The Hierarchy and Absolute Values of Neutrino Masses

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Abstract. In neutrino physics, the hierarchy of neutrino masses as well as the absolute values of neutrino masses are still open-ended questions that seek answers.The greatest obstacle in the neutrino physics related to the indeterminacy in the difference of the squared neutrino masses was avoided by introducing two conjectures:one presents all three neutrinos in the form of plane waves, and the other introduces the relation between interconnections of masses of neutrinos. Defining the mathematical models of the mass hierarchy, it is proven in the theoretical discussion that the same number *n* is present on two levels: on the macroscopic level between corresponding oscillation wavelengths and on the microscopic level between corresponding squared masses differences.The mathematical model was also developed to determine the absolute values of neutrino masses. Absolute values of neutrino masses were calculated, and then, their sum was shown to be consistent with the sum of neutrino masses in the cosmology.

Key words: Special relativity, Leptons, Ordinary neutrino, Neutrino mass and mixing.
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1 Introduction

The nature of neutrinos related to neutrino flavour oscillations was experimentally resolved [1,2,3,4],which made it clear that neutrinos could possess masses.The existence of neutrino masses naturally raised the question of the position of these masses in their mass hierarchy. And the hierarchy of neutrino masses, in itself, raises the question of the absolute values of neutrino masses.Owing to that, a special attention was paid to the phase factor of the mass eigenstate,which is simple and straightforward from the perspective of mathematical presentation, but which is, essentially, a key element in defining the nature of neutrinos.The entire theory on the oscillations of neutrino flavour states is based on the application of the unitary PMNS mixing matrix that contains parameters which connect flavor eigenstates with mass eigenstates. These parameters provide mismatches between flavor states and mass eigenstates that are necessary for establishing oscillations between certain neutrino flavor states.

In the standard scenario, the three neutrinos ν_1, ν_2, ν_3 are known to have relative masses measured as $\delta m_{21}^2 = m_2^2 - m_1^2$ and $\Delta m_{31}^2 = \left| m_3^2 - m_1^2 \right|$.In neutrino physics, the sign of Δm_{31}^2 is still debatable and unspecifiedas it has not been measured yet, and that allows two different configurations for the masses:either $m_1 < m_2 < m_3$ (normal mass ordered) or $m_3 < m_1 < m_{2.}$ (inverted mass ordered).The dilemmas in neutrino physics related to the sign of Δm_{31}^2 still present an obstacle for defining the neutrino mass ordering.Further studies have shown how this dilemma was avoided, leading to the emergence of the number *n*. On the one hand, the number *n* connects the oscillation wavelengths as macroscopic magnitudes. On the other hand, the number *n* defines the relation between the corresponding squared masses differences of neutrinos, which by their nature represent subatomic magnitudes.

It should be noted that there are some more open questions in neutrino physics that are waiting for answers [5,6,7,8,9,10,11,12]:

1.Are squared neutrino masses ordered normally:

$m_1^2 < m_2^2 < m_3^2$ or are they inverted: $m_3^2 < m_1^2 < m_2^2$?

2. What is the lightest neutrino mass?

- Is one type of neutrinos much heavier than the other two or much lighter?

- What is the absolute mass of a neutrino?
- Do neutrinos and antineutrinos behave differently?
- Is neutrino its own antiparticle?

2 Mathematical model for three flavour neutrino oscillations: normal mass hierarchy

Based on the mathematical model for the three flavour neutrino oscillations, the following mathematical identity is proposed:

$$\left(\Psi_1\Psi_1^*\right)\left(\Psi_2\Psi_2^*\right)\left(\Psi_3\Psi_3^*\right)\equiv 1 \tag{1}$$

Where neutrinos are presented in the form of plane waves

$$\Psi_k = e^{i\Phi_k} = e^{\frac{i}{\hbar}(p_k x - E_k t)}; k = 1,2,3 \tag{2}$$

that represent solutions of the Klein-Gordon equations.

Comment. All further research will be based on the normal hierarchy of neutrino masses**.**

The normal mass hierarchy is defined by relations $m_1 < m_2 < m_3$ with the corresponding momentums

$$p_3 < p_2 < p_1 \tag{3}$$

Which are obtained from the relations

$$\begin{aligned} &E_1 = E_2 = E_3 = E;\ p_1 = \frac{E}{c^2}v_1 = \frac{E}{c}\left(1-\delta_1\right) = \frac{E}{c}\left(1-\frac{m_1^2c^4}{2E^2}\right);\delta_j << 1,\ j = 1,2,3. \\ &p_2 = \frac{E}{c^2}v_2 = \frac{E}{c}\left(1-\delta_2\right) = \frac{E}{c}\left(1-\frac{m_2^2c^4}{2E^2}\right);\ p_3 = \frac{E}{c^2}v_3 = \frac{E}{c}\left(1-\delta_3\right) = \frac{E}{c}\left(1-\frac{m_3^2c^4}{2E^2}\right) \end{aligned} \tag{4}$$

Because of the normal mass hierarchy conditions (3), it is needed to modify the identity (1):

$$\left(\Psi_1\Psi_2^*\right)\left(\Psi_2\Psi_3^*\right)\left(\Psi_3\Psi_1^*\right) = e^{i[(\Phi_1-\Phi_2)+(\Phi_2-\Phi_3)-(\Phi_1-\Phi_3)]} = e^{\frac{ix}{\hbar}[(p_1-p_2)+(p_2-p_3)-(p_1-p_3)]} \equiv 1 \tag{5}$$

Comment. The presented differences between phase factors represent the following neutrino flavour oscillation processes:

$$\left(\nu_e \to \nu_\mu\right) \to \Phi_1 - \Phi_2,$$

$$\left(\nu_\mu \to \nu_\tau\right) \to \Phi_2 - \Phi_3, \qquad (6)$$

$$\left(\nu_\tau \to \nu_e\right) \to \Phi_3 - \Phi_1$$

This identity is satisfied for the arbitrary chosen distance ***x*****.** That means that, if x is observed as the current coordinate of the direction the neutrino beam is travelling along, then this identity is always satisfied in the continuity. Namely,

$$\Phi_1 - \Phi_2 + \Phi_2 - \Phi_3 - \Phi_1 + \Phi_3 = 0$$
$$\frac{x}{\hbar}\left[\left(p_1 - p_2\right) + \left(p_2 - p_3\right) - \left(p_1 - p_3\right)\right] = 0 \qquad (7)$$

As can be seen, the satisfying of this identity is present at every moment and at every arbitrary place. If this identity is observed as a discrete set of partial identities, which are interdependent, and which occur at different moments and at different places, then the identity (5) can be presented as the product of those partial identities.

$$e^{\left[i\frac{L_{12}}{\hbar}\left(p_1 - p_2\right)\right]} e^{\left[i\frac{L_{23}}{\hbar}\left(p_2 - p_3\right)\right]} e^{\left[-i\frac{L_{13}}{\hbar}\left(p_1 - p_3\right)\right]} = 1 \qquad (8)$$

And the solutions of the equations (8) are:

$$\left(p_1 - p_2\right)L_{12} = h;$$
$$\left(p_1 - p_3\right)L_{13} = h; \qquad (9)$$
$$\left(p_2 - p_3\right)L_{23} = h.$$

where: $h = 6.67 \times 10^{-34}\, Js$ is the Planck constant and $L_{12}\left(\nu_e \to \nu_\mu\right), L_{13}\left(\nu_e \to \nu_\tau\right), L_{23}\left(\nu_\mu \to \nu_\tau\right)$ are the appropriate oscillation wavelengths.

Based on the equation (9), and inserting (9) into (7), the same result is obtained:

$$\frac{1}{L_{13}} = \frac{1}{L_{12}} + \frac{1}{L_{23}} \qquad (10)$$

and direct expressions for oscillation wavelengths:

$$L_{12} = \frac{h}{\left(p_1 - p_2\right)} = \frac{2hE}{c^3\left(m_2^2 - m_1^2\right)};\; L_{13} = \frac{h}{\left(p_1 - p_3\right)} = \frac{2hE}{c^3\left(m_3^2 - m_1^2\right)}\; L_{23} = \frac{h}{\left(p_2 - p_3\right)} = \frac{2hE}{c^3\left(m_3^2 - m_2^2\right)} \qquad (11)$$

Based on (10,11), it follows:

$$L_{13} < L_{23} < L_{12} \qquad (12)$$

Using the corresponding neutrino oscillation probabilities,under the condition $x = L_{12}/2$,the following can be shown

$$\sin^2\left(\frac{L_{12}}{2L_{23}}\pi\right) = 1 \tag{13}$$

From here,there is a unique solution

$$\frac{L_{12}}{2L_{23}}\pi = n\frac{\pi}{2} \tag{14}$$

And

$$\frac{L_{12}}{L_{23}} = n \tag{15}$$

Where *n* must be an odd number.

3 Absolute values of neutrino masses

In order to obtain absolute values of neutrino masses m_1, m_2, m_3, the following assumption is introduced, which interconnects them:

$$m_3 = m_1 + m_2 \tag{16}$$

It is obvious that such interconnection is possible only in the case when $m_1 < m_2 < m_3$.And such interrelationships fall into the normal mass ordering.

Comment. The correctness of the introduced assumption (16) can be evaluated if the following mathematical expressions are introduced

$$\frac{m_3^2 - m_1^2}{m_2^2 - m_1^2} = \frac{L_{12}}{L_{13}} = n + 1 \tag{17}$$

and

$$\frac{m_3^2 - m_2^2}{m_2^2 - m_1^2} = \frac{L_{12}}{L_{23}} = n \tag{18}$$

Based on the conjecture (16), it can be written:

$$m_3^2 = (m_1 + m_2)^2 = m_1^2 + 2m_1m_2 + m_2^2 \tag{19}$$

And from here,one can find:

$$m_3^2 - m_1^2 = m_2^2 + 2m_1m_2 = \frac{A}{L_{13}}; A = \frac{2Eh}{c^3} \tag{20}$$

$$m_3^2 - m_2^2 = m_1^2 + 2m_1m_2 = \frac{A}{L_{23}} \tag{21}$$

Then, it is written:

$$\begin{aligned} m_2^2 - m_1^2 &= m_3^2 - 2m_1^2 - 2m_1m_2 = m_3^2 - m_1^2 - \left(m_1^2 + 2m_1m_2\right) = \left(m_3^2 - m_1^2\right) - \left(m_3^2 - m_2^2\right) \\ &= \frac{A}{L_{13}} - \frac{A}{L_{23}} = \frac{A}{L_{12}} \end{aligned} \tag{22}$$

And from here, it follows

$$\frac{1}{L_{13}} = \frac{1}{L_{23}} + \frac{1}{L_{12}} \tag{23}$$

On the other hand, there are:

$$m_1m_2 = \frac{1}{2}\left(\frac{A}{L_{13}} - m_2^2\right) \tag{24}$$

$$m_1m_2 = \frac{1}{2}\left(\frac{A}{L_{23}} - m_1^2\right) \tag{25}$$

From equations (24) and (25)

$$\left(\frac{A}{L_{13}} - m_2^2\right) = \left(\frac{A}{L_{23}} - m_1^2\right) \tag{26}$$

It is found

$$\frac{A}{L_{13}} = \frac{A}{L_{23}} + m_2^2 - m_1^2 = \frac{A}{L_{23}} + \frac{A}{L_{12}} \tag{27}$$

Obviously, by including relations (20) to (27),the justification of the assumption (16) was shown.And,it describes only the normal hierarchy of neutrino masses.Therefore, in further calculation, it should be expected that the assumption made will yield the expected results for neutrino masses.

4 Procedure for determining absolute values of neutrino masses

Two quadratic equations are formed from the relations (20) and (21):

$$m_2^2 + 2m_1m_2 - \frac{A}{L_{13}} = 0 \tag{28}$$

$$m_1^2 + 2m_1m_2 - \frac{A}{L_{23}} = 0 \tag{29}$$

From these equations, there are the solutions for the corresponding squares of masses:

$$\begin{gathered} m_1^2 = -\frac{A}{3}\left(\frac{1}{L_{13}} + \frac{1}{L_{12}}\right) + \frac{A}{3}\sqrt{\left(\frac{1}{L_{13}} + \frac{1}{L_{12}}\right)^2 + 3\frac{1}{L_{23}^2}} = \frac{2}{3}\left(m_2^2 - m_1^2\right)\left(\sqrt{n^2+n+1} - \frac{n+2}{2}\right); \\ \frac{L_{12}}{L_{13}} = n+1, \frac{L_{12}}{L_{23}} = n \end{gathered} \tag{30}$$

while the absolute value for m_1 equals

$$m_1 = \sqrt{\frac{2A}{3L_{12}L_{13}}\left[\sqrt{\left(L_{12} - L_{13}\right)^2 + L_{12}L_{13}} - \frac{L_{12} + L_{13}}{2}\right]} = \sqrt{\frac{2}{3}\left(m_2^2 - m_1^2\right)\left(\sqrt{n^2+n+1} - \frac{n+2}{2}\right)} \tag{31}$$

The solution of the equation (28) for the square of m_2^2 is:

$$\begin{gathered} m_2^2 = -\frac{A}{3}\left(\frac{1}{L_{23}} - \frac{1}{L_{12}}\right) + \frac{A}{3}\sqrt{\left(\frac{1}{L_{23}} - \frac{1}{L_{12}}\right)^2 + 3\frac{1}{L_{13}^2}} = \frac{2}{3}\left(m_2^2 - m_1^2\right)\left(\sqrt{n^2+n+1} - \frac{n-1}{2}\right); \\ \frac{L_{12}}{L_{13}} = n+1, \frac{L_{12}}{L_{23}} = n \end{gathered} \tag{32}$$

The validation of formulas (30) and (32) is checked if the difference between $m_2^2 - m_1^2$ is made:

$$m_2^2 - m_1^2 = \frac{2A}{3L_{12}}\left(-\frac{n-1}{2} + \frac{n+2}{2}\right) = \frac{A}{L_{12}} \tag{33}$$

While the absolute value for m_2 is:

$$m_2 == \sqrt{\frac{2}{3}\left(m_2^2 - m_1^2\right)\left(\sqrt{n^2+n+1} - \frac{n-1}{2}\right)} \tag{34}$$

Including (20,21,30,32), the value for m_3^2 is calculated:

$$m_3^2 = m_1^2 + 2m_1m_2 + m_2^2 = \frac{2}{3}\left(m_2^2 - m_1^2\right)\left(\sqrt{n^2+n+1} + \frac{2n+1}{2}\right) \tag{35}$$

On the other hand, the absolute value of mass m_3 is:

$$m_3 = m_1 + m_2 = \sqrt{\frac{2}{3}\left(m_2^2 - m_1^2\right)\left(\sqrt{n^2+n+1} + \frac{2n+1}{2}\right)} \tag{36}$$

5 Calculation of the absolute value of the neutrino masses:the case of the stable neutrinos

It is important to note that the formulas for absolute values of neutrino masses depend on two parameters, which are experimentally measured.One parameter is $m_2^2 - m_1^2 = \left(\Delta m_{21}^2\right)_{\exp}$,while the other parameter is the number *n*.As it is known, the number *n* is calculated as a quotient in two ways:between two oscillation wavelengths, or between the corresponding squared mass differences of neutrinos.The following measurement data will be used to calculate neutrino mass values[13]:

$$\left(m_2^2 - m_1^2\right)_{\exp} = \left(\Delta m_{21}^2\right)_{\exp} = 7.56 \times 10^{-5}\, ev^2/c^4 \tag{37}$$

$$\left(m_3^2 - m_1^2\right)_{\exp} = \left(\Delta m_{31}^2\right)_{\exp} = 2.55 \times 10^{-3}\, ev^2/c^4 \tag{38}$$

First, based on the data (37) and (38),an orientation value for the number *n* is calculated:

$$n+1 = \frac{\left(\Delta m_{31}^2\right)_{\exp}}{\left(\Delta m_{21}^2\right)_{\exp}} = \frac{2.55 \times 10^{-3}}{7.56 \times 10^{-5}} = 33.730 \tag{39}$$

The calculated value (39), according to theoretical analyses (13,14,15), must be an even number

$$n+1 = 34 \tag{40}$$

Therefore, it is:

$$n = 33 \tag{41}$$

Which as an odd number satisfies theoretical criterion given in the relations (13,14,15). Inserting the number *n=33* into the expressions (31,34,36) provides the following absolute values for neutrino masses:

$$m_1 = \sqrt{\frac{2}{3}\left(m_2^2 - m_1^2\right)_{\exp}\left(\sqrt{n^2+n+1} - \frac{n+2}{2}\right)} = 0.028407113\, ev/c^2 \tag{42}$$

$$m_2 = \sqrt{\frac{2}{3}\left(m_2^2 - m_1^2\right)_{\exp}\left(\sqrt{n^2+n+1} - \frac{n-1}{2}\right)} = 0.029707980\, ev/c^2 \tag{43}$$

$$m_3 = \sqrt{\frac{2}{3}\left(m_2^2 - m_1^2\right)_{\exp}\left(\sqrt{n^2+n+1} + \frac{2n+1}{2}\right)} = 0.058115093\, ev/c^2 = m_1 + m_2 \tag{44}$$

The sum of neutrino masses, (42)+(43)+(44), is

$$\sum_{j=1}^{3} m_j = m_1 + m_2 + m_3 = 2m_3 = 0.1162301869ev/c^2 \qquad (45)$$

Based on the measurement in cosmology [14], it was found that the sum of the neutrino mass is tightly constrained to

$$\sum_{j=1}^{3} m_j = m_1 + m_2 + m_3 < 0.12\,ev/c^2 \qquad (46)$$

Based on the agreement of theoretical results (45) with the experimental results (46), it follows:

1.Neutrino mass hierarchy is normal: $m_3 > m_2 > m_1$.

2. The neutrino masses are interconnected by the relation:

$$m_3 = m_1 + m_2$$

3.The quotients between the oscillation wavelengths and the quotients between the corresponding differences of the squares of mass are connected by the same odd number:*n=33*.

6 Summary and Conclusions

In neutrino physics, mass eigenstates are written as solutions of the Klein-Gordon equations, which are described using the appropriate phase factors which contain current coordinates *(x,t)*. The crucial parameter in neutrino physics is the difference between such phase factors. They are in the function of current coordinates ***(x,t)***, and their difference gives the formula that satisfies the further development of neutrino physics.
The goals set in this paper are to find the answers to an open question in neutrino physics which is related to the neutrino mass hierarchy.Then, based on the choosen hierarchy of neutrino masses, to introduce the assumption of a possible interconnection of neutrino masses.
Defining the mathematical models of the mass hierarchy, it is proven in the theoretical discussion that the number *n=33* is present on two levels:
1.On the macroscopic level between oscillation wavelengths through the number ***n***: $L_{12}/L_{13} = L_{12}/L_{23} + 1 = n + 1$
2. On the microscopic level between squared masses differences through the same number:
$\Delta m_{31}^2/\Delta m_{21}^2 = \Delta m_{32}^2/\Delta m_{21}^2 + 1 = n + 1$.
The simplest way to determine the number n is by measuring the corresponding wavelength of oscillation:
1.Only in the case when n is an odd number defined by the relations (13,14,15)
$L_{12}/L_{23} = n$
is the hierarchy of neutrino masses normal.
Based on the proposed mathematical model, the following neutrino properties are derived:

1.In the case when neutrino beam consists of three mass eigenstates,then their wave functions satisfy the following identity:

$$\left(\Psi_1\Psi_2^*\right)\left(\Psi_2\Psi_3^*\right)\left(\Psi_3\Psi_1^*\right) = e^{[i(\Phi_1-\Phi_2)]}e^{[i(\Phi_2-\Phi_3)]}e^{[-i(\Phi_1-\Phi_3)]} \equiv 1$$

2. On the basis of 1. it follows that the sum of the differences of the corresponding phase factors, at each instant of time and at each place, is always equal to zero: $(\Phi_1 - \Phi_2) + (\Phi_2 - \Phi_3) - (\Phi_1 - \Phi_3) = 0$

3.The differences between momentums multiplied by oscillation wavelength is equal to the Planck constant

$$\left| p_j - p_k \right| L_{jk} = h;\ j,k = 1,2,3.$$

4. $m_3 = m_1 + m_2$

5. $m_1 + m_2 + m_3 = 2m_3 = 0.1162301869 ev / c^2$

Based on the experimental measurement [14], it follows that the sum of neutrinos is tightly constrained to

$$\sum_{j=1}^{3} m_j < 0.12 ev / c^2$$

Which is in agreement with the theoretical calculation.

And finally, at the end of this study, it is emphasized that researches in neutrino mass hierarchy [5,6,7,8,9,10,11,12] preferred normal mass hierarchy to the inverted one.

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