) and (12) and removes the axial anomaly via a mechanism that bypasses the instanton and axion paradigms.

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