Singular semi-Riemannian geometry using Colombeau approuch.

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Abstract This book is an exposition of "Singular Semi-Riemannian Geometry"- the study of a smooth manifold furnished with a degenerate (singular) metric tensor of arbitrary signature. This book also dealing with Colombeau extension of the Einstein field equations using apparatus of the Colombeau generalized function and contemporary generalization of the classical Lorentzian geometry named in literature Colombeau distributional geometry.The regularizations of singularities present in some Colombeau solutions of the Einstein equations is an important part of this approach. Any singularities present in some solutions of the Einstein equations recognized only in the sense of Colombeau generalized functions and not classically.In this paper essentially new class Colombeau solutions to Einstein fild equations is obtained. We leave the neighborhood of the singularity at the origin and turn to the singularity at the horizon.Using nonlinear distributional geometry and Colombeau generalized functions it seems possible to show that the horizon singularity is not only a coordinate singularity without leaving Schwarzschild coordinates.However the Tolman formula for the total energy E_T of a static and asymptotically flat spacetime,gives $E_T = m,$ as it should be. The vacuum energy density of free scalar quantum field Φ with a distributional background spacetime also is considered.It has been widely believed that, except in very extreme situations, the influence of gravity on quantum fields should amount to just small, sub-dominant contributions. Here we argue that this belief is false by showing that there exist well-behaved spacetime evolutions where the vacuum energy density of free quantum fields is forced, by the very same background distributional spacetime such distributional BHs, to become dominant over any classical energy density component. This semiclassical gravity effect finds its roots in the singular behavior of quantum fields on curved distributional spacetimes. In particular we obtain that the vacuum fluctuations $\langle\Phi^2\rangle$ has a singular behavior on BHs horizon r_+: $\langle\Phi^2(r)\rangle \sim |r - r_+|^{-2}$. A CHALLENGE TO THE BRIGHTNESS TEMPERATURE LIMIT OF THE QUASAR 3C273 explained successfully.

Keywords: Colombeau nonlinear generalized functions,Distributional Riemannian Geometry,Distributional Schwarzschild Geometry,Schwarzschild singularity, Schwarzschild Horizon,smooth regularization,nonsmooth regularization,quantum fields on curved spacetime,vacuum fluctuations,vacuum dominance

Mathematics Subject Classification : Primary: 83C15; secondary: 83C15, 83C75, 46F30, 46T30.

Physics and Astronomy Classification Scheme: Primary: 04.20.Cv; secondary: 04.20.Jb, 04.20.Dw, 02.30.Sa, 02.40.Vh

This paper is divided into six sections.

Shapter 1 contains a brief introduction in contemporary theory of the degenerate and singular semi Riemannian manifolds using the linear distributionel geometry based on L.Schwartz distributions $\mathcal{D}'(M)$ on smooth manifold M and Colombeau distributional geometry based on full algebra $\mathcal{G}(M)$ of the Colombeau generalized functions on smooth manifold M.

Subsection 1.1 contains a brief review the results in General Relativity obtained by using the linear and nonlinear distributional approuch in order to resolve problems with canonical Schwarzschild geometry.

Subsection 1.2 contains the elementary explanation of the basic notions of Colombeau generalized functions and Colombeau generalized numbers.

Subsection1.3 contains a brief introduction in point free classical Colombeau geometry.

Subsection1.4 contains a brief introduction in point-free loop quantum gravity.

Subsection1.5 contains description of the classical point-like phase space variables corresponding to the classical Schwarzschild metric.

Subsection1.6 contains explanation why classical point-like Loop Quantum Gravity contradict with a well established the linear Colombeau geometry.

Subsection1.6.1-1.6.2 contains a brief introduction in point free quantum Schwarzschild geometry.

Subsection1.7 contains Generalized Stokes' theorem corresponding to nonlinear Colombeau geometry.

Subsection1.8 contains derivation of the Colombeau Generalized Curvature Tensor.

Subsection1.9 contains derivation of the Generalized Einstein's field equations obtained in [37].

Subsection1.10 contains explanation why the canonical densitized Einstein field equations from contemporary literature wrong and revision is needed,etc.

Subsection1.10.1 contains some importent remarks on the A. Einstein and N. Rosen paper from 1935.

Subsection1.10.2 contains some importent remarks on Möller abnormal famous paper from 1943

Subsection1.10.3 contains derivation of the revisited densitized Einstein field equations by using the special kind of the linear distributional geometry.

Subsection1.11 contains the nonlinear distributional Möller geometry using the full algebra of the Colombeau generalized functions originally derived in [5] in order to avoid the Polchinski's paradox.

Subsection1.12 contains a brief reviev of the distributional Schwarzschild geometry by using the linear L.Schwartz distributions and by using the full algebra of the Colombeau generalized functions.

Subsection1.12.1 contains a brief reviev of the distributional Schwarzschild geometry at the origin by using the smooth regularization of the singularity at the origin.

Subsection1.12.2 contains a brief reviev of the nonsmooth regularization of the singularity at the origin.

Subsection1.12.3 contains a brief reviev of the smooth regularization via Horizon in Schwarzschild coordinates and the smooth regularization at horizon in isotropic coordinates.

Subsection1.12.4 contains a brief reviev of the nonsmooth regularization via Gorizon.

Subsection1.12.5 Distributional Eddington-Finkelstein Space-time is considered.

Shapter 2 contains a brief introduction in contemporary generalized Colombeau calculus

Subsection 2.1-2.2 contains basic notions of the canonical Colombeau theory [1],[2]

Subsection 2.3 contains the full derivation of the nonlinear distributional Schwarzschild geometry. The truncated distributional Schwarzschild geometry and Gravitational singularity is considered.

Subsection 2.4 contains the full derivation of the nonlinear distributional Schwarzschild geometry in isotropic coordinates.

Subsection 2.5 contains a brief introduction in generalized Colombeau calculus.

Subsections 2.5.1-2.5.3 contains basic definitions of the super generalized functions.

Subsection 2.5.4 contains the full derivation of the super distributional geometry of the Schwarzschild space-time.

Shapter 3 contains the full derivation of the linear distributional Schwarzschild geometry from nonsmooth regularization via horizon

Subsection 3.1 contains the full calculation of the Stress-tensor by using nonsmooth regularization via horizon.

Subsection 3.2 contains examples of distributional geometries.

Shapter 4 contains quantum field theory of the scalar field in curved distributional space-time.

Subsection 4.1 contains a brief introduction in canonical quantization in curved distributional Space-time.

Subsection 4.2 contains example of the distributional outgoing modes.

Subsection 4.2.1 contains a boundary conditions at the horizon of the distributional SAdS BH geometry.

Shapter 5 contains the full calculation of th energy-momentum tensor by using Colombeau Distributional Modes.

Shapter 6 contains the main theoretical result of this paper: Distributional SAdS

BH Space-time-induced Vacuum Dominance.
Subsection 6.1 contains the full derivation of the adiabatic expansion of Green functions in curved distributional space-time.
Subsection 6.2 contains effective action for the scalar quantum matter fields in curved distributional space-time.
Subsection 6.3 contains stress-tensor renormalization.
Shapter 7 contains novel explanation of the active galactic nuclei.
Subsection 7.1 contains a brief introduction in the current paradigm for active galactic nuclei and resalts of the observations of the high energy emission from galactic jets.
Subsection 7.1 contains Novel Explanation of the Active Galactic Nuclei based on the Colombeau distributional Kerr Space-time in Boyer - Lindquist form.

1.Introduction. Classical, semiclassical and nonclassical semi-Riemannian manifolds (M,g).

The classical Cartan's structural equations show in a compact way the relation between a connection and its curvature, and reveals their geometric interpretation in terms of moving frames. In order to study the mathematical properties of singularities, we need to study the geometry of manifolds endowed on the tangent bundle with a symmetric bilinear form which is allowed to become degenerate or singular (or both degenerate and singular) on semi Riemannian manifold (M,g) or on submanifolds of semi Riemannian manifold (M,g). But if the fundamental tensor is allowed to be degenerate or singular, there are some obstructions in constructing the geometric objects normally associated to the fundamental tensor. Also, local orthonormal frames and coframes no longer exist, as well as the metric connection and its curvature operator.

Definition 1.1. (i) Semi Riemannian manifold (M,g) is nonclassical if the fundamental tensor g is allowed to be degenerate or singular, (ii) semi Riemannian manifold (M,g) is
internally nonclassical if the fundamental tensor g is not allowed to be degenerate or singular but there exists semi Riemannian submanifold $(M',g'),M' \subsetneq M,g' = g|_{M'}$ such that the fundamental tensor g' is allowed to be degenerate or singular,(iii) otherwise we
will be say that (M,g) is classical.

Remark 1.1. In the nonclassical case the main problem arises from the degeneracy of the $\det(g_{ij}(\hat{x}))$ on some isolated points: $\det(g_{ij}(\hat{x}^0)) = 0,\hat{x}^0 \in M$ or some submanifold $\det(g_{ij}(\hat{x})) = 0$ for all $\hat{x} \in M' \subsetneq M$ and consequantly the corresponding Christoffel symbols
bicome infinity.

In mathematical literature more than 50 yers accepted that a nonclassical semi Riemannian manifold mentioned above impossible treated classically, i.e. by using canonical apparatus of the Riemannian geometry. However in the contemporary mathematical literature, manifolds with degenerate metric tensors have been studied only fore some special case called a Reinhart manifold (see for example [3] Refferences B).

Remark 1.1. In this paper we studied nonclassical semi Riemannian manifold (M,g) extrinsically,i.e. as degenerate submanifolds of Colombeau generalized semi Riemannian manifolds furnished with non degenerate Colombeau generalized or super generalized fundamental tensor $(g_{ij,\varepsilon})_\varepsilon \in \mathcal{G}_0^2(M,\Sigma)$, see subsect.2.5,Definition 2.5.8.

Definition 1.2.We shall say that Colombeau generalized semi Riemannian manifold $(M,(g_{i,j,\varepsilon}(\hat{x}))_\varepsilon)$ is the Colombeau extension of the nonclassical semi Riemannian manifold $(M,g_{i,j,0}(\hat{x}))$ if there exists the canonical imbedding $(M,g_{i,j,0}(\hat{x})) \hookrightarrow (M,(g_{i,j,\varepsilon}(\hat{x}))_\varepsilon)$, see subsect.1.1,where such imbedding is obtained for Schwarzschild space-time with metric (1.6) in canonical Schwarzschild coordinates (t,r,θ,ϕ).

Remark 1.2. The canonical imbedding $(M,g_{i,j,0}(\hat{x})) \hookrightarrow (M,(g_{i,j,\varepsilon}(\hat{x}))_\varepsilon)$ easily obtained by aproporiate ε-regularization of the tensor $g_{i,j,0}(\hat{x})$, see subsections 1.11.1-1.11.5.

Remark 1.3. (i) Note that the notion of the classical Rimannian curvature comes from the study of parallel transport on a classical Rimannian manifold,see Fig.1.1. For instance, if a vector $A_i(\hat{x})$ is moved around closed contour (or a loop) on the surface of a sphere keeping parallel throughout the motion,then the final position of the vector may not be the same as the initial position of the vector. This phenomenon is known as holonomy.

The classical holonomy presented by the classical formula (1.1) for the change ΔA_k in a smooth vector $A_i(\hat{x})$ after parallel displacement around infinitesimal closed contour Γ.

The classical formula for the change in a smooth vector $A_i(\hat{x})$ after parallel displacement around infinitesimal closed contour Γ or integral measure of the classical holonomy of the surface Σ_Γ spanning by Γ reads [4]:

$$\Delta A_k(\Gamma) = \oint_\Gamma \delta A_k = \oint_\Gamma \Gamma^i_{kl}(\hat{x}) A_k dx^l. \tag{1.1}$$

Now applying classical Stokes' theorem (see Theorem 1.10.1) to the integral (1.1) and considering that the area enclosed by the contour has the infinitesimal value (Δf^{lm}), one obtains, (see subsection 1.10.3) [4]:

$$\Delta A_k = \frac{1}{2} R^i_{klm}(x) A_i(x) \Delta f^{lm}, \tag{1.2}$$

where $R^i_{klm}(x)$ is a tensor field of the fourth rank:

$$R^i_{klm}(x) = \frac{\partial(\Gamma^i_{km}(x))}{\partial x^l} - \frac{\partial(\Gamma^i_{kl}(x))}{\partial x^m} + \Gamma^i_{ni}(x)\Gamma^n_{km}(x) - \Gamma^i_{nm}(x)\Gamma^n_{kl}(x). \tag{1.3}$$

The tensor field $R^l_{kim}(x)$ is called the classical curvature tensor or the classical Riemann tensor.

The classical Riemann tensor that is a tensorial measure of holonomy

(ii) Various generalizations capture in an abstract form this idea of curvature as a measure of holonomy.

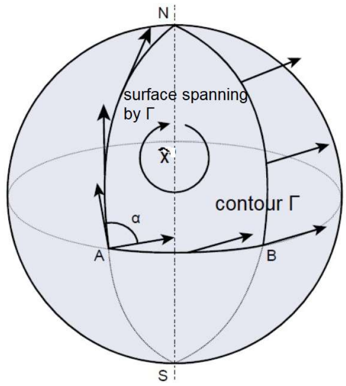

Fig.1.1.Parallel transporting a vector from
A → N → B → A yields a different vector.
This failure to return to the initial vector
is measured by the change in asmooth
vector $A_i(\hat{x})$ after parallel displacement around
infinitesimal closed contour Γ or
by the classical integral measure $\Delta A_k(\Gamma)$ of the
holonomy of the surface Σ_Γ spanning by contour Γ.

(iv) Note that in classical case the change $\Delta A_k(\Gamma)$ always finite,i.e. $\Delta A_k(\Gamma) < \infty$
(v) If semi Riemannian manifold (M,g) is nonclassical the condition $\Delta A_k(\Gamma) < \infty$ is not always holds and we shall say that contour Γ is regular if $\Delta A_k(\Gamma) < \infty$,see Fig.1.2 if $\Delta A_k(\Gamma) = \infty$ we shall say that contour Γ is a singular contour or singular loop,see Fig.1.3.

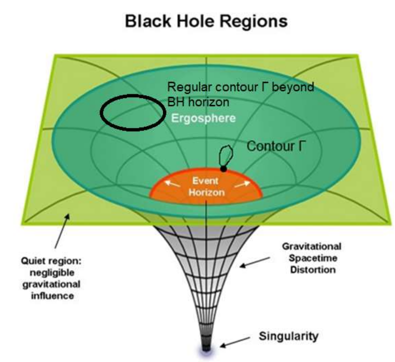

Fig.1.2.Regular contour Γ
beyond BH horizon.

Definition 1.3.We shall say that a point $\hat{x}_0 \in \Sigma_\Gamma$ is a singular point of the surface $\Sigma_\Gamma \subset M$
if the Levi-Civitá connection is not available at point $\hat{x}_0$, i.e. some of the Christoffel symbols become infinity at point $\hat{x}_0$.

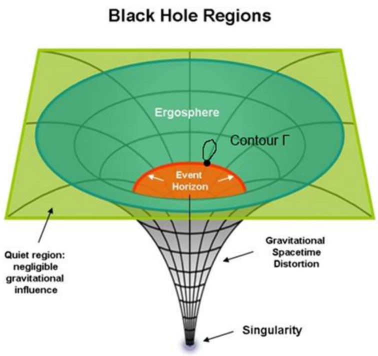

Fig.1.3. Singular point $\hat{x}_0$ at BH horizon
and corresponding singular countur $\Gamma_{\mathbf{sing}}$.

Remark 1.4. Note that any singular countur $\Gamma_{\mathbf{sing}}$ contains at least one singular point $\hat{x}_0 \in \Gamma_{\mathbf{sing}}$, see Fig.1.3 and we shall abbreviate $\Gamma^{\hat{x}_0}_{\mathbf{sing}}$ if $\hat{x}_0 \in \Gamma^{\hat{x}_0}_{\mathbf{sing}}$.
Remark 1.5.(i) Note that the classical formula (1.1) holds only for regular loops but obviously breaks down on singular loops $\Gamma_{\mathbf{sing}}$, since $\Delta A_k(\Gamma_{\mathbf{sing}}) = \infty$.
(ii) Note that Eqs.(1.2)-(1.3) again no longer hold for singular loops $\Gamma_{\mathbf{sing}}$ (see sect.1.8). and therefore the classical Ricci scalar and the classical Kretschman scalar is not holds
at BH horizon. However in classical literature (see [3],[4]) it was just pulled to the BH horizon by the ears.
Remark 1.6.(i) Note that for trunketed singular contour $\Gamma^{\#}_{\mathbf{sing}} = \Gamma^{\hat{x}_0}_{\mathbf{sing}} \backslash \{\hat{x}_0\}$ see Fig.1.4, the
Levi-Civit'a connection is available at whole contour $\Gamma^{\#}_{\mathbf{sing}}$ and classical formula (1.1) reads:

$$\Delta A_k(\Gamma^{\#}_{\mathbf{sing}}) = \oint\limits_{\Gamma^{\hat{x}_0}_{\mathbf{sing}} \backslash \{\hat{x}_0\}} \delta A_k = \oint\limits_{\Gamma^{\hat{x}_0}_{\mathbf{sing}} \backslash \{\hat{x}_0\}} \Gamma^i_{kl}(\hat{x}) A_k dx^l.$$

Obviously for trunketed singular contour $\Gamma^{\#}_{\mathbf{sing}} = \Gamma^{\hat{x}_0}_{\mathbf{sing}} \backslash \{\hat{x}_0\}$ again we get
$\Delta A_k(\Gamma_{\hat{x}_0} \backslash \{\hat{x}_0\}) = \infty$ and therefore the Eqs.(1.2)-(1.3) again no longer hold for trunketed singular loops $\Gamma^{\#}_{\mathbf{sing}}$.
(ii) The semi Riemannian manifold (M,g) which contain trunketed singular loops $\Gamma^{\hat{x}_0}_{\mathbf{sing}}$ such that $\Gamma^{\hat{x}_0}_{\mathbf{sing}} \subset M, \hat{x}_0 \notin M$, seems as classical since the
Levi-Civit'a connection is available at whole

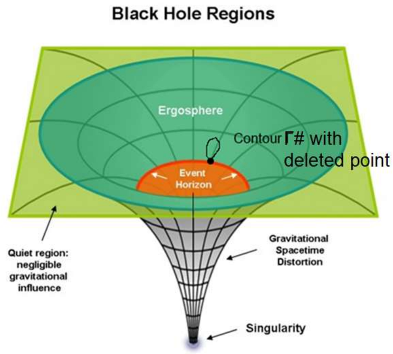

Fig.1.4.Trunketed singular contour
(loop) $\Gamma^{\#}_{\mathbf{sing}} = \Gamma_{\hat{x}_0}\backslash\{\hat{x}_0\}$ with deleted
singular point $\hat{x}_0$.

(iii) Note that in order to obtain the full geometrical properties of BH horizon one needs non classical definition of the integral measure $\widetilde{\Delta A}_k(\Gamma_{\mathbf{sing}})$ of the holonomy of the surface

$\Sigma_{\Gamma_{\mathbf{sing}}}$ spanning by singular loop $\Gamma_{\mathbf{sing}}$ such that $\widetilde{\Delta A}_k(\Gamma_{\mathbf{sing}})$ is well defined quantity.

(iv) By using contemporary Colombeau approach [1] one obtains appropriate definition of

the non classical integral measure $\widetilde{\Delta A}_k(\Gamma_{\mathbf{sing}})$ of the holonomy as direct generalization in

natural way of the Eq.(1.1) (see subsection 1.8 Remark 1.8.8,Eq.(1.8.10))

$$\widetilde{\Delta A}_k(\Gamma_{\mathbf{sing}}) \triangleq (\Delta A_{k,\varepsilon})_\varepsilon = \left(\oint_{\Gamma^{\hat{x}^0}_{\mathbf{sing}}} \delta A_{k,\varepsilon}\right)_\varepsilon = \left(\oint_{\Gamma^{\hat{x}^0}_{\mathbf{sing}}} \Gamma^i_{kl,\varepsilon}(\hat{x}) A_{k,\varepsilon} dx^l\right)_\varepsilon. \tag{1.4}$$

where point $\hat{x}^0$ belongs to BH horizon and $\hat{x}^0 \in \Gamma^{\hat{x}^0}_{\mathbf{sing}}$,$\widetilde{\Delta A}_k(\Gamma_{\mathbf{sing}}) \in \widetilde{\mathbb{R}}$,

Definition 1.4.We shall say that a surface $\Sigma^{\mathbf{sing}}_{\Gamma} \subset M$ is a singular surface if $\Sigma^{\mathbf{sing}}_{\Gamma}$ it has at least one singular point point $\hat{x}_0 \in \Sigma^{\mathbf{sing}}_{\Gamma}$, see Fig.1.5.

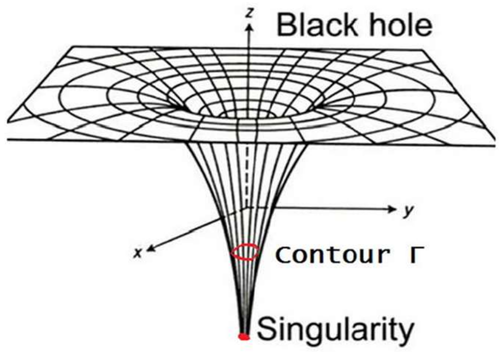

Fig.1.5.Singular surface $\Sigma_{\Gamma}^{\mathbf{sing}}$ with singular
point $r = 0$,
spanning by regular contour Γ.

Remark 1.7.Note that for the case of the surface $\Sigma_{\Gamma}^{\mathbf{sing}}$ again problem arises, (see sect.1.8.Remark 1.8.3) even if it spanned by regular contour Γ, see Fig.1.5. In this case the non classical integral measure $\widetilde{\Delta A}_k\left(\Gamma, \Sigma_{\Gamma}^{\mathbf{sing}}\right)$ of the holonomy reads:

$$\widetilde{\Delta A}_k\left(\Gamma, \Sigma_{\Gamma}^{\mathbf{sing}}\right) \triangleq (\Delta A_{k,\varepsilon})_{\varepsilon} = \left(\oint_{\Gamma} \delta A_{k,\varepsilon}\right)_{\varepsilon} = \left(\oint_{\Gamma} \Gamma^{i}_{kl,\varepsilon}(\hat{x},\hat{x}_0) A_{k,\varepsilon} dx^l\right)_{\varepsilon}. \qquad (1.5)$$

Definition 1.5.Let M be a differentiable manifold equipped with it canonical topology $\Im$.

Let $M^{\#}$ be a submanifold $M^{\#} \subsetneqq M$. The closure $M^{\#}_{\Im}$ of a submanifold $M^{\#} \subsetneqq M$ of points in a topological space $(M, \Im)$ consists of all points in $M^{\#}$ together with all limit points of $M^{\#}$.

The closure of $M^{\#}_{\Im}$ may equivalently be defined as the union of $M^{\#}$ and its boundary $\partial M^{\#}$, and also as the intersection of all closed sets containing $M^{\#}$. Intuitively, the closure can be thought of as all the points that are either in $M^{\#}$ or "near" $M^{\#}$. A point which is in the closure of $M^{\#}$ is a point of closure of $M^{\#}$.

Definition 1.6. Let (M,g) and $(M^{\#},g^{\#})$ semi Riemannian manifolds such that (i) $M^{\#} \subsetneqq M$, (ii) $M^{\#}_{\Im} = M$ and (iii) $g^{\#} = g|_{M^{\#}}$. Assume that: (1) the Levi-Civit‘a connection $\Gamma^{\#l}_{kj} = \frac{1}{2}[g^{\#lm}][(g^{\#}_{mk,j} + g^{\#}_{mj,k} - g^{\#}_{kj,m}]$ corresponding to metric tensor $g^{\#}$ is available at whole semi Riemannian manifold $(M^{\#},g^{\#})$ and (2) the Levi-Civit‘a connection $\Gamma^{l}_{kj} = \frac{1}{2}[g^{lm}][(g_{mk,j} + g_{mj,k} - g_{kj,m}]$ corresponding to metric tensor g is not available at its boundary $\partial M^{\#}$, i.e. (M,g) is a nonclassical semi Riemannian manifold.Then we shall say that $(M^{\#},g^{\#})$ is a semiclassical semi Riemannian manifold.

Remark 1.8.It follows from consideration above that semiclassical semi Riemannian manifolds obviously impossible treated classically as nonclassical semi Riemannian manifolds mentioned above.

Example 1.1. Obviously the Levi-Civita connection is available at whole Schwarzschild spacetime (see Remark 1.10.1) $(\mathbf{Sch}, g_{ij}^{\mathbf{Sch}}(t,r,\theta,\phi))$,

$$\mathbf{Sch} = (\mathbf{S}^2 \times \{r > 2m\} \cup \{0 < r < 2m\}) \times \mathbb{R},$$

but spacetime $\mathbf{Sch}$ contains truncated singular loops $\Gamma^{\#}_{\mathbf{sing}} = \Gamma_{\hat{x}_0} \backslash \{\hat{x}_0\} \subset \mathbf{Sch}$, see Fig.1.4,

and in particular the Levi-Civita connectionis is not available at whole its topological closure $(\overline{\mathbf{Sch}}, g_{ij}^{\mathbf{Sch}}(t,r,\theta,\phi))$, where

$$\hat{x}_0 \in \overline{\mathbf{Sch}} = (\mathbf{S}^2 \times (\{r \geq 2m\} \cup \{0 \leq r \leq 2m\})) \times \mathbb{R}.$$

Thus Schwarzschild spacetime is not classical but exactly is a semiclassical semi Riemannian manifold. Similarly the Levi-Civita connection is available at the open semi Riemannian manifold $(\mathbf{Sch}^+, g_{ij}^{\mathbf{Sch}^+}(t,r,\theta,\phi))$ above Schwarzschild horizon: $\mathbf{Sh}^+ = (\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}$ but is not available at whole its topological closure $\left(\overline{\mathbf{Sch}^+}, g_{ij}^{\overline{\mathbf{Sch}^+}}(t,r,\theta,\phi)\right), \overline{\mathbf{Sch}^+} = (\mathbf{S}^2 \times \{r \geq 2m\}) \times \mathbb{R}$.

Thus semi Riemannian manifold $\mathbf{Sh}^+$ is not classical but exactly is a semiclassical semi Riemannian manifold.

Remark 1.9. Note that in physical literature the spacetime $\mathbf{Sch}^+ = (\mathbf{S}^2 \times \{r > 2m\}) \times \mathbb{R}$ mistakenly considered as classical semi Riemannian manifold. Note that only spacetimes $\left(\mathbf{Sch}^+_\delta, g_{ij}^{\overline{\mathbf{Sch}^+_\delta}}\right)$, $\mathbf{Sch}^+_\delta = (\mathbf{S}^2 \times \{r > 2m + \delta\}) \times \mathbb{R}, \delta > 0$ is a classical semi Riemannian manifolds.

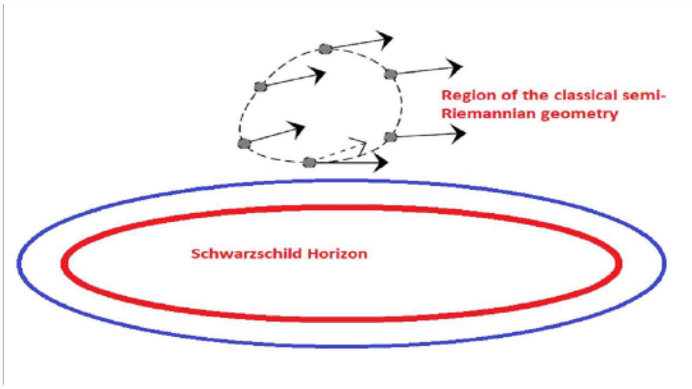

Fig.1.6.Region of the classical semi Riemannian geometry above Schwarzschild horizon $\mathbf{Sh}^+_\delta = (\mathbf{S}^2 \times \{r > 2m + \delta\}) \times \mathbb{R}, \delta > 0.$

Remark 1.10. Note that in physical literature

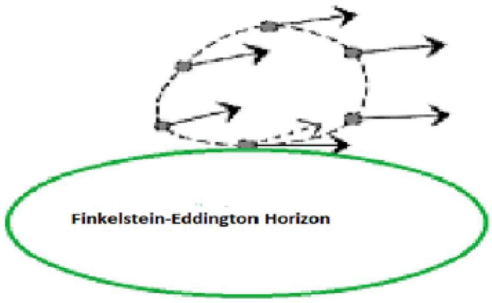

Fig.1.7.The classical semi Riemannian Geometry
presented by Eddington-Finkelstein spacetime

Remark 1.10. There exist examples of the nonclassical semi Riemannian manifolds. However in physical literature these nonclassical semi Riemannian manifolds mistakenly were considered as classical semi Riemannian manifolds.

1.The Schwarzschild metric represented by Schwarzschild coordinates (t,r,θ,ϕ) reads

$$ds^2 = -\left(1-\frac{r_s}{r}\right)(dt)^2 + \left(1-\frac{r_s}{r}\right)^{-1}(dr)^2 + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \quad (1.6)$$

The Christoffel symbols in Schwarzschild coordinates (t,r,θ,ϕ) reads

$$\begin{gathered}\Gamma^r_{tt} = \frac{c^2 r_s(r-r_s)}{2r^3}, \Gamma^t_{tr} = \frac{r_s}{2r(r-r_s)}, \Gamma^r_{rr} = -\frac{r_s}{2r(r-r_s)},\\ \Gamma^\theta_{r\theta} = \frac{1}{r}, \Gamma^\phi_{r\phi} = \frac{1}{r}, \Gamma^r_{\theta\theta} = -(r-r_s), \Gamma^\phi_{\theta\phi} = \cot\theta,\\ \Gamma^r_{\phi\phi} = -(r-r_s)\sin 2\theta, \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta.\end{gathered} \quad (1.7)$$

Remark 1.3. Note that:(i) the Schwarzschild metric is allowed to become degenerate and singular at Schwarzschild horizon $\{r = r_s\}$, (ii) the Schwarzschild metric is allowed to become degenerate and singular at Schwarzschild singularity $\{r = 0\}$, (iii) the Christoffel symbols Γ^t_{tr} and Γ^r_{rr} become infinity at Schwarzschild horizon $\{r = r_s\}$.
Thus the Schwarzschild space-time is nonclassical semi Riemannian manifold. The full nonlinear distributional Schwarzschild geometry at horizon is obtained in subsect. 2.3.

2.The singular transformation of the Schwarzschild metric (1.6) from the usual Schwarzschild time coordinate t to the advanced null coordinate v with

$$cv = ct + r + r_s \ln(r - r_s) \quad (1.8)$$

leads to the ingoing Eddington-Finkelstein metric (1.9) with coordinates (v,r,ϑ,ϕ).The Schwarzschild metric (1.6) in Eddington-Finkelstein coordinates (v,r,θ,ϕ), reads

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dv^2 + 2cdvdr + r^2 d\Omega^2. \quad (1.9)$$

The Christoffel symbols in Eddington-Finkelstein coordinates (v,r,θ,ϕ), reads

$$\Gamma^{v}_{vv} = \frac{cr_s}{2r^2}, \Gamma^{r}_{vv} = \frac{c^2 r_s(r - r_s)}{2r^3}, \Gamma^{r}_{vr} = -\frac{cr_s}{2r^2}, \Gamma^{\theta}_{r\theta} = \frac{1}{r},$$
$$\Gamma^{\phi}_{r\phi} = \frac{1}{r}, \Gamma^{v}_{\theta\theta} = -\frac{r}{c}, \Gamma^{r}_{\theta\theta} = -(r - r_s), \Gamma^{\phi}_{\theta\phi} = \cot\theta, \quad (1.10)$$
$$\Gamma^{v}_{\phi\phi} = \frac{-r\sin^2\theta}{c}, \Gamma^{r}_{\phi\phi} = -(r - r_s)\sin^2\theta, \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta.$$

Remark 1.9. The Eddington-Finkelstein space-time is classical semi Riemannian manifold except any finite neighborhood of singularity $\{r = 0\}$.

Remark 1.10.Note that Eddington-Finkelstein space-time is not equivalent with Schwarzschild space-time, since the transform (1.8) is singular at Schwarzschild horizon $\{r = r_s\}$.

3.The Schwarzschild metric (1.6) in isotropic coordinates (t, ρ, θ, ϕ) reads,

$$ds^2 = -\left(\frac{1 - r_s/4\rho}{1 + r_s/4\rho}\right)^2 c^2 dt^2 + \left(1 + \frac{r_s}{4\rho}\right)^4 d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1.11)$$

where

$$r = \rho\left(1 + \frac{r_s}{4\rho}\right)^2 \quad (1.12)$$

is the coordinate transformation between the Schwarzschild radial coordinate r and the isotropic radial coordinate ρ. The Christoffel symbols reads

$$\Gamma^{\rho}_{tt} = 2048\rho^4 r_s c^2 \frac{(4\rho - r_s)}{(4\rho + r_s)^7}, \Gamma^{t}_{t\rho} = \frac{8r_s}{16\rho^2 - r_s^2}, \Gamma^{\rho}_{\rho\rho} = -\frac{2r_s}{(4\rho + r_s)\rho},$$
$$\Gamma^{\theta}_{\rho\theta} = \frac{4\rho - r_s}{(4\rho + r_s)\rho}, \Gamma^{\phi}_{\rho\phi} = \frac{4\rho - r_s}{(4\rho + r_s)\rho}, \Gamma^{\rho}_{\theta\theta} = -\rho\frac{4\rho - r_s}{4\rho + r_s}, \quad (1.13)$$
$$\Gamma^{\phi}_{\theta\phi} = \cot\theta, \Gamma^{\rho}_{\phi\phi} = -\frac{(4\rho - rs)\rho\sin^2\theta}{4\rho + rs}, \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta.$$

Remark 1.11. (i) Note that "Schwarzschild space-time in isotropic coordinates" is nonclassical semi Riemannian manifold, since the metric (1.6) is allowed to become degenerate at horizon $\rho^* = r_s/4$, (ii) note that "Schwarzschild space-time in isotropic coordinates" is not equivalent with Schwarzschild space-time in Schwarzschild coordinates (t, r, θ, ϕ). The full nonlinear distributional "Schwarzschild geometry in isotropic coordinates" at horizon is obtained in subsect. 2.4.

Remark 1.10.Note that the space-times mentioned above in physical literature mistakenly were considered as the same geometrical object in different coordinates only.

1.1.Remarks on linear and nonlinear distributional geometry in general relativity.Why

A degenerate (singular) semi-Riemannian manifold $(M; g)$ is a differentiable manifold M endowed with a symmetric bilinear form $g \in T^0_2 M$ named metric. Note that the metric g is not required to be non-degenerate. In particular, if the metric is non-degenerate, $(M; g)$ is a semi-Riemannian manifold. If in addition g is positive definite, $(M; g)$ is a Riemannian manifold.

This paper dealing with Colombeau extension of the Einstein field equations using apparatus of the Colombeau generalized function [1]-[2] and contemporary generalization of the classical Lorentzian geometry named in literature Colombeau distributional geometry. The regularizations of singularities present in some solutions of the Einstein equations is an important part of this approach. Any singularities present in some solutions such that Schwarzschild solution etc. of the Einstein equations recognized only in the sense of Colombeau generalized functions [1]-[2] and not classically. Note that in physical literature these singular solutions many years were mistakenly considered as vacuum solutions of the Einstein field equations, see for example [26],[30].

During last 30 years the applications classical linear distributional geometry in general relativity was many developed [5]-[31].

Remark 1.1.1. Let $(R^a_{bcd,\varepsilon})_\varepsilon \in \mathcal{G}_\delta(\mathbb{R}^4)$ be Colombeau generalized function obtained using

the standard definition of the Riemann curvature in a coordinate basis,i.e.

$$(R^a_{bcd,\varepsilon})_\varepsilon = (\Gamma^a_{db,c,\varepsilon})_\varepsilon - (\Gamma^a_{cb,d,\varepsilon})_\varepsilon + \left(\Gamma^a_{cf,\varepsilon}\Gamma^f_{ab,\varepsilon}\right)_\varepsilon - \left(\Gamma^a_{df,\varepsilon}\Gamma^f_{cb,\varepsilon}\right)_\varepsilon, \tag{1.1.1}$$

where $(\Gamma^a_{bc,\varepsilon})_\varepsilon \in \mathcal{G}_\delta(\mathbb{R}^n)$ and $\Gamma^a_{bc,\varepsilon}, \varepsilon \in (0,\delta], \delta \leq 1$ is the regularized Levi-Civita connection

coefficients in terms of the regularized metric $g_{ab,\varepsilon}, \varepsilon \in (0,1]$ such that $(g_{ab,\varepsilon})_\varepsilon$, $(g^{ab}_\varepsilon)_\varepsilon$ $\in \mathcal{G}(\mathbb{R}^4), (\det(g_{ab,\varepsilon}))_\varepsilon \neq 0_{\widetilde{\mathbb{R}}}$ It has been shown by many authors (see for example [22]) that under apropriate regularization using the Eq.(1.1.1) one can defines the curvature scalar as a classical Schwartz distribution in $\mathcal{D}'(\mathbb{R}^n)$, [18],[19].

Remark 1.1.2. This is the case even for the well-known Schwarzschild spacetime, which

is given in the Schwarzschild coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$, by the metric

$$ds^2 = -\left(1 - \frac{a}{\hat{r}}\right)(d\hat{x}^0)^2 + \left(1 - \frac{a}{\hat{r}}\right)^{-1}(d\hat{r})^2 + \hat{r}^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \tag{1.1.2}$$

Here, a is the Schwarzschild radius $a = 2GM/c^2$ with G, M and c being the Newton gravitational constant, mass of the source, and the light velocity in vacuum Minkowski space-time, respectively. Obviously the fundamental tensor corresponding to ds^2 has the components which is degenerate or singular: (i) at $\hat{r} = 0$ and (ii) at $\hat{r} = a$.

Remark 1.1.3. Note that in classical papers [5]-[31],etc. (i) the Colombeau distributional metric tensor $(g_{ab,\varepsilon})_\varepsilon \in \mathcal{G}_\delta(\mathbb{R}^4)$ related to $(R^a_{bcd,\varepsilon})_\varepsilon \in \mathcal{G}_\delta(\mathbb{R}^4)$ by Eq.(1.1.1) never is not considered as the Colombeau solution of the Einstein field equations, (ii) Colombeau nonlinear distributional geometry never is not considered as the rigorous mathematical model related to really physical spacetime but only as useful purely mathematical tools in order to obtain related to $(R^a_{bcd,\varepsilon})_\varepsilon \in \mathcal{G}_\delta(\mathbb{R}^4)$ classical Schwartz distributions in $\mathcal{D}'(\mathbb{R}^n)$, (iii) there is no any important physical applications of the classical linear distributional geometry were obtained.

Remark 1.1.4. Originally fundamental physical applications of the Colombeau nonlinear distributional geometry has been obtained in author paper [33]-[37].

By using now the Cartesian coordinates $(\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)$, which are related to $(\hat{x}^0, \hat{r}, \theta, \phi)$ through the canonical relations: $\hat{x}^1 = \hat{r}\cos\phi\sin\theta$, $\hat{x}^2 = \hat{r}\sin\phi\sin\theta$, $\hat{x}^3 = \hat{r}\cos\theta$,the metric (1.1.2) reads $ds^2 = \hat{g}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu$,where at points $\hat{r} \neq 0, \hat{r} \neq a$ the metric $\hat{g}_{\mu\nu}$ is given by

[29]:

$$\hat{g}_{00} = -(1-h)\ ,\ \ \hat{g}_{0\alpha} = 0\ ,$$
$$\hat{g}_{\alpha\beta} = \delta^{\alpha\beta} + h(1-h)^{-1}\frac{\hat{x}^{\alpha}\hat{x}^{\beta}}{\hat{r}^2}\ ,\quad \alpha,\beta = 1,2,3 \tag{1.1.3}$$

with $h \triangleq a/\hat{r}$. Well known that at points $\hat{r} \neq 0, \hat{r} \neq a$:

$$\kappa\hat{\widetilde{T}}_0^{\,0} = -\frac{h'}{\hat{r}} - \frac{h}{\hat{r}^2}\ ,$$
$$\kappa\hat{\widetilde{T}}_0^{\,\alpha} = 0\ ,\quad \kappa\hat{\widetilde{T}}_\alpha^{\,0} = 0\ ,$$
$$\kappa\hat{\widetilde{T}}_\alpha^{\,\beta} = \delta_\alpha^\beta\left(-\frac{h''}{2} - \frac{h'}{\hat{r}}\right) + \frac{\hat{x}^{\alpha}\hat{x}^{\beta}}{\hat{r}^2}\left(\frac{h''}{2} - \frac{h}{\hat{r}^2}\right), \tag{1.1.4}$$

where the hatted symbols $\hat{\widetilde{T}}_\mu^{\,\nu}$ represent the quantity $\widetilde{T}_\mu^{\,\nu}$ in the coordinate system $\{\hat{x}^\mu; \mu = 0,1,2,3\}$. Also, we have defined $h' \triangleq dh/d\hat{r}$ and $h'' \triangleq d^2h/d\hat{r}^2$.

Remark 1.1.5. We extend now the quantity (1.1.3)-(1.1.4) in point $\hat{r} = 0$ as Colombeau generalized functions from Colombeau algebra $\mathcal{G}_\delta(\mathbb{R}^3)$.Regularizing now the function $h = a/\hat{r}$ as $(h_\varepsilon)_\varepsilon = a/\left(\sqrt{\hat{r}^2+\varepsilon^2}\right)_\varepsilon$ and the function $\frac{\hat{x}^{\alpha}\hat{x}^{\beta}}{\hat{r}^2}$ as $\frac{\hat{x}^{\alpha}\hat{x}^{\beta}}{(\hat{r}^2+\varepsilon^2)_\varepsilon}$ with $\varepsilon \in (0,1]$,

we

replace now the the singular metric (1.1.3) by the Colombeau generalized metric

$$ds^2 = \left(\hat{g}_{\mu\nu,\varepsilon}d\hat{x}^\mu d\hat{x}^\nu\right)_\varepsilon\ , \tag{1.1.5}$$

where

$$\left(\hat{g}_{00,\varepsilon}\right)_\varepsilon = -(1-h_\varepsilon)\ ,\ \ \left(\hat{g}_{0\alpha,\varepsilon}\right)_\varepsilon = 0_{\widetilde{\mathbb{R}}}\ ,$$
$$\left(\hat{g}_{\alpha\beta,\varepsilon}\right)_\varepsilon = \delta^{\alpha\beta} + ((h_\varepsilon(1-h_\varepsilon)^{-1})_\varepsilon)\left(\frac{\hat{x}^{\alpha}\hat{x}^{\beta}}{\hat{r}^2+\varepsilon^2}\right)_\varepsilon,\quad \alpha,\beta = 1,2,3 \tag{1.1.6}$$

and therefore

$$\kappa\left(\hat{\widetilde{T}}_{0,\varepsilon}^{\,0}(\hat{x})\right)_\varepsilon = -\left(\frac{a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon, \kappa\left(\hat{\widetilde{T}}_0^{\,\alpha}(\hat{x};\varepsilon)\right)_\varepsilon = 0_{\widetilde{\mathbb{R}}},\quad \kappa\left(\hat{\widetilde{T}}_\alpha^{\,0}(\hat{x};\varepsilon)\right)_\varepsilon = 0_{\widetilde{\mathbb{R}}},$$
$$\kappa\left(\hat{\widetilde{T}}_\alpha^{\,\beta}(\hat{x};\varepsilon)\right)_\varepsilon = \delta_\alpha^\beta\left(\frac{3a\varepsilon^2}{2(\hat{r}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon - \tag{1.1.7}$$
$$\left(\left(\frac{\hat{x}^{\alpha}\hat{x}^{\beta}}{\hat{r}^2+\varepsilon^2}\right)_\varepsilon\right)\left(\frac{a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon\left(\frac{5}{2} + \left(\frac{\varepsilon^2}{\hat{r}^2+\varepsilon^2}\right)_\varepsilon\right).$$

Note that from Eq. (1.1.7) one obtains

$$\hat{\widetilde{T}}_\mu^{\,\nu}(\hat{x}) \triangleq w\text{-}\lim_{\epsilon\to 0}\hat{\widetilde{T}}_{\mu,\varepsilon}^{\,\nu}(\hat{x}) \sim -Mc^2\delta_\mu^{\,0}\delta_0^{\,\nu}\delta^{(3)}(\hat{x}). \tag{1.1.8}$$

Remark 1.1.6.Note that $\left(\hat{\widetilde{T}}_0^{\,0}(\hat{x};\varepsilon)\right)_\varepsilon, \left(\hat{\widetilde{T}}_\alpha^{\,\beta}(\hat{x};\varepsilon)\right)_\varepsilon \in \mathcal{G}(\mathbb{R}^4)$.Thus the generalized

Einstein

equation [37] related to Eq.(1.1.6)-Eq.(1.1.7) in Colombeau notations reads:

$$\left(\hat{G}_{\mu,\varepsilon}{}^{\nu}\right)_\varepsilon = \left(\hat{R}_{\mu,\varepsilon}{}^{\nu}\right)_\varepsilon - \frac{1}{2}\delta_\mu{}^\nu\left(\hat{R}_\varepsilon\right)_\varepsilon = \kappa\left(\widetilde{T}_{\mu,\varepsilon}{}^{\nu}\right)_\varepsilon, \tag{1.1.9}$$

where

$$\left(\hat{R}_\varepsilon(\hat{r})\right)_\varepsilon = \left(\hat{R}^\mu{}_{\mu,\varepsilon}(\hat{r})\right)_\varepsilon =$$
$$-\left(\frac{3a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon + \left(\frac{2a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon = -\left(\frac{a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon \quad (1.1.10)$$

Remark 1.1.7. Note that the regularized scalar curvature $\hat{R}_\varepsilon$ has the well-defined weak limit $\hat{R}_w$ in $\mathcal{D}'(\mathbb{R}^n)$

$$\hat{R}_w \triangleq w\text{-}\lim_{\epsilon\to 0}\hat{R}_\varepsilon = -\frac{4}{3}\pi a\delta^{(3)}(\hat{x}) \ . \quad (1.1.11)$$

Remark 1.1.8. Note that: (i) for any $(\hat{r}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}$ such that $\mathbf{st}((\hat{r}_\varepsilon)_\varepsilon) = \hat{r}_{\mathbf{fin}} \neq 0$,(see Definition
1.2.5) where $\hat{r}_{\mathbf{fin}} \in \mathbb{R}$ from Eq.(1.1.10) it follows that

$$\mathbf{st}\left(\left(\hat{R}_\varepsilon(\hat{r}_\varepsilon)\right)_\varepsilon\right) = -\mathbf{st}\left(\left(\frac{a\varepsilon^2}{(\hat{r}_\varepsilon^2+\varepsilon^2)^{5/2}}\right)_\varepsilon\right) = -\mathbf{st}\left(\left(\frac{a\varepsilon^2}{(\hat{r}_{\mathbf{fin}}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon\right) = 0, \quad (1.1.12)$$

(ii) for any $(\hat{r}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}$ such that $(\hat{r}_\varepsilon) \leq_{\widetilde{\mathbb{R}}} (\varepsilon)_\varepsilon$ (see Definition 1.2.4) from Eq.(1.1.10) it follows that

$$\left(\left(\hat{R}_\varepsilon(\hat{r})\right)_\varepsilon\right) \approx_{\widetilde{\mathbb{R}}} -\widetilde{\infty}, \quad (1.1.13)$$

(iii) at origin $(\hat{r}_\varepsilon^O)_\varepsilon =_{\widetilde{\mathbb{R}}} 0_{\widetilde{\mathbb{R}}}$ (see Definition 1.2.4) one obtains

$$\left(\left(\hat{R}_\varepsilon(\hat{r}_\varepsilon^O)\right)_\varepsilon\right) =_{\widetilde{\mathbb{R}}} -\left(\frac{a\varepsilon^2}{(\varepsilon^2)^{5/2}}\right)_\varepsilon =_{\widetilde{\mathbb{R}}} -\frac{a}{(\varepsilon^3)_\varepsilon}, \quad (1.1.14)$$

where $\varepsilon \in (0,\delta]$.

Remark 1.1.9. Note that the Eq.(1.1.12) in accordance with Eq.(1.1.11) and by Eqs.(1.1.12)-(1.1.14) we have recovered the intuitive meaning about δ-function.
For the regularized quadratic scalars one obtains [29]:

$$\hat{R}_\varepsilon^{\mu\nu}(\hat{r})\hat{R}_{\mu\nu,\varepsilon}(\hat{r}) = \frac{1}{2}\left[\frac{3a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right]^2 + 2\left[\frac{a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right]^2 =$$
$$\frac{13}{2}\left[\frac{a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right]^2 \quad (1.1.15)$$

$$\hat{R}_\varepsilon^{\rho\sigma\mu\nu}(\hat{r})\hat{R}_{\rho\sigma\mu\nu,\varepsilon}(\hat{r}) = \frac{4a^2}{\hat{r}^2+\varepsilon^2}\left[\frac{3}{(\hat{r}^2+\varepsilon^2)^2}\right] -$$
$$\frac{12a^2\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^4} + \frac{9a^2\varepsilon^4}{(\hat{r}^2+\varepsilon^2)^5}.$$

Remark 1.1.10. Note that in contrast with the regularized scalar curvature $\hat{R}_\varepsilon$ the regularized quadratic scalars do not have the weak limits, which can be symbolically written as

$$\hat{R}^{\mu\nu}(\hat{x})\hat{R}_{\mu\nu}(\hat{x}) \triangleq \lim_{\varepsilon\to 0}\hat{R}^{\mu\nu}(\hat{x};\varepsilon)\hat{R}_{\mu\nu}(\hat{x};\varepsilon) \sim 40\pi^2 a^2[\delta^{(3)}(\hat{x})]^2 \ ,$$
$$\hat{R}^{\rho\sigma\mu\nu}(\hat{x})\hat{R}_{\rho\sigma\mu\nu}(\hat{x}) \triangleq \lim_{\epsilon\to 0}\hat{R}^{\rho\sigma\mu\nu}(\hat{x};\varepsilon)\hat{R}_{\rho\sigma\mu\nu}(\hat{x};\varepsilon) \quad (1.1.16)$$
$$\sim \frac{12a^2}{\hat{r}^6} + \frac{16\pi a^2}{3}\frac{1}{\hat{r}^3}\delta^{(3)}(\hat{x}) + 16\pi^2 a^2[\delta^{(3)}(\hat{x})]^2.$$

Remark 1.1.11. However Colombeau quadratic scalars $\left(\hat{R}_\varepsilon^{\mu\nu}(\hat{r}_\varepsilon)\hat{R}_{\mu\nu,\varepsilon}(\hat{r}_\varepsilon)\right)_\varepsilon$ and

$\left(\hat{R}_{\varepsilon}^{\rho\sigma\mu\nu}(\hat{r}_{\varepsilon})\hat{R}_{\rho\sigma\mu\nu,\varepsilon}(\hat{r}_{\varepsilon})\right)_{\varepsilon}$ well defined as Colombeau generalized functions in $\mathcal{G}_{\delta}\left(\widetilde{\mathbb{R}}^{3}\right)$.

$$\left(\hat{R}_{\varepsilon}^{\mu\nu}(\hat{r}_{\varepsilon})\hat{R}_{\mu\nu,\varepsilon}(\hat{r}_{\varepsilon})\right)_{\varepsilon} = \frac{1}{2}\left[\left(\frac{3a\varepsilon^{2}}{(\hat{r}_{\varepsilon}^{2}+\varepsilon^{2})^{5/2}}\right)_{\varepsilon}\right]^{2} + 2\left[\left(\frac{a\varepsilon^{2}}{(\hat{r}_{\varepsilon}^{2}+\varepsilon^{2})^{5/2}}\right)_{\varepsilon}\right]^{2},$$

$$\left(\hat{R}_{\varepsilon}^{\rho\sigma\mu\nu}(\hat{r})\hat{R}_{\rho\sigma\mu\nu,\varepsilon}(\hat{r})\right)_{\varepsilon} = \frac{12a^{2}}{((\hat{r}_{\varepsilon}^{2}+\varepsilon^{2})^{3})_{\varepsilon}} - \left(\frac{12a^{2}\varepsilon^{2}}{(\hat{r}_{\varepsilon}^{2}+\varepsilon^{2})^{4}}\right)_{\varepsilon} + +\left(\frac{9a^{2}\varepsilon^{4}}{(\hat{r}_{\varepsilon}^{2}+\varepsilon^{2})^{5}}\right)_{\varepsilon}. \quad (1.1.17)$$

Remark 1.1.12. Note that Colombeau quadratic scalars $\left(\hat{R}_{\varepsilon}^{\mu\nu}(\hat{r})\hat{R}_{\mu\nu,\varepsilon}(\hat{r})\right)_{\varepsilon}$ and $\left(\hat{R}_{\varepsilon}^{\rho\sigma\mu\nu}(\hat{r})\hat{R}_{\rho\sigma\mu\nu,\varepsilon}(\hat{r})\right)_{\varepsilon}$ can be triating only nonclassically as Colombeau generalized functions extended on $\widetilde{\mathbb{R}}^{3} = \widetilde{\mathbb{R}}\times\widetilde{\mathbb{R}}\times\widetilde{\mathbb{R}}$, since at origin $(\hat{r}_{\varepsilon}^{O})_{\varepsilon} = 0_{\widetilde{\mathbb{R}}}$ we get

$$\left(\hat{R}_{\varepsilon}^{\mu\nu}(\hat{r}_{\varepsilon}^{O})\hat{R}_{\mu\nu,\varepsilon}(\hat{r}_{\varepsilon}^{O})\right)_{\varepsilon} = \frac{1}{2}\left[\left(\frac{3a\varepsilon^{2}}{(\varepsilon^{2})^{5/2}}\right)_{\varepsilon}\right]^{2} + 2\left[\left(\frac{a\varepsilon^{2}}{(\varepsilon^{2})^{5/2}}\right)_{\varepsilon}\right]^{2},$$

$$\left(\hat{R}_{\varepsilon}^{\rho\sigma\mu\nu}(\hat{r}_{\varepsilon}^{O})\hat{R}_{\rho\sigma\mu\nu,\varepsilon}(\hat{r}_{\varepsilon}^{O})\right)_{\varepsilon} = \frac{12a^{2}}{(\varepsilon^{6}))_{\varepsilon}} - \left(\frac{12a^{2}\varepsilon^{2}}{(\varepsilon^{8})}\right)_{\varepsilon} + +\left(\frac{9a^{2}\varepsilon^{4}}{(\varepsilon^{10})}\right)_{\varepsilon}. \quad (1.1.18)$$

Remark 1.1.13. In the usual Schwarzschild coordinates (t,r,θ,ϕ), $r \neq a$ the Schwarzschild metric (1.1.2) takes the form above horizon $r > a$ and below horizon $r < a$
correspondingly

$$\begin{cases} \text{above horizon } r > 2m : ds^{+2} = h^{+}(r)dt^{2} - [h^{+}(r)]^{-1}dr^{2} + r^{2}d\Omega^{2}, \\ h^{+}(r) = -1 + \frac{a}{r} = -\frac{r-a}{r} \\ \text{below horizon } r < 2m : ds^{-2} = h^{-}(r)dt^{2} - h^{-}(r)^{-1}dr^{2} + r^{2}d\Omega^{2}, \\ h^{-}(r) = -1 + \frac{a}{r} = \frac{a-r}{r} \end{cases} \quad (1.1.19)$$

Following the above discussion we consider the metric coefficients $h^{+}(r), [h^{+}(r)]^{-1}$ $h^{-}(r)$, and $[h^{-}(r)]^{-1}$ as an element of $\mathcal{D}'(\mathbb{R}^{3})$ and embed it into $\mathcal{G}_{\delta}(\mathbb{R}^{3})$ by replacements above horizon $r \geqslant 2m$ and below horizon $r \leqslant 2m$ correspondingly

$$r \geqslant 2m : r - 2m \mapsto \sqrt{(r-2m)^{2}+\varepsilon^{2}}\,; r < 2m : 2m - r \mapsto \sqrt{(2m-r)^{2}+\varepsilon^{2}}. \quad (1.1.20)$$

Inserting (1.1.20) into (1.1.2) we obtain Colombeau generalized object modeling the singular Schwarzschild metric above (below) gorizon, i.e.,

$$\begin{gathered} (ds_{\varepsilon}^{+2})_{\varepsilon} = (h_{\varepsilon}^{+}(r)dt^{2})_{\varepsilon} - \left([h_{\epsilon}^{+}(r)]^{-1}dr^{2}\right)_{\varepsilon} + r^{2}d\Omega^{2}, \\ (ds_{\varepsilon}^{-2})_{\varepsilon} = (h_{\varepsilon}^{-}(r)dt^{2})_{\varepsilon} - \left([h_{\varepsilon}^{-}(r)]^{-1}dr^{2}\right)_{\varepsilon} + r^{2}d\Omega^{2} \end{gathered} \quad (1.1.21)$$

The generalized Ricci tensor above horizon $\left([\mathbf{R}_{\varepsilon}^{+}]_{\alpha}^{\beta}\right)_{\varepsilon}$ may now be calculated componentwise using the classical formulae

$$\left([\mathbf{R}_{\varepsilon}^{+}(r)]_{0}^{0}\right)_{\varepsilon} = \left([\mathbf{R}_{\varepsilon}^{+}(r)]_{1}^{1}\right)_{\varepsilon} = \frac{1}{2}\left((h_{\varepsilon}^{+\prime\prime})_{\varepsilon} + \frac{2}{r}(h_{\varepsilon}^{+\prime})_{\varepsilon}\right)$$

$$\left([\mathbf{R}_{\varepsilon}^{+}(r)]_{2}^{2}\right)_{\varepsilon} = \left([\mathbf{R}_{\varepsilon}^{+}(r)]_{3}^{3}\right)_{\varepsilon} = \frac{(h_{\varepsilon}^{+\prime})_{\varepsilon}}{r} + \frac{1+(h_{\varepsilon}^{+})_{\varepsilon}}{r^{2}}, \tag{1.1.22}$$

where

$$(h_{\varepsilon}^{+\prime}(r))_{\varepsilon} = -\frac{r-2m}{r\left(\left[(r-2m)^{2}+\varepsilon^{2}\right]^{1/2}\right)_{\varepsilon}} + \frac{\left(\left[(r-2m)^{2}+\varepsilon^{2}\right]^{1/2}\right)_{\varepsilon}}{r^{2}},$$

$$(h_{\varepsilon}^{\prime\prime}(r))_{\varepsilon} = -\frac{1}{\left(r\left[(r-2m)^{2}+\varepsilon^{2}\right]^{1/2}\right)_{\varepsilon}} + \frac{(r-2m)^{2}}{r\left(\left[(r-2m)^{2}+\varepsilon^{2}\right]^{3/2}\right)_{\varepsilon}} +$$

$$+\frac{r-2m}{r^{2}\left(\left[(r-2m)^{2}+\varepsilon^{2}\right]^{1/2}\right)_{\varepsilon}} + \frac{r-2m}{r^{2}\left(\left[(r-2m)^{2}+\varepsilon^{2}\right]^{1/2}\right)_{\varepsilon}} - \tag{1.1.23}$$

$$-\frac{2\left(\left[(r-2m)^{2}+\varepsilon^{2}\right]^{1/2}\right)_{\varepsilon}}{r^{3}}.$$

From Eq.(1.1.22)- Eq.(1.1.23) we obtain (see sect.3)

$$w\text{-}\lim_{\epsilon\to 0}\,[\mathbf{R}_{\epsilon}^{+}(r)]_{1}^{1} = w\text{-}\lim_{\epsilon\to 0}\,[\mathbf{R}_{\epsilon}^{+}(r)]_{0}^{0} = -2m\delta(r-2m). \tag{1.1.24}$$

Remark 1.1.14. Note that the ε-regularization of degenerate and singular metric fields originally has been proposed in A. Einstein and N. Rosen paper [32].

Remark 1.1.15. The full non-linear theory of Colombeau distributional geometry based on
Colombeau algebras in general relativity and its various applications to fundamental problems of the quantum gravity in curved Colombeau distributional spacetime originally
has been obtained in authors papers [33]-[37].

1.2.Basic notions of Colombeau generalized functions and Colombeau generalized numbers.Point values of Colombeau generalized functions.

1.2.1.Basic notions of Colombeau generalized functions

In contemporary mathematics, a Colombeau algebra of Colombeau generalized functions is an algebra of a certain kind containing the space of Schwartz distributions. While in classical distribution theory a general multiplication of distributions is not possible, Colombeau algebras provide a rigorous framework for this.

Remark 1.2.1. Such a multiplication of distributions has been a long time mistakenly believed to be impossible because of Schwartz' impossibility result, which basically states that there cannot be a differential algebra containing the space of distributions and preserving the product of continuous functions. However, if one only wants to preserve the product of smooth functions instead such a construction becomes possible, as demonstrated first by J.F.Colombeau [1],[2].

As a mathematical tool, Colombeau algebras can be said to combine a treatment of singularities, differentiation and nonlinear operations in one framework, lifting the limitations of distribution theory. These algebras have found numerous applications in the fields of partial differential equations, geophysics, microlocal analysis and general relativity so far.

Basic idea.

Definition 1.2.1. The algebra moderate functions $C_M^\infty(\mathbb{R}^n)$ on $\mathbb{R}^n$ is the algebra of families of smooth functions $(f_\varepsilon(x))_\varepsilon \triangleq (f_\varepsilon(x))_\varepsilon, x \in \mathbb{R}^n, \varepsilon \in (0,\delta], \delta \leq 1$ (smooth ε-regularisations, where ε is the regularization parameter), such that: (i) for all compact subsets K of $\mathbb{R}^n$ and all multiindices α, there is an $N > 0$ such that

$$\sup_{x \in K} \left| \frac{\partial^{|\alpha|} f_\varepsilon(x)}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}} \right| = O(\varepsilon^{-N}), \varepsilon \to 0, \tag{1.2.1}$$

with addition and multiplication defined by natural way:

$$(f_\varepsilon(x))_\varepsilon + (g_\varepsilon(x))_\varepsilon = (f_\varepsilon(x) + g_\varepsilon(x))_\varepsilon \tag{1.2.2}$$

and

$$(f_\varepsilon(x))_\varepsilon \times (g_\varepsilon(x))_\varepsilon = (f_\varepsilon(x) \times g_\varepsilon(x))_\varepsilon. \tag{1.2.3}$$

Definition 1.2.2. The ideal $\mathcal{N}_\delta(\mathbb{R}^n)$ of negligible functions is defined in the same way but with the partial derivatives instead bounded by $O(\varepsilon^N)$ for all $N > 0$, i.e.

$$\sup_{x \in K} \left| \frac{\partial^{|\alpha|} f_\varepsilon(x)}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}} \right| = O(\varepsilon^{N}), \varepsilon \to 0. \tag{1.2.4}$$

Definition 1.2.3. The Colombeau Algebra $\mathcal{G}_\delta(\mathbb{R}^n)$ [1],[2] is defined as the quotient algebra

$$\mathcal{G}_\delta(\mathbb{R}^n) = C_M^\infty(\mathbb{R}^n)/\mathcal{N}_\delta(\mathbb{R}^n). \tag{1.2.5}$$

Elements of $\mathcal{G}_\delta(\mathbb{R}^n)$ are denoted by:

$$u = \mathbf{cl}[(u_\varepsilon)_\varepsilon] \triangleq (u_\varepsilon)_\varepsilon + \mathcal{N}_\delta(\mathbb{R}^n). \tag{1.2.6}$$

Embedding of distributions

The space of Schwartz distributions $\mathcal{D}'(\mathbb{R}^n)$ can be embedded into the Colombeau algebra $\mathcal{G}_\delta(\mathbb{R}^n)$ by (component-wise) convolution with any element $(\varphi_\varepsilon)_\varepsilon$ of the algebra $\mathcal{G}_\delta(\mathbb{R}^n)$ having as representative a δ-net, i.e. a family of smooth functions φ_ε such that $\varphi_\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$.

Remark 1.2.2. Note that the embedding $\iota : \mathcal{D}'(\mathbb{R}^n) \hookrightarrow \mathcal{G}_\delta(\mathbb{R}^n)$ is non-canonical, because it

depends on the choice of the δ-net.

Example 1.2.1. Delta function $\delta(x) \in \mathcal{D}'(\mathbb{R})$ for example has the following different representatives in Colombeau algebra $\mathcal{G}_\delta(\mathbb{R})$:

$$\begin{gathered} \frac{1}{2}\left(\frac{1}{\sqrt{\pi\varepsilon}} \exp\left(-\frac{x^2}{4\varepsilon}\right)\right)_\varepsilon \in \mathcal{G}_\delta(\mathbb{R}), \frac{1}{\pi}\left(\frac{1}{x}\sin\left(\frac{x}{\varepsilon}\right)\right)_\varepsilon \in \mathcal{G}_\delta(\mathbb{R}), \\ \frac{1}{\pi}\left(\frac{\varepsilon}{x^2+\varepsilon^2}\right)_\varepsilon \in \mathcal{G}_\delta(\mathbb{R}), \frac{1}{\pi}\left(\frac{1}{x^2}\sin^2\left(\frac{x}{\varepsilon}\right)\right)_\varepsilon \in \mathcal{G}_\delta(\mathbb{R}), \end{gathered} \tag{1.2.7}$$

since

$$\frac{1}{2}\frac{1}{\sqrt{\pi\varepsilon}}\exp\left(-\frac{x^2}{4\varepsilon}\right) \to \delta(x), \frac{1}{\pi}\frac{1}{x}\sin\left(\frac{x}{\varepsilon}\right) \to \delta(x),$$
$$\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} \to \delta(x), \frac{1}{\pi}\frac{1}{x^2}\sin^2\left(\frac{x}{\varepsilon}\right) \to \delta(x) \quad (1.2.8)$$

in $\mathcal{D}'$ as $\varepsilon \to 0$.

Remark 1.2.2. However note that embedding $\mathcal{D}'(\mathbb{R}^n) \hookrightarrow \mathcal{G}(\mathbb{R}^n)$ does not meant the full equivalence of the Schwartz distributions and corresponding by embedding Colombeau generalized functions. In contrast with the Schwartz distributions Colombeau generalized functions has well defined walue at any point $x \in \mathbb{R}^n$ these point values of the Colombeau generalized functions is the Colombeau generalized numbers.

Example 1.2.2. Delta function $\delta(x)$ ill defined at point $x = 0$ since $\delta(0) = \infty$. However any Colombeau generalized function defined by Eq.(1.2.7) has well defined point value at point $x = 0$. For example

$$\frac{1}{2}\left(\frac{1}{\sqrt{\pi\varepsilon}}\exp\left(-\frac{x^2}{4\varepsilon}\right)\right)_\varepsilon\Bigg|_{x=0} = \frac{1}{2\sqrt{\pi\varepsilon}}\left(\frac{1}{\sqrt{\varepsilon}}\right)_\varepsilon \in \widetilde{\mathbb{R}}_\delta,$$
$$\frac{1}{\pi}\left(\frac{\varepsilon}{x^2+\varepsilon^2}\right)_\varepsilon\Bigg|_{x=0} = \frac{1}{\pi}\left(\frac{1}{\varepsilon}\right)_\varepsilon \in \widetilde{\mathbb{R}}_\delta. \quad (1.2.9)$$

Here $\widetilde{\mathbb{R}}$ is the ring of real Colombeau generalized numbers [34].

1.2.2. The ring of Colombeau generalized numbers $\widetilde{\mathbb{R}}_\delta$. Point values of Colombeau generalized functions.

Designation 1.2.1. (**I**) We denote by $\widetilde{\mathbb{R}}_\delta, \delta \leq 1$ the ring of real Colombeau generalized numbers. Recall that by definition $\widetilde{\mathbb{R}}_\delta = \mathbf{E}_{\mathbb{R},\delta}(\mathbb{R})/\mathbf{N}_\delta(\mathbb{R})$ where [34],[36],[37]:

$$\mathbf{E}_{\mathbb{R},\delta}(\mathbb{R}) = \{(x_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,\delta)} | (\exists a \in \mathbb{R}_+)(\exists \varepsilon_0 \in (0,1))(\forall \varepsilon \leq \varepsilon_0)[|x_\varepsilon| \leq \varepsilon^{-a}]\},$$
$$\mathbf{N}_\delta(\mathbb{R}) = \{(x_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,\delta)} | (\forall a \in \mathbb{R}_+)(\exists \varepsilon_0 \in (0,1))(\forall \varepsilon \leq \varepsilon_0)[|x_\varepsilon| \leq \varepsilon^{a}]\}. \quad (1.2.10)$$

(**II**) In this subsection we will be write for short $\widetilde{\mathbb{R}}$ instead $\widetilde{\mathbb{R}}_\delta$.

Notice that the ring $\widetilde{\mathbb{R}}$ arises naturally as the ring of constants of the Colombeau algebras $\mathcal{G}_\delta(\Omega)$. Recall that there exists natural embedding $\widetilde{r} : \mathbb{R} \hookrightarrow \widetilde{\mathbb{R}}$ such that for all $r \in \mathbb{R}, \widetilde{r} = (r_\varepsilon)_\varepsilon$ where $r_\varepsilon \equiv r$ for all $\varepsilon \in (0,1]$. We say that r is standard number and abbreviate $r \in \mathbb{R}$ for short. The ring $\widetilde{\mathbb{R}}$ can be endowed with the structure of a partially ordered ring: for $r,s \in \widetilde{\mathbb{R}}$ $\eta \in \mathbb{R}_+, \eta \leq \delta$ we abbreviate $r \leq_{\widetilde{\mathbb{R}},\eta} s$ or simply $r \leq_{\widetilde{\mathbb{R}}} s$ if and only if there are representatives $(r_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ with $r_\varepsilon \leq s_\varepsilon$ for all $\varepsilon \in (0,\eta]$. Colombeau generalized number $r \in \widetilde{\mathbb{R}}$ with representative $(r_\varepsilon)_\varepsilon$ we abbreviate $\mathbf{cl}[(r_\varepsilon)_\varepsilon]$.

Definition 1.2.4. (i) Let $\breve{\delta} = \mathbf{cl}[(\delta_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}$. We say that $\breve{\delta}$ is infinite small Colombeau generalized number and abbreviate $\breve{\delta} \approx_{\widetilde{\mathbb{R}}} \widetilde{0}$ if there exists representative $(\delta_\varepsilon)_\varepsilon$ and some $q \in \mathbb{N}$ such that $|\delta_\varepsilon| = O(\varepsilon^q)$ as $\varepsilon \to 0$. (ii) Let $\Delta \in \widetilde{\mathbb{R}}$. We say that Δ is infinite large

Colombeau generalized number and abbreviate $\Delta =_{\widetilde{\mathbb{R}}} \widetilde{\infty}$ if $\Delta^{-1}_{\widetilde{\mathbb{R}}} \approx_{\widetilde{\mathbb{R}}} \widetilde{0}$. (iii) Let $\mathbb{R}_{\pm\infty}$ be $\mathbb{R} \cup \{\pm\infty\}$ We say that $\Theta \in \widetilde{\mathbb{R}}_{\pm\infty}$ is infinite Colombeau generalized number and abbreviate $\Theta =_{\widetilde{\mathbb{R}}} \pm\infty_{\widetilde{\mathbb{R}}}$ if there exists representative $(\Theta_\varepsilon)_\varepsilon$ where $|\Theta_\varepsilon| = \infty$ for all $\varepsilon \in (0,1]$. Here we abbreviate $\mathcal{E}_M(\mathbb{R}_{\pm\infty}) = \mathcal{E}_M(\mathbb{R} \cup \{\pm\infty\})$, $\mathcal{N}(\mathbb{R}_{\pm\infty}) = \mathcal{N}(\mathbb{R} \cup \{\pm\infty\})$ and $\widetilde{\mathbb{R}}_{\pm\infty} = \mathcal{E}_M(\mathbb{R}_{\pm\infty})/\mathcal{N}(\mathbb{R}_{\pm\infty})$

Definition 1.2.5. (Standard Part Mapping). (i) The standard part mapping $\mathbf{st} : \widetilde{\mathbb{R}} \to \mathbb{R}$ is defined by the formula:

$$\mathbf{st}(x) = \sup\left\{r \in \mathbb{R} | r \leq_{\widetilde{\mathbb{R}}} x\right\}. \tag{1.2.11}$$

If $x \in \widetilde{\mathbb{R}}$, then $\mathbf{st}(x)$ is called the standard part of x.

(ii) The mapping $\mathbf{st} : \widetilde{\mathbb{R}} \to \mathbb{R} \cup \{\pm\infty\}$ is defined by (i) and by $\mathbf{st}(x) = \pm\infty$ for $x \in \widetilde{\mathbb{R}}$ and for $x \in \widetilde{\mathbb{R}}_{\pm\infty}$, respectively.

Definition 1.2.6.[37]. Let $(f_\varepsilon(x))_\varepsilon \in \mathcal{G}(\mathbb{R})$ and $\breve{x} \in \mathbb{R}$, then $\mathbf{cl}[(f_\varepsilon(\breve{x}))_\varepsilon] \in \widetilde{\mathbb{R}}$. We will say that Colombeau generalized number $\mathbf{cl}[(f_\varepsilon(\breve{x}))_\varepsilon]$ is a point values of Colombeau generalized function $(f_\varepsilon(x))_\varepsilon$ at point $\breve{x} \in \mathbb{R}$.

Definition 1.2.7.(Principal value mapping) The principal value mapping $\mathbf{p.v.} : \widetilde{\mathbb{R}} \to \mathbb{R}$ of Colombeau generalized function $(f_\varepsilon(x))_\varepsilon$ at point $\breve{x} \in \mathbb{R}$ is defined by the formula:

$$\mathbf{p.v.}\{\mathbf{cl}[(f_\varepsilon(\breve{x}))_\varepsilon]\} = \sup_{\varepsilon\in(0,1]} |f_\varepsilon(\breve{x})|. \tag{1.2.12}$$

We will be write for short $\mathbf{p.v.}[(f_\varepsilon(\breve{x}))_\varepsilon]$.

Example 1.2.3. The principal value of the curvature scalar $\left(\hat{R}_\varepsilon(r,a)\right)_\varepsilon$ (1.1.10) at point $\breve{r} \in \mathbb{R}$ reads

$$\mathbf{p.v.}\left[\left(\hat{R}_\varepsilon(\breve{r},a)\right)_\varepsilon\right] = \sup_{\varepsilon\in(0,1]} \frac{a\varepsilon^2}{(\breve{r}^2+\varepsilon^2)^{5/2}}. \tag{1.2.13}$$

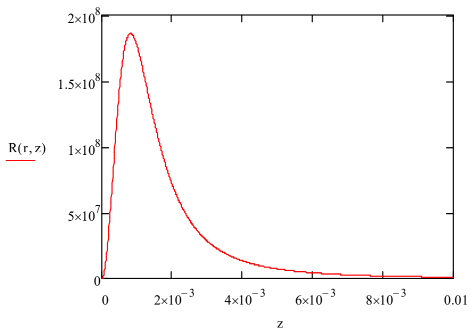

Fig.1.Plot of the function

$$R(a,\breve{r},\varepsilon) = \frac{a\varepsilon^2}{(\breve{r}^2+\varepsilon^2)^{5/2}},$$

$$a = 1, \breve{r} = 10^{-3}, z = \varepsilon \in (0, 0.01].$$

$$R(10^{-3}, 7\times10^{-4}) = 1.808\times10^8.$$

$$\mathbf{p.v.}\left[\left(\hat{R}_\varepsilon(\breve{r},a)\right)_\varepsilon\right] \simeq 1.808\times10^8.$$

1.3.The point free classical Colombeau geometry.

The first definition (prior to the well-known five postulates) of Euclides describes the point as "that of which there is no part" [40].

A huge portion of our mathematics of the physical world is based on the amazingly simple Euclidean geometry. Indeed, starting from very straightforward assumptions and theorems such as those found in Euclid's geometry, it is feasible to build also non-Euclidean geometries and complex manifolds able to explain issues such as those in quantum mechanics. One of the main components of Euclidean geometry is the point, that stands for the most fundamental object. The first definition of a point (prior to Euclid) is given by the Pythagoreans: a point is a monad having position. Euclid begins his geometry with the definition of a point [that of which there is no part] (Def.1, Euclid, 300 BCE) and the extremities of a line are points (Def.2). Euclid's Def.1 is interpreted by T.L. Heath to mean that a point is that which is indivisible in parts. Therefore, we are confronted with a primitive notion defined only by axioms that it must satisfy, i.e., the point upon which the whole apparatus is built, meaning that geometry cannot be described in terms of previously defined real objects or structures. Here we ask whether the zero-substance point holds true in our physical world and extend our analysis also to other Euclidean objects, such as lines, surfaces, volumes and so on [41].

Definition 1.3.1. Let $(f_\varepsilon(x))_\varepsilon \in \mathcal{G}_\delta(\mathbb{R})$ and $\mathbf{cl}[(\check{x}_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}_\delta$. Assum that $\mathbf{cl}[(f_\varepsilon(\check{x}_\varepsilon))_\varepsilon] \in \widetilde{\mathbb{R}}_\delta$. We will say that Colombeau generalized number $\mathbf{cl}[(f_\varepsilon(\check{x}_\varepsilon))_\varepsilon]$ is a point values of Colombeau generalized function $(f_\varepsilon(x))_\varepsilon$ at point $(\check{x}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}_\delta$.

Example 1.3.1. For any $(\hat{r}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}_\delta, \varepsilon \in (0,\delta]$ the point values $\left(\hat{R}_\varepsilon(\hat{r}_\varepsilon)\right)_\varepsilon$ of Colombeau generalized function $\left(\hat{R}_\varepsilon(\hat{r})\right)_\varepsilon$ (see Eq.(1.1.10) reads

$$\begin{gathered}\left(\hat{R}_\varepsilon(\hat{r}_\varepsilon)\right)_\varepsilon = \left(\hat{R}^\mu{}_{\mu,\varepsilon}(\hat{r}_\varepsilon)\right)_\varepsilon = -\left(\frac{a\varepsilon^2}{(\hat{r}_\varepsilon^2+\varepsilon^2)^{5/2}}\right)_\varepsilon = -a\frac{(\varepsilon^2)_\varepsilon}{\left[(\hat{r}_\varepsilon^2)_\varepsilon+(\varepsilon^2)_\varepsilon\right]^{5/2}} = \\ -a\frac{\delta^2(\varepsilon_1^2)_{\varepsilon_1}}{\left[(\hat{r}_\varepsilon^2)_\varepsilon+\delta^2(\varepsilon_1^2)_{\varepsilon_1}\right]^{5/2}},\end{gathered} \tag{1.3.1}$$

where $\varepsilon_1 \in (0,1]$.

Remark 1.3.2. We choose now $(\hat{r}_\varepsilon)_\varepsilon = (\hat{r}_\varepsilon^*)_\varepsilon = \eta(\varepsilon)_\varepsilon = \eta\delta(\varepsilon_1)_{\varepsilon_1}$and from Eq.(1.3.1) we get

$$\begin{gathered}\left(\hat{R}_\varepsilon(\hat{r}_\varepsilon^*)\right)_\varepsilon = -a\frac{\delta^2(\varepsilon_1^2)_{\varepsilon_1}}{\left[(\hat{r}_\varepsilon^{*2})_\varepsilon+\delta^2(\varepsilon_1^2)_{\varepsilon_1}\right]^{5/2}} = -a\frac{\delta^2(\varepsilon_1^2)_{\varepsilon_1}}{\left[\eta\delta(\varepsilon_1^2)_{\varepsilon_1}+\delta^2(\varepsilon_1^2)_{\varepsilon_1}\right]^{5/2}} = \\ -a\frac{\delta^2(\varepsilon_1^2)_{\varepsilon_1}}{\left[\eta^2\delta^2(\varepsilon_1^2)_{\varepsilon_1}+\delta^2(\varepsilon_1^2)_{\varepsilon_1}\right]^{5/2}} = -a\frac{\delta^2(\varepsilon_1^2)_{\varepsilon_1}}{\delta^5(\varepsilon_1^5)_{\varepsilon_1}[\eta^2+1]^{5/2}} = \\ \frac{-a}{\delta^3(\varepsilon_1^3)_{\varepsilon_1}[\eta^2+1]^{5/2}}.\end{gathered} \tag{1.3.2}$$

Thus in "point limit" $\delta \asymp 0$ the curvature scalar $\left(\hat{R}_\varepsilon(\hat{r}_\varepsilon^*)\right)_\varepsilon$ diverges as $\delta^{-3}(\varepsilon_1^{-3})_{\varepsilon_1}$.

Remark 1.3.2. In order prevent the divergence mentioned above, we assume now that there exist fundamental generalized length $l^* = \mathbf{cl}\left[(l_{\varepsilon_1}^*)_{\varepsilon_1}\right] = \eta\mathbf{cl}\left[(\varepsilon_1)_{\varepsilon_1}\right]$, $\varepsilon_1 \in (0,1]$,

$\eta \in \mathbb{R}, \quad \delta \ll \eta \ll 1$, such that: $(\hat{r}_{\varepsilon_1})_{\varepsilon_1} \geq l^*$,see [36] sec.2.Thus from Eq.(1.3.1) we get now instead Eq.(1.3.2)

$$\begin{gathered}
\left|\left(\hat{R}_\varepsilon(\hat{r}_\varepsilon)\right)_\varepsilon\right| \leq \left(\frac{a\varepsilon^2}{(l_{\varepsilon_1}^{*2}+\varepsilon^2)^{5/2}}\right)_\varepsilon = a\frac{a(\varepsilon^2)_\varepsilon}{\left[(l_{\varepsilon_1}^{*2})_\varepsilon+(\varepsilon^2)_\varepsilon\right]^{5/2}} = \\
a\frac{\delta^2(\varepsilon_1^2)_{\varepsilon_1}}{\left[\eta^2(\varepsilon_1^2)_{\varepsilon_1}+\delta^2(\varepsilon_1^2)_{\varepsilon_1}\right]^{1/2}}\frac{1}{\left[\eta^2(\varepsilon_1^2)_{\varepsilon_1}+\delta^2(\varepsilon_1^2)_{\varepsilon_1}\right]^{1/2}}\frac{1}{\left[(\hat{r}_{\varepsilon_1})_{\varepsilon_1}+(\varepsilon^2)_\varepsilon\right]^{3/2}} = \\
\frac{a\delta^2}{[\eta^2+\delta^2]}\frac{1}{\left[(\hat{r}_{\varepsilon_1}^2)_{\varepsilon_1}+(\varepsilon^2)_\varepsilon\right]^{3/2}} = \frac{a\delta^2}{[\eta^2+\delta^2]}\frac{1}{\left[(\hat{r}_{\hat{r}}^2)_{\hat{r}\in(0,1]}+(\varepsilon^2)_\varepsilon\right]^{3/2}}.
\end{gathered} \tag{1.3.3}$$

1.4.The Point-Free Loop Quantum Gravity.

We remind that canonical quantization of GRT can be expressed as an $SU(2)$ gauge theory on the 3 dimensional manifold Σ furnished by canonical point-like geometry, where a topology of space-time M of the form $M \cong \mathbb{R} \times \Sigma$ is assumed, in a background independent manner. In such formulation of GR, the gravitational field is described by a pair of conjugate variables (A,E), where $A_a^i(x)$ is an $SU(2)$ connection and $E_i^a(x)$ is the densitised triad vector field, conjugate to A :

$$\{A_a^i(x), E_j^b(x')\} = 8\pi\gamma\delta_j^i\delta_a^b\delta(x-x'), \tag{1.4.1}$$

with G the gravitational constant and γ the Immirzi parameter.The conjugate pair are constraint to satisfy the system

$$\begin{gathered}
G_i = D_aE_i^a = 0, H_b = E_i^aF_{ab}^i = 0, \\
H = \epsilon_{ijk}F_{ab}^iE_j^aE_k^b - 2(1+\gamma^2)K_{[a}^iK_{b]}^jE_i^aE_j^b = 0,
\end{gathered} \tag{1.4.2}$$

which are called Gauss, spatial diffeomorphism and Hamiltonian constraints respectively.In fact, the task of finding a metric satisfying the Einstein's equations, describing configuration of a gravitating system, is now replaced by finding a conjugate pair (A,E) satisfying the constraint system (1.4.1). On quantization, one smears the basic fields (A,E) to holonomies of A_i^a along a curve γ, defined by $h_\gamma[A] = P\left[\exp\left(\int_\gamma A\right)\right]$, and fluxes of E_i^a through the surface S, defined by $E_i(S) = \int_S d^2\sigma n_a E_i^a$. They form the holonomy-flux algebra in which holonomies act by multiplication, and fluxes act by derivation. Using a functional representation of quantum field theory and representing states as functionals of the cylindrical functions of holonomies, the kinematical Hilbert space of the theory is constructed. After imposing Gauss and diffeomorphism constraints as operators on such states, the true gauge and diffeomorphism invariant states of the theory turns out to be spin networks acted upon by holonomies and fluxes operators which form a unique representation. More precisely, a spin network is a triplet (Γ, j_l, i_n) consisting of a graph Γ with nodes in Σ, labeled by intertwiners i_n, and links connecting different nodes, labeled by $SU(2)$ representations j_l.

Remark 1.4.1.(**I**)The quantum geometrical picture suggested by canonical LQG [44] is manifest in quantization of geometrical observables, such as area and volume, as quantum operators acting on spin network states which result in discrete spectra and reflect the discrete nature of space-time.

(**II**) In fact singularity resolution occurs as a result of fundamental discreteness of space; while in a classical continuum, divergences emerge as distance goes to zero, there is no room for divergences in quantum level since there is no zero distance below the Planck length.

Canonical quantization of nonlinear distributional GRT can be expressed as an $\widetilde{SU}(2)$ gauge theory on the 3 dimensional Colombeau distributional manifold $\widetilde{\Sigma}$ furnished by Colombeau point-free geometry, where a topology of space-time $\widetilde{M}$ of the form $\widetilde{M} \cong \widetilde{\mathbb{R}}_{\delta} \times \widetilde{\Sigma}$ is assumed, in a background independent manner. In such formulation of GRT, the gravitational field is described by a pair of conjugate variables $((A_{\varepsilon})_{\varepsilon}, (E_{\varepsilon})_{\varepsilon})$, where $(A^{i}_{a,\varepsilon}(x_{\varepsilon}))_{\varepsilon}$ is an $\widetilde{SU}(2)$ Colombeau distributional connection and $(E^{a}_{i,\varepsilon}(x_{\varepsilon}))_{\varepsilon}$ is the distributional densitised triad vector field, conjugate to $(A_{\varepsilon})_{\varepsilon}$:

$$\left\{(A^{i}_{a,\varepsilon}(x_{\varepsilon}))_{\varepsilon}, (E^{b}_{j}(x'_{\varepsilon}))_{\varepsilon}\right\} = 8\pi\gamma\delta^{i}_{j}\delta^{b}_{a}(\delta(x_{\varepsilon} - x'_{\varepsilon}))_{\varepsilon}, \tag{1.4.3}$$

with G the gravitational constant and γ the Immirzi parameter.The conjugate pair are constraint to satisfy the system

$$\begin{gathered}(G_{i,\varepsilon})_{\varepsilon} =_{\widetilde{\mathbb{R}}} (D_{a,\varepsilon}E^{a}_{i,\varepsilon})_{\varepsilon} = 0, (H_{b,\varepsilon})_{\varepsilon} = (E^{a}_{i,\varepsilon}F^{i}_{ab,\varepsilon})_{\varepsilon} =_{\widetilde{\mathbb{R}}} 0_{\widetilde{\mathbb{R}}},\\ (H_{\varepsilon})_{\varepsilon} =_{\widetilde{\mathbb{R}}} \epsilon_{ijk}(F^{i}_{ab,\varepsilon}E^{a}_{j,\varepsilon}E^{b}_{k,\varepsilon})_{\varepsilon} - 2(1+\gamma^{2})\left(K^{i}_{[a,\varepsilon}K^{j}_{b,\varepsilon]}E^{a}_{i,\varepsilon}E^{b}_{j,\varepsilon}\right)_{\varepsilon} =_{\widetilde{\mathbb{R}}} 0_{\widetilde{\mathbb{R}}},\end{gathered} \tag{1.4.4}$$

In fact, the task of finding Colombeau metric satisfying the generalized Einstein's field equations (see subsect.1.8), describing configuration of a gravitating system, is now replaced by finding a conjugate pair $((A_{\varepsilon})_{\varepsilon}, (E_{\varepsilon})_{\varepsilon})$ satisfying the constraint system (1.4.3). On quantization, one smears the basic Colombeau generalized fields $((A_{\varepsilon})_{\varepsilon}, (E_{\varepsilon})_{\varepsilon})$ to holonomies of $(A^{a}_{i,\varepsilon})_{\varepsilon}$ along a curve $\gamma = (\gamma_{\varepsilon})_{\varepsilon}$, defined by $(h_{\gamma}[A_{\varepsilon}])_{\varepsilon} = P\left[\exp\left(\int_{\gamma} A_{\varepsilon}\right)_{\varepsilon}\right]$, and fluxes of $(E^{a}_{i,\varepsilon})_{\varepsilon}$ through the surface $\widetilde{S}$, defined by $\left(E_{i,\varepsilon}(\widetilde{S})\right)_{\varepsilon} = \left(\int_{\widetilde{S}} d^{2}\sigma n_{a}E^{a}_{i,\varepsilon}\right)_{\varepsilon}$.

A spin network is a triplet $(\widetilde{\Gamma}, j_{l}, i_{n})$ consisting of a graph $\widetilde{\Gamma}$ with nodes in $\widetilde{\Sigma}$, labeled by intertwiners i_{n}, and links connecting different nodes, labeled by $\widetilde{SU}(2)$ representations j_{l}.

1.4.1.Classical point-free phase space

Definition 1.4.1.(1) The general linear group over Colombeau algebras $\widetilde{\mathbb{R}}, \widetilde{\mathbb{C}}$ (the set of real,complex Colombeau numbers) is the group of $n \times n$ invertible matrices of real (complex) Colombeau numbers, and is denoted by $GL_{n}\left(\widetilde{\mathbb{R}}\right), GL_{n}\left(\widetilde{\mathbb{C}}\right)$ or $GL\left(n, \widetilde{\mathbb{R}}\right), GL\left(n, \widetilde{\mathbb{C}}\right)$.

(2) The unitary group of degree n over Colombeau algebra $\widetilde{\mathbb{C}}$, denoted $\widetilde{U}(n)$, or $U(n, \widetilde{\mathbb{C}})$ is the group of $n \times n$ unitary matrices over $\widetilde{\mathbb{C}}$.

(3) The unitary group is a subgroup of the general linear group $GL\left(n, \widetilde{\mathbb{C}}\right)$.

(4) In the simple case $n = 1$, the group $U(n, \widetilde{\mathbb{C}})$ corresponds to the circle group $\widetilde{\mathbf{T}}$, consisting of all Colombeau complex numbers with absolute value 1 under

multiplication,i.e. $\widetilde{\mathbf{T}} = \left\{z \in \widetilde{\mathbb{C}} \,\middle|\, |z| = 1\right\}$.

(5) The special unitary group of degree n, denoted $\widetilde{SU}(n)$, is the Lie group of $n \times n$ unitary matrices over Colombeau algebra $\widetilde{\mathbb{R}}$ with determinant 1.

The Colombeau distributional manifold $\widetilde{\Sigma}$ over Colombeau algebra $\widetilde{\mathbb{R}}$ having the symmetry group $\widetilde{S}$ with an isotropy subgroup $\widetilde{F}$, can be decomposed as $\widetilde{\Sigma} \cong \widetilde{\Sigma}/\widetilde{S} \times \widetilde{S}/\widetilde{F}$. The connection can generally be written as $(A_\varepsilon)_\varepsilon = \left(A_{\widetilde{\Sigma}/\widetilde{S},\varepsilon}\right)_\varepsilon + \left(A_{\widetilde{S}/\widetilde{F},\varepsilon}\right)_\varepsilon$. Then $\left(A_{\widetilde{\Sigma}/\widetilde{S},\varepsilon}\right)_\varepsilon$ can be considered as the connection of the reduced theory and its holonomies along curves in $\widetilde{\Sigma}/\widetilde{S}$ can be quantized.For the spherically symmetric case, $\widetilde{\Sigma} \cong \widetilde{\mathbb{R}}_\delta \times \widetilde{S}^2$, and the symmetry group is $\widetilde{S} = \widetilde{SU}(2)$. This implies identifying $\widetilde{\Sigma}/\widetilde{S}$ with $\widetilde{\mathbb{R}}_\delta$ and the gauge group of the reduced theory F with $U(1)$. Therefore, reduced connections are $\widetilde{U}(1)$ gauge fields on $\widetilde{\mathbb{R}}$. Roughly speaking, spherical symmetry implies that our basic fields, in the spherical coordinate $((x_\varepsilon)_\varepsilon, \theta, \phi)$, are independent of angular variables. Thus, the Colombeau generalized connection $(A_\varepsilon(\vec{x}_\varepsilon))_\varepsilon$ is just a function of the radial coordinate; $(A_\varepsilon)_\varepsilon = (A_\varepsilon(x_\varepsilon))_\varepsilon$.These connections and triads of the reduced spherically symmetric phase space have the general form:

$$\begin{aligned}(A_\varepsilon)_\varepsilon = {} & [(A_{x_\varepsilon}(x_\varepsilon))_\varepsilon]\tau_3[(dx_\varepsilon)_\varepsilon] + ([(A_1(x_\varepsilon))_\varepsilon]\tau_1 + [(A_2(x_\varepsilon))_\varepsilon]\tau_2)[(d\theta_\varepsilon)_\varepsilon] + \\ & (((A_{1,\varepsilon}(x_\varepsilon)\tau_2)_\varepsilon - (A_{2,\varepsilon}(x_\varepsilon)\tau_1 \sin\theta_\varepsilon)_\varepsilon) + \tau_3(\cos\theta_\varepsilon)_\varepsilon)[(d\phi_\varepsilon)_\varepsilon]\end{aligned} \tag{1.4.5}$$

and

$$\begin{aligned}(E_\varepsilon)_\varepsilon = {} & (E_\varepsilon^{x_\varepsilon}(x_\varepsilon)\tau_3 \sin\theta_\varepsilon \partial_{x_\varepsilon})_\varepsilon + ((E_\varepsilon^1(x_\varepsilon)\tau_1)_\varepsilon + \\ & (E_\varepsilon^2(x_\varepsilon)\tau_2)\sin\theta_\varepsilon \partial_{\theta_\varepsilon})_\varepsilon + ((E_\varepsilon^1(x_\varepsilon)\tau_2)_\varepsilon - (E_\varepsilon^2(x_\varepsilon)\tau_1)\partial_{\phi_\varepsilon})_\varepsilon\end{aligned} \tag{1.4.6}$$

correspondingly, where $\tau_i = -\frac{i}{2}\sigma_i$ are the generators of $\widetilde{su}(2)$ algebra. They define the Colombeau generalized symplectic structure:

$$(\Omega_\varepsilon)_\varepsilon = \frac{1}{2\gamma G}\left(\int dx_\varepsilon (dA_{x_\varepsilon,\varepsilon} \wedge dE_\varepsilon^{x_\varepsilon} + 2dA_{1,\varepsilon} \wedge dE_\varepsilon^1 + 2dA_{2,\varepsilon} \wedge dE_\varepsilon^2)\right)_\varepsilon. \tag{1.4.7}$$

However, a suitable canonical transformation can be made resulting in Colombeau generalized canonical variables $((A_{x_\varepsilon}(x_\varepsilon))_\varepsilon, (E_\varepsilon^{x_\varepsilon}(x_\varepsilon))_\varepsilon)$, $(\gamma(K_{\phi_\varepsilon,\varepsilon}(x_\varepsilon))_\varepsilon, \left(E_\varepsilon^{\phi_\varepsilon}(x_\varepsilon)\right)_\varepsilon)$ and $((\eta_\varepsilon(x_\varepsilon))_\varepsilon, (P_\varepsilon^{\eta_\varepsilon}(x_\varepsilon))_\varepsilon)$:

$$(\Omega_\varepsilon)_\varepsilon = \frac{1}{2\gamma G}\left(\int dx_\varepsilon (dA_{x_\varepsilon} \wedge dE_\varepsilon^{x_\varepsilon} + d(\gamma K_{\phi_\varepsilon,\varepsilon}) \wedge dE_\varepsilon^{\phi_\varepsilon} + 2d\eta_\varepsilon \wedge dP_\varepsilon^{\eta_\varepsilon})\right)_\varepsilon, \tag{1.4.8}$$

with $(K_{\phi_\varepsilon,\varepsilon})_\varepsilon$ being the $(\phi_\varepsilon)_\varepsilon$ component of the extrinsic Colombeau generalized curvature.The Gauss constraint, generating $\widetilde{U}(1)$ gauge transformations, takes the form:

$$(G_\varepsilon[\lambda_\varepsilon])_\varepsilon = \left(\int dx_\varepsilon \lambda_\varepsilon (E_\varepsilon^{x'_\varepsilon} + P_\varepsilon^{\eta_\varepsilon})\right)_\varepsilon =_{\widetilde{\mathbb{R}}} 0_{\widetilde{\mathbb{R}}}, \tag{1.4.9}$$

where prime denotes differentiation with respect to x_ε.

Note that in terms of these variables, conjugate pair is not simply the connection-flux pair which suggests a different situation than the full theory.The Colombeau generalized Hamiltonian constraint can be written as

$$(H_\varepsilon[N_\varepsilon])_\varepsilon = -\frac{1}{2G} \times$$
$$\left(\int dx_\varepsilon N_\varepsilon(x_\varepsilon)\frac{1}{\sqrt{|E_\varepsilon^{x_\varepsilon}|}}\left((1-\Gamma_{\phi_\varepsilon,\varepsilon}^2+K_{\phi_\varepsilon,\varepsilon}^2)E_\varepsilon^{\phi_\varepsilon}+\frac{2}{\gamma}K_{\phi_\varepsilon,\varepsilon}E_\varepsilon^{x_\varepsilon}(A_{x_\varepsilon,\varepsilon}+\eta'_\varepsilon)+2E_\varepsilon^{x_\varepsilon}\Gamma'_{\phi_\varepsilon,\varepsilon}\right)\right)_\varepsilon. \quad (1.4.10)$$

1.4.2. Quantization

Along the standard lines of constructing basic operators and states in the kinematical Hilbert space of classical LQG, we start with holonomies of the connections. Holonomies of $(A_{x,\varepsilon})_\varepsilon$ along curves $(\gamma_\varepsilon)_\varepsilon$ in $\widetilde{\mathbb{R}}$ are defined as $\left(h_\varepsilon^{(\gamma_\varepsilon)}\right)_\varepsilon \equiv \exp\left[\frac{i}{2}\left(\int_{\gamma_\varepsilon} A_{x_\varepsilon}(x_\varepsilon)\right)_\varepsilon\right]$ which are elements in $\widetilde{U}_\delta(1) = \widetilde{\mathbb{R}}_\delta/\widetilde{\mathbb{Z}}$. For $(A_{\phi_\varepsilon,\varepsilon})_\varepsilon$ point holonomies $\exp[i\mu(A_{\phi_\varepsilon,\varepsilon}(x_\varepsilon))_\varepsilon]$ are used which belongs to the space of continuous almost periodic functions on the Bohr compactification of real line $\widetilde{\mathbb{R}}_\delta$, and point holonomies of $(\eta_\varepsilon)_\varepsilon \in \widetilde{S}^1$, have the form $\exp[(i\eta_\varepsilon(x_\varepsilon))_\varepsilon]$ which are elements of $\widetilde{U}(1)$.

The kinematical Hilbert space of the present reduced theory is the space spanned by spin network state $(T_{g,k,\mu,\varepsilon})_\varepsilon$:

$$(T_{g,k,\mu,\varepsilon})_\varepsilon =$$
$$\prod_{e\in g}\exp\left(\frac{i}{2}k_e\left(\int_e dx_\varepsilon A_{x_\varepsilon}(x_\varepsilon)\right)_\varepsilon\right)\left(\prod_{v\in V(g)}\exp(i\mu_{v,\varepsilon}\gamma K_{\phi_\varepsilon,\varepsilon}(v))\exp(ik_v\eta_\varepsilon(v))\right)_\varepsilon. \quad (1.4.11)$$

For a given graph g, these are cylindrical functions of holonomies along edges e of g. Vertices $V(g)$ of such spin networks are labeled by irreducible $\widetilde{\widetilde{\mathbb{R}}}_{Bohr}$ representations $(\mu_{v,\varepsilon})_\varepsilon \in \widetilde{\mathbb{R}}_\delta$ and irreducible $\widetilde{S}^1$ representation $k_v \in \widetilde{\mathbb{Z}}$, while edges are labeled by irreducible representations of $\widetilde{U}_\delta(1)$.

Holonomies act on spin network states by multiplication. Their corresponding momenta, on the other hand, act by differentiation

$$\left(\hat{E}_\varepsilon^{x_\varepsilon}(x_\varepsilon)T_{g,k,\mu,\varepsilon}\right)_\varepsilon = \gamma\frac{\ell_p^2}{2}((k_{e^+(x_\varepsilon)}+k_{e^-(x_\varepsilon)})T_{g,k,\mu,\varepsilon})_\varepsilon, \quad (1.4.12)$$

$$\int dx_\varepsilon \hat{E}_\varepsilon^{\phi_\varepsilon}(x_\varepsilon)T_{g,k,\mu} = \gamma\ell_p^2\sum_v \mu_{v,\varepsilon}T_{g,k,\mu,\varepsilon}, \quad (1.4.13)$$

$$\int dx_\varepsilon \hat{P}_\varepsilon^{\eta_\varepsilon}(x_\varepsilon)T_{g,k,\mu,\varepsilon} = 2\gamma\ell_p^2\sum_v k_{v,\varepsilon}T_{g,k,\mu,\varepsilon}. \quad (1.4.14)$$

The generalized volume operator can be expressed as

$$\left(\hat{V}_\varepsilon\right)_\varepsilon = 4\pi\int dx_\varepsilon|\hat{E}_\varepsilon^{\phi_\varepsilon}(x_\varepsilon)|\sqrt{|\hat{E}_\varepsilon^{x_\varepsilon}(x_\varepsilon)|} \quad (1.4.15)$$

which is diagonal in spin network representation

$$\left(\hat{V}_\varepsilon T_{g,k,\mu,\varepsilon}\right)_\varepsilon = (V_{k,m,\varepsilon}T_{g,k,\mu,\varepsilon})_\varepsilon, \quad (1.4.16)$$

where

$$(V_{k,m,\varepsilon})_\varepsilon = 4\pi\gamma^{3/2}\ell_p^3\left(\sum_v|\mu_v|\sqrt{\frac{1}{2}|k_{e^+(x_\varepsilon)}+k_{e^-(x_\varepsilon)}|}\right)_\varepsilon. \quad (1.4.17)$$

Implementing the Gauss constraint as an operator on spin networks to select the gauge invariant states, leads to a restriction on labels

$$\left(\hat{G}_\varepsilon[\lambda_\varepsilon]T_{g,k,\mu,\varepsilon}\right)_\varepsilon = \gamma\ell_p^2\left(\sum_v \lambda_\varepsilon(v)(k_{e^+(x_\varepsilon)} - k_{e^-(x_\varepsilon)} + 2k_v)T_{g,k,\mu,\varepsilon}\right)_\varepsilon \quad (1.4.18)$$

$$\left(\hat{G}_\varepsilon[\lambda_\varepsilon]T_{g,k,\mu}\right)_\varepsilon = 0_{\widetilde{\mathbb{R}}} \Rightarrow k_v = -\frac{1}{2}(k_{e^+(x_\varepsilon)} - k_{e^-(x_\varepsilon)})_\varepsilon. \quad (1.4.19)$$

Imposing now this on (1.4.11) results in the gauge invariant states

$$(T_{g,k,\mu,\varepsilon})_\varepsilon = \prod_{e\in g}\exp\left[\frac{i}{2}k_e\left(\int_e dx_\varepsilon(A_{x_\varepsilon}(x_\varepsilon) + \eta_\varepsilon^{\prime})\right)_\varepsilon\right]\left(\prod_{v\in V(g)}\exp(i\mu_v\gamma K_{\phi_\varepsilon,\varepsilon}(v))\right)_\varepsilon. \quad (1.4.20)$$

1.5. Schwarzschild Black Hole

Remind that the Schwarzschild metric is a spherically symmetric solution to Einstein equations describing the space-time of a source with mass m in coordinate system (x,θ,ϕ) reads

$$ds^2 = -\left(1 - \frac{2m}{x}\right)dt^2 + \left(1 - \frac{2m}{x}\right)^{-1}dx^2 + x^2 d\Omega^2. \quad (1.5.1)$$

Horizon x_+ appear where $g_{00} = 0$:

$$x_+ - 2m = 0. \quad (1.5.2)$$

The event horizons partition space-time into 2 regions: **I** $(x > x_+)$, and **II** $(0 < x < x+)$. By inspecting the sign of g_{00}, one observes that in region **II**, x and t interchange their roles and becomes time-like and space-like respectively.

Classical point-like phase space variables

In region **II**, the metric of space-time takes the form

$$ds^2 = -\left(\frac{2m}{t} - 1\right)^{-1}dt^2 + \left(\frac{2m}{t} - 1\right)dx^2 + t^2 d\Omega^2. \quad (1.5.3)$$

According to definition of tetrad (frame) fields $g_{\mu\nu} = \eta_{IJ}e_\mu^I e_\nu^J$, they can be determined only up to a Lorentz transformation. This leaves us with an $SO(3,1)$ freedom in choosing tetrad. In fact, given the metric $g_{\mu\nu} = \eta_{IJ}e_\mu^I e_\nu^J$ we are free to choose their sign and Minkowski indices, which can be viewed as sort of a labeling 4 tetrad fields. However, in order to serve as the fundamental fields for constructing the conjugate pair (A,E), a particular labeling must be chosen which will be clear below.The suitable choice for labeling 4 orthogonal frame fields reads

$$e^0 = \pm\left(\frac{2m}{t} - 1\right)^{-1/2}dt\ ;\ e^1 = \pm t\sin\theta d\phi; e^2 = \pm t d\theta\ ; e^3 = \pm\left(\frac{2m}{t} - 1\right)^{1/2}dx, \quad (1.5.4)$$

which gives the compatible spin connection components

$$\omega^{30} = -\omega^{03} = \left(-\frac{m}{t^2}\right)dx\ ;\ \omega^{20} = -\omega^{02} = \left(\frac{2m}{t} - 1\right)^{1/2}d\theta,$$
$$\omega^{10} = -\omega^{01} = \left(\frac{2m}{t} - 1\right)^{1/2}\sin\theta d\phi\ ;\ \omega^{12} = -\omega^{21} = \cos\theta d\phi. \quad (1.5.5)$$

The A field can be constructed using spin connections:

$$A^3 = \pm\gamma\left(-\frac{m}{t^2}\right)dx, A^2 = \pm\gamma\left(\frac{2m}{t} - 1\right)^{1/2} d\theta,$$
$$A^1 = \pm\gamma\left(\frac{2m}{t} - 1\right)^{1/2} \sin\theta d\phi, A^3 = \pm\cos\theta d\phi. \quad (1.5.6)$$

To construct the E field on Σ_{in} we choose a gauge in which $e^0_\mu = n_\mu$, the normal vector field to the spatial slice. This way we are in fact breaking the $SO(3,1)$ symmetry into $SO(3)$ on a hypersurface with topology $\Sigma = \mathbb{R} \times S^2$. The 3 triad fields become:

$$e^1 = \pm t\sin\theta d\phi \ ; \ e^2 = \pm t d\theta \ ; \ e^3 = \pm\left(\frac{2m}{t} - 1\right)^{1/2} dx, \quad (1.5.7)$$

with determinant

$$\det(e) = t^2 \sin\theta\left(\frac{2m}{t} - 1\right)^{1/2}, \quad (1.5.8)$$

and inverse triad

$$e_1 = \pm\frac{1}{t\sin\theta}\partial_\phi \ ; \ e_2 = \pm\frac{1}{t}\partial_\theta \ ; \ e_3 = \pm\left(\frac{2m}{t} - 1\right)^{-1/2}\partial_x, \quad (1.5.9)$$

The E fields become

$$E_1 = \pm t\left(\frac{2m}{t} - 1\right)^{1/2}\partial_\phi, E_2 = \pm t\left(\frac{2m}{t} - 1\right)^{1/2}\sin\theta\partial_\theta, E_3 = \pm t^2\sin\theta\partial_x. \quad (1.5.10)$$

The 3 triad fields (1.5.7) define their compatible spin connection, $\Gamma^{ij} \wedge e^j + de^i = 0$:

$$\Gamma^{12} = -\Gamma^{21} = \cos\theta d\phi, \quad (1.5.11)$$

and

$$\Gamma^3 = \tfrac{1}{2}(\epsilon^{312}\Gamma^{12} + \epsilon^{321}\Gamma^{21}) = \cos\theta d\phi. \quad (1.5.12)$$

Extrinsic curvature is related to A via $\gamma K = A - \Gamma$ reads

$$K^3_r = \frac{1}{\gamma}A^3_r = \pm\left(-\frac{m}{t^2}\right)dx, K^2_\theta = \frac{1}{\gamma}A^2_\theta = \pm\left(\frac{2m}{t} - 1\right)^{1/2}, \quad (1.5.13)$$

$$K^1_\phi = \tfrac{1}{\gamma}A^1_\phi = \pm\left(\frac{2m}{t} - 1\right)^{1/2}\sin\theta. \quad (1.5.14)$$

Note that had we chosen other Minkowski indices for tetrad (1.5.4) we would not have obtained the conjugate pair(A,E) with correct indices satisfying $\{A^i_a(x), E^b_j(x'_j)\} = \delta^i_j\delta^b_a\delta(x - x')$.

The phase space variables are determined up to a sign freedom. By demanding E and A to satisfy the diffeomorphism, Gauss and Hamiltonian constraints, their signs can be fixed relative to each other. All components of diffeomorphism and Gauss constraints are zero except

$$H_\theta = \gamma t\left(\frac{2m}{t} - 1\right)\cos\theta\{\mathbf{sgn}(A^1_\phi) + \mathbf{sgn}(A^2_\theta A^3_\phi)\}, \quad (1.5.15)$$

$$G_2 = t\left(\frac{2m}{t} - 1\right)^{1/2}\cos\theta\{\mathbf{sign}(E^\theta_2) + \mathbf{sign}(A^3_\phi E^{\phi 1})\}, \quad (1.5.16)$$

and Hamiltonian constraint gives:

$$C = t\left(\frac{2m}{t} - 1\right)\sin^2\theta\left\{\mathbf{sign}(E^\theta_2) + \mathbf{sign}(E^\phi_1)\right\}. \quad (1.5.17)$$

For the above constraints to be zero we must have

$$sign(E^\theta_2) = -\mathbf{sign}(E^\phi_1), \mathbf{sign}(A^3_\phi) = +1, \mathbf{sign}(A^1_\phi) = -\mathbf{sign}(A^2_\theta). \quad (1.5.18)$$

This leaves us with two alternatives corresponding to the residual gauge freedom

$(b,p_b) \to (-b,-p_b)$.

$$A^i_a = c\tau_3 dr + b\tau_2 d\theta + (\cos\theta\tau_3 - b\sin\theta\tau_1)d\phi$$
$$E^a_i = p_c\tau_3\sin\theta\partial_r + p_b\tau_2\sin\theta\partial_\theta - p_b\tau_1\partial_\phi, \tag{1.5.19}$$

and

$$A^i_a = c\tau_3 dr - b\tau_2 d\theta + (\cos\theta\tau_3 + b\sin\theta\tau_1)d\phi$$
$$E^a_i = p_c\tau_3\sin\theta\partial_r - p_b\tau_2\sin\theta\partial_\theta + p_b\tau_1\partial_\phi, \tag{1.5.20}$$

where,

$$b = \pm\gamma\left(\frac{2m}{t} - 1\right)^{1/2} ; c = \pm\gamma\left(-\frac{m}{t^2}\right), \tag{1.5.21}$$

$$p_c = \pm t^2 ; p_b = t\left(\frac{2m}{t} - 1\right)^{1/2}. \tag{1.5.22}$$

The momentum $p_c = \pm t^2$ is a monotonic function and can be interpreted as an internal time parameter (as is interpreted in [44] for the case of the Kantowski-Sachs minisuperspace of Schwarzshild black hole).

Region I.

The analoguos calculations for region I with line element (1.5.1) leads to the folowing phase space coordinates

$$\tilde{A}^i_a = \tilde{c}\tau_3 dr + \tilde{b}\tau_2 d\theta + (cos\theta\tau_3 - \tilde{b}sin\theta\tau_1)d\phi$$
$$\tilde{E}^a_i = \tilde{p}_c\tau_3 sin\theta\partial_r + \tilde{p}_b\tau_2 sin\theta\partial_\theta - \tilde{p}_b\tau_1\partial_\phi, \tag{1.5.23}$$

$$\tilde{A}^i_a = \tilde{c}\tau_3 dr - \tilde{b}\tau_2 d\theta + (\cos\theta\tau_3 + \tilde{b}\sin\theta\tau_1)d\phi$$
$$\tilde{E}^a_i = \tilde{p}_c\tau_3\sin\theta\partial_r - \tilde{p}_b\tau_2\sin\theta\partial_\theta + \tilde{p}_b\tau_1\partial_\phi, \tag{1.5.24}$$

where,

$$\tilde{b} = \pm\gamma\left(1 - \frac{2m}{x}\right)^{1/2} ; \tilde{c} = \mp\gamma\left(\frac{m}{x^2}\right); \tilde{p}_c = \pm x^2 ; \tilde{p}_b = x\left(1 - \frac{2m}{x}\right)^{1/2}. \tag{1.5.25}$$

This defines variables (1.4.5)-(1.4.6) introduced above in subsection 1.4 as

$$A_x = \tilde{c}\ , E^x = \tilde{p}_c; \gamma K_\phi = \tilde{b}\ ,\ E^\phi = \tilde{p}_b;\ \eta = (2n+1)\pi, P^\eta = 0 \tag{1.5.27}$$

which constitute a 4 dimensional phase space.

1.6.Classical point-like Loop Quantum Gravity contradict with a linear Colombeau geometry.

1.6.1.The point free quantum Schwarzschild geometry.

We remind that in accordance with a linear Colombeau geometry approach [30], the Schwarzschild black hole,etc. has a distributional source $\sim \delta(\hat{x}) \in \mathcal{D}'(\mathbb{R}^3)$, see Eq. (1.1.8) and Eq. (1.1.11). This result as well established and acceptet by scientific community as physical reality [29]-[31].

Remark 1.6.1. However under local singularity resolution based on canonical LQG approach [44], these distributional sources vanishes and we go bak to ubnormal and mistaken results from classical handbooks, see for example [3],[4].Obviously this is a contradiction. Thus by using canonical LQG approach we can not quantized the well established classically distributional Schwarzschild black hole,etc.

Viewing LQG as a method to quantize connections, one would be able to impose a symmetry through two avenues: (i) to pick, in the classical level, only those connections which are invariant under symmetry group action and consequently reduce the phase space, and (ii) to restrict the distributional states of the quantum theory, at the kinematical level, only to invariant connection [42]-[45].

We will consider the simplest case of a spin network that is equispaced in normal coordinates with lattice spacing $\Delta \sim l_{Pl}$.

Remind that under naive formal calculation the Kretschmann scalar curvature of the Schwarzschild black hole reads [43]:

$$\hat{R}^{\rho\sigma\mu\nu}(\hat{r})\hat{R}_{\rho\sigma\mu\nu}(\hat{r}) = \frac{48M^2}{t^6}. \tag{1.6.1}$$

Obviously (1.6.1) indicates that in this case the singularity of space-time lies at $\hat{r} = 0$ as well. The classical phase space variables calculated in subsect.1.5 $\{c, p_c\}$ used in this section are given by Eq.(1.5.21)-Eq.(1.5.22) and therefore

$$b = \pm\gamma\left(\frac{2m}{t} - 1\right)^{1/2} ; \; c = \mp\gamma\left(\frac{m}{t^2}\right); p_c = \pm t^2 \; ; \; p_b = t\left(\frac{2m}{t} - 1\right)^{1/2}. \tag{1.6.2}$$

Let us consider the following quantity on the classical point-like phase space [43]:

$$\mathcal{R} \equiv \frac{1}{2\pi\gamma G}\left\{c, \sqrt{|p_c|}\right\} = \frac{sgn(p_c)}{\sqrt{|p_c|}} = \frac{1}{t}. \tag{1.6.3}$$

Following the methods presented in [44], we expand now the holonomy along x direction of $\Sigma = \mathbb{R} \times S^2$ with oriented length τ as

$$h_x^{(\tau)} = 1 + \epsilon \int_0^\tau dx c \tau_3 + \mathcal{O}(\epsilon^2), \tag{1.6.4}$$

and rewrite $\mathcal{R}$ as

$$\mathcal{R} = \frac{1}{2\pi\gamma G}\mathbf{tr}\left(\tau_3 h_x^{(\tau)}\left\{h_x^{(\tau)-1}, \sqrt{|p_c|}\right\}\right). \tag{1.6.5}$$

Now, quantization would be straightforward:

$$\begin{aligned}\hat{\mathcal{R}} &= \frac{1}{2\pi\gamma\ell_{Pl}^2}\mathbf{tr}\left(\tau_3 \hat{h}_x^{(\tau)}\left[\hat{h}_x^{(\tau)-1}, \sqrt{|\hat{p}_c|}\,\right]\right) \\ &= \frac{1}{2\pi\gamma\ell_{Pl}^2}\left(\cos\left(\frac{\tau c}{2}\right)\sqrt{|\hat{p}_c|}\sin\left(\frac{\tau c}{2}\right) - \sin\left(\frac{\tau c}{2}\right)\sqrt{|\hat{p}_c|}\cos\left(\frac{\tau c}{2}\right)\right).\end{aligned} \tag{1.6.6}$$

Its action on $|\tau, \mu\rangle$ which are the simplified version of the spin network states in this reduced model (with μ being the oriented length along the equator of S^2), then becomes:

$$\hat{\mathcal{R}}|\tau, \mu\rangle = \frac{1}{2\pi\sqrt{\gamma}\,\ell_{Pl}}\left(\sqrt{|\tau + 1|} - \sqrt{|\tau - 1|}\right)|\tau, \mu\rangle. \tag{1.6.7}$$

Such operator $\hat{\mathcal{R}}|\tau, \mu\rangle$ has a bounded spectrum with maximum value of $\left(\sqrt{2}\,\pi\sqrt{\gamma}\,\ell_{Pl}\right)^{-1}$. Thus the Kretschmann scalar curvature, which is classically divergent, at quantum level has a maximum value of [43]:

$$\hat{R}^{\rho\sigma\mu\nu}(\hat{r})\hat{R}_{\rho\sigma\mu\nu}(\hat{r})\Big|_{\max} = \frac{48M^2}{\hat{r}^6}\Big|_{\max} = \frac{48M^2}{\gamma^3\pi^6 l_{Pl}^6}. \tag{1.6.8}$$

Remark 1.6.2. Note that a quantity $\left(\hat{R}_\varepsilon(\hat{r})\right)_\varepsilon$ which is classically has a weak distributional

limit, at quantum level obtained by canonical LQG has a maximum value of

$$\left|\hat{R}_\varepsilon(\hat{r})\right|\Big|_{\max} \sim \left.\frac{M\varepsilon^2}{l_{Pl}^5}\right|_{\max}. \tag{1.6.9}$$

Thus $\lim_{\varepsilon\to 0}\hat{R}_\varepsilon(\hat{r}) = 0$ since RHS of the Eq.(1.6.9) wanishes in the limit $\varepsilon \to 0$.

1.6.2.The point free quantum Schwarzschild geometry.

Classical point-free phase space variables

In region **II**, the Colombeau metric of point-free Schwarzschild space-time takes the form

$$(ds_\varepsilon^2)_\varepsilon = -\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{-1}(dt_\varepsilon^2)_\varepsilon + \left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right](dx_\varepsilon^2)_\varepsilon + (t_\varepsilon^2 d\Omega_\varepsilon^2)_\varepsilon, \tag{1.6.10}$$

where $\mathbf{cl}[(t_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}$. According to definition of Colombeau tetrad (frame) fields $(g_{\mu\nu,\varepsilon})_\varepsilon = \eta_{IJ}(e^I_{\mu,\varepsilon}e^J_{\nu,\varepsilon})_\varepsilon$, they can be determined only up to a Lorentz transformation. This leaves us with an $SO(3,1)$ freedom in choosing tetrad. In fact, given the Colombeau metric $(g_{\mu\nu,\varepsilon})_\varepsilon = \eta_{IJ}(e^I_{\mu,\varepsilon}e^J_{\nu,\varepsilon})_\varepsilon$ we are free to choose their sign and Minkowski indices, which can be viewed as sort of a labeling 4 tetrad fields. However, in order to serve as the fundamental fields for constructing the conjugate pair $((A_\varepsilon)_\varepsilon,(E_\varepsilon)_\varepsilon)$, a particular labeling must be chosen which will be clear below.The suitable choice for labeling 4 orthogonal Colombeau generalized frame fields reads

$$\begin{gathered}(e_\varepsilon^0)_\varepsilon = \pm\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{-1/2}(dt_\varepsilon)_\varepsilon\,;(e_\varepsilon^1)_\varepsilon = \pm[(t_\varepsilon)_\varepsilon][(\sin\theta_\varepsilon)_\varepsilon][(d\phi_\varepsilon)_\varepsilon];\\ (e_\varepsilon^2)_\varepsilon = \pm[(t_\varepsilon)_\varepsilon][(d\theta_\varepsilon)_\varepsilon]\,;(e_\varepsilon^3)_\varepsilon = \pm\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}[(dx_\varepsilon)_\varepsilon],\end{gathered} \tag{1.6.11}$$

which gives the compatible Colombeau generalized spin connection components

$$\begin{gathered}(\omega_\varepsilon^{30})_\varepsilon = -(\omega_\varepsilon^{03})_\varepsilon = -\frac{m}{[(t_\varepsilon^2)_\varepsilon]}[(dx_\varepsilon)_\varepsilon]\,;\\ (\omega_\varepsilon^{20})_\varepsilon = -(\omega_\varepsilon^{02})_\varepsilon = \left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}[(d\theta_\varepsilon)_\varepsilon],\\ (\omega_\varepsilon^{10})_\varepsilon = -(\omega_\varepsilon^{01})_\varepsilon = \left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}[(\sin\theta_\varepsilon)_\varepsilon][(d\phi_\varepsilon)_\varepsilon]\,;\\ (\omega_\varepsilon^{12})_\varepsilon = -(\omega_\varepsilon^{21})_\varepsilon = [(\cos\theta_\varepsilon)_\varepsilon][(d\phi_\varepsilon)_\varepsilon].\end{gathered} \tag{1.6.12}$$

The $(A_\varepsilon)_\varepsilon$ field can be constructed using spin connections:

$$\begin{gathered}(A_\varepsilon^3)_\varepsilon = \pm\gamma\left(-\frac{m}{(t_\varepsilon^2)_\varepsilon}\right)[(dx_\varepsilon)_\varepsilon],(A_\varepsilon^2)_\varepsilon = \pm\gamma\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}[(d\theta_\varepsilon)_\varepsilon],\\ (A_\varepsilon^1)_\varepsilon = \pm\gamma\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}[(\sin\theta_\varepsilon)_\varepsilon][(d\phi_\varepsilon)_\varepsilon],(A_\varepsilon^3)_\varepsilon = \pm[(\cos\theta_\varepsilon)_\varepsilon][(d\phi_\varepsilon)_\varepsilon].\end{gathered} \tag{1.6.13}$$

To construct Colombeau generalized field $(E_\varepsilon)_\varepsilon$ on $\widetilde{\Sigma}$ we choose a gauge in which $e^0_\mu = n_\mu$, the normal vector field to the spatial slice. This way we are in fact breaking the $SO(3,1)$ symmetry into $SO(3)$ on a hypersurface with topology $\widetilde{\Sigma} = \widetilde{\mathbb{R}} \times \widetilde{S}^2$. The

Colombeau generalized 3 triad fields become:

$$(e_\varepsilon^1)_\varepsilon = \pm\left[(t_\varepsilon^{1/2})_\varepsilon\right][(\sin\theta_\varepsilon)_\varepsilon][(d\phi_\varepsilon)_\varepsilon]\ ;\ (e_\varepsilon^2)_\varepsilon = \pm[(t_\varepsilon)_\varepsilon][(d\theta_\varepsilon)_\varepsilon]\ ;$$
$$(e_\varepsilon^3)_\varepsilon = \pm\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}[(dx_\varepsilon)_\varepsilon], \tag{1.6.14}$$

with determinant

$$(\det(e_\varepsilon))_\varepsilon = [(t_\varepsilon^2)_\varepsilon][(\sin\theta_\varepsilon)_\varepsilon]\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}, \tag{1.6.15}$$

and inverse triad

$$(e_{1,\varepsilon})_\varepsilon = \pm\frac{1}{[(t_\varepsilon)_\varepsilon][(\sin\theta_\varepsilon)_\varepsilon]}[(\partial_{\phi_\varepsilon})_\varepsilon]\ ;\ (e_{2,\varepsilon})_\varepsilon = \pm\frac{1}{[(t_\varepsilon)_\varepsilon]}[(\partial_{\theta_\varepsilon})_\varepsilon]\ ;$$
$$(e_{3,\varepsilon})_\varepsilon = \pm\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{-1/2}[(\partial_{x,\varepsilon})_\varepsilon]. \tag{1.6.16}$$

The $(E_\varepsilon)_\varepsilon$ fields become

$$(E_{1,\varepsilon})_\varepsilon = \pm[(t_\varepsilon)_\varepsilon]\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right][^{1/2}(\partial_{\phi_\varepsilon})_\varepsilon],$$
$$(E_{2,\varepsilon})_\varepsilon = \pm[(t_\varepsilon)_\varepsilon]\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}[(\sin\theta_\varepsilon)_\varepsilon][(\partial_{\theta_\varepsilon})_\varepsilon], \tag{1.6.17}$$
$$(E_{3,\varepsilon})_\varepsilon = \pm[(t_\varepsilon^2)_\varepsilon][(\sin\theta_\varepsilon)_\varepsilon][(\partial_{x,\varepsilon})_\varepsilon].$$

The Colombeau generalized 3 triad fields (1.6.14) define their generalized compatible spin connection, $\left(\Gamma_\varepsilon^{ij} \wedge e_\varepsilon^j\right)_\varepsilon + (de_\varepsilon^i)_\varepsilon = 0_{\widetilde{\mathbb{R}}}$:

$$(\Gamma_\varepsilon^{12})_\varepsilon = -(\Gamma_\varepsilon^{21})_\varepsilon = [(\cos\theta_\varepsilon)_\varepsilon][(d\phi_\varepsilon)_\varepsilon], \tag{1.6.18}$$

and

$$(\Gamma_\varepsilon^3)_\varepsilon = \frac{1}{2}(\epsilon^{312}\Gamma_\varepsilon^{12} + \epsilon^{321}\Gamma_\varepsilon^{21})_\varepsilon = [(\cos\theta_\varepsilon)_\varepsilon][(d\phi_\varepsilon)_\varepsilon]. \tag{1.6.19}$$

Extrinsic distributional curvature is related to $(A_\varepsilon)_\varepsilon$ via $\gamma(K_\varepsilon)_\varepsilon = (A_\varepsilon)_\varepsilon - (\Gamma_\varepsilon)_\varepsilon$ reads

$$(K_{r_\varepsilon,\varepsilon}^3)_\varepsilon = \frac{1}{\gamma}(A_{r_\varepsilon,\varepsilon}^3)_\varepsilon = \pm\left(-\frac{m}{(t_\varepsilon^2)_\varepsilon}\right)[(dx_\varepsilon)_\varepsilon],$$
$$(K_{\theta_\varepsilon,\varepsilon}^2)_\varepsilon = \frac{1}{\gamma}(A_{\theta_\varepsilon,\varepsilon}^2)_\varepsilon = \pm\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}, \tag{1.6.20}$$

$$(K_{\phi_\varepsilon,\varepsilon}^1)_\varepsilon = \frac{1}{\gamma}(A_{\phi_\varepsilon,\varepsilon}^1)_\varepsilon = \pm\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}[(\sin\theta_\varepsilon)_\varepsilon]. \tag{1.6.21}$$

Note that had we chosen other Minkowski indices for tetrad (1.6.11) we would not have obtained the conjugate pair(A,E) with correct indices satisfying $(\{A_{a,\varepsilon}^i(x_\varepsilon), E_j^b(x'_\varepsilon)\})_\varepsilon = \delta_j^i(\delta_a^b\delta(x_\varepsilon - x'_\varepsilon))_\varepsilon$.

The phase space variables are determined up to a sign freedom. By demanding E and A to satisfy the diffeomorphism, Gauss and Hamiltonian constraints, their signs can be fixed relative to each other. All components of diffeomorphism and Gauss constraints are zero except

$$(H_{\theta_\varepsilon,\varepsilon})_\varepsilon = \gamma[(t_\varepsilon)_\varepsilon]\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right][(\cos\theta_\varepsilon)_\varepsilon]\{\mathbf{sign}[(A^1_{\phi_\varepsilon,\varepsilon})_\varepsilon] + \mathbf{sign}[(A^2_{\theta_\varepsilon,\varepsilon}A^3_{\phi_\varepsilon,\varepsilon})_\varepsilon]\}, \quad (1.6.22)$$

$$(G_{2,\varepsilon})_\varepsilon = [(t_\varepsilon)_\varepsilon]\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2}[(\cos\theta_\varepsilon)_\varepsilon]\left\{\mathbf{sign}[(E^{\theta_\varepsilon}_{2,\varepsilon})_\varepsilon] + \mathbf{sign}\left[(A^3_{\phi_\varepsilon}E^{\phi_\varepsilon 1}_\varepsilon)_\varepsilon\right]\right\}, \quad (1.6.23)$$

and Hamiltonian constraint gives:

$$(C_\varepsilon)_\varepsilon = [(t_\varepsilon)_\varepsilon]\left(\frac{2m}{[(t_\varepsilon)_\varepsilon]} - 1\right)\left[(\sin^2\theta_\varepsilon)_\varepsilon\right]\left\{\mathbf{sign}[(E^{\theta_\varepsilon}_{2,\varepsilon})_\varepsilon] + \mathbf{sign}\left[(E^{\phi_\varepsilon}_{1,\varepsilon})_\varepsilon\right]\right\}. \quad (1.6.24)$$

For the above constraints to be zero we must have

$$\begin{gathered}\mathbf{sign}[(E^{\theta_\varepsilon}_{2,\varepsilon})_\varepsilon] = -\mathbf{sign}\left[(E^{\phi_\varepsilon}_{1,\varepsilon})_\varepsilon\right],\\ \mathbf{sign}[(A^3_{\phi_\varepsilon,\varepsilon})_\varepsilon] = +1, \mathbf{sign}[(A^1_{\phi_\varepsilon,\varepsilon})_\varepsilon] = -\mathbf{sign}[(A^2_{\theta_\varepsilon,\varepsilon})_\varepsilon].\end{gathered} \quad (1.6.25)$$

This leaves us with two alternatives corresponding to the residual gauge freedom $((b_\varepsilon)_\varepsilon, (p_{b_\varepsilon,\varepsilon})_\varepsilon) \to (-(b_\varepsilon)_\varepsilon, -(p_{b_\varepsilon,\varepsilon})_\varepsilon)$.

$$\begin{gathered}(A^i_{a,\varepsilon})_\varepsilon = [(c_\varepsilon)_\varepsilon]\tau_3[(dr_\varepsilon)_\varepsilon] + [(b_\varepsilon)_\varepsilon]\tau_2[(d\theta_\varepsilon)_\varepsilon] +\\ ([(\cos\theta_\varepsilon)_\varepsilon]\tau_3 - [(b_\varepsilon)_\varepsilon][(\sin\theta_\varepsilon)_\varepsilon]\tau_1)[(d\phi_\varepsilon)_\varepsilon]\\ (E^a_{i,\varepsilon})_\varepsilon = [(p_{c_\varepsilon,\varepsilon})_\varepsilon]\tau_3[(\sin\theta_\varepsilon)_\varepsilon][(\partial_{r_\varepsilon})_\varepsilon] +\\ [(p_{b_\varepsilon,\varepsilon})_\varepsilon]\tau_2[(\sin\theta_\varepsilon)_\varepsilon][(\partial_{\theta_\varepsilon})_\varepsilon] - [(p_{b_\varepsilon,\varepsilon})_\varepsilon]\tau_1[(\partial_{\phi_\varepsilon})_\varepsilon],\end{gathered} \quad (1.6.26)$$

and

$$\begin{gathered}(A^i_{a,\varepsilon})_\varepsilon = [(c_\varepsilon)_\varepsilon]\tau_3[(dr_\varepsilon)_\varepsilon] - [(b_\varepsilon)_\varepsilon]\tau_2[(d\theta_\varepsilon)_\varepsilon] +\\ ([(\cos\theta_\varepsilon)_\varepsilon]\tau_3 + [(b_\varepsilon)_\varepsilon][(\sin\theta_\varepsilon)_\varepsilon]\tau_1)d\phi\\ (E^a_{i,\varepsilon})_\varepsilon = [(p_{c_\varepsilon,\varepsilon})_\varepsilon]\tau_3[(\sin\theta_\varepsilon)_\varepsilon][(\partial_{r_\varepsilon})_\varepsilon] - [(p_{c_\varepsilon,\varepsilon})_\varepsilon]\tau_2[(\sin\theta_\varepsilon)_\varepsilon][(\partial_{\theta_\varepsilon})_\varepsilon] +\\ +[(p_{b_\varepsilon,\varepsilon})_\varepsilon]\tau_1[(\partial_{\phi_\varepsilon})_\varepsilon],\end{gathered} \quad (1.6.27)$$

where,

$$(b_\varepsilon)_\varepsilon = \pm\gamma\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]^{1/2};\ (c_\varepsilon)_\varepsilon = \mp\gamma\frac{m}{[(t^2_\varepsilon)_\varepsilon]}, \quad (1.6.28)$$

$$(p_{c_\varepsilon,\varepsilon})_\varepsilon = \pm(t^2_\varepsilon)_\varepsilon;\ (p_{b_\varepsilon,\varepsilon})_\varepsilon = [(t_\varepsilon)_\varepsilon]\left(\left[\frac{2m}{(t_\varepsilon)_\varepsilon} - 1\right]\right)^{1/2}. \quad (1.6.29)$$

The momentum $(p_{c_\varepsilon,\varepsilon})_\varepsilon = \pm(t^2_\varepsilon)_\varepsilon$ is a monotonic generalized function on $\widetilde{\mathbb{R}}$ and can be interpreted as an internal generalized time parameter.

Region I.

The analoguos calculations for region **I** with Colombeau generalized line element (1.6.10) leads to the folowing phase space coordinates

$$\begin{gathered}(\tilde{A}^i_{a,\varepsilon})_\varepsilon = [(\tilde{c}_\varepsilon)_\varepsilon]\tau_3[(dr_\varepsilon)_\varepsilon] + \left[\left(\tilde{b}_\varepsilon\right)_\varepsilon\right]\tau_2[(d\theta_\varepsilon)_\varepsilon] +\\ ([(\cos\theta_\varepsilon)_\varepsilon]\tau_3 - \left[\left(\tilde{b}_\varepsilon\right)_\varepsilon\right][(\sin\theta_\varepsilon)_\varepsilon]\tau_1)[(d\phi_\varepsilon)_\varepsilon],\\ (\tilde{E}^a_{i,\varepsilon})_\varepsilon = [(\tilde{p}_{c_\varepsilon,\varepsilon})_\varepsilon]\tau_3[(\sin\theta_\varepsilon)_\varepsilon][(\partial_{r_\varepsilon})_\varepsilon] +\\ [(\tilde{p}_{b_\varepsilon,\varepsilon})_\varepsilon]\tau_2[(\sin\theta_\varepsilon)_\varepsilon][(\partial_{\theta_\varepsilon})_\varepsilon] - [(\tilde{p}_{b_\varepsilon,\varepsilon})_\varepsilon]\tau_1[(\partial_{\phi_\varepsilon})_\varepsilon],\end{gathered} \quad (1.6.30)$$

$$
\begin{aligned}
(\tilde{A}^i_{a,\varepsilon})_\varepsilon &= [(\tilde{c}_\varepsilon)_\varepsilon]\tau_3[(dr_\varepsilon)_\varepsilon] - \left[\left(\tilde{b}_\varepsilon\right)_\varepsilon\right]\tau_2[(d\theta_\varepsilon)_\varepsilon] + \\
&([(\cos\theta_\varepsilon)_\varepsilon]\tau_3 + \left[\left(\tilde{b}_\varepsilon\right)_\varepsilon\right][(\sin\theta_\varepsilon)_\varepsilon]\tau_1)[(d\phi_\varepsilon)_\varepsilon], \\
(\tilde{E}^a_{i,\varepsilon})_\varepsilon &= [(\tilde{p}_{c_\varepsilon,\varepsilon})_\varepsilon]\tau_3[(\sin\theta_\varepsilon)_\varepsilon][(\partial_{r_\varepsilon})_\varepsilon] - \\
&[(\tilde{p}_{b_\varepsilon,\varepsilon})_\varepsilon]\tau_2[(\sin\theta_\varepsilon)_\varepsilon][(\partial_{\theta_\varepsilon})_\varepsilon] + [(\tilde{p}_{b_\varepsilon,\varepsilon})_\varepsilon]\tau_1[(\partial_{\phi_\varepsilon})_\varepsilon],
\end{aligned}
\tag{1.6.31}
$$

where,

$$
\begin{gathered}
\left(\tilde{b}_\varepsilon\right)_\varepsilon = \pm\gamma\left[1 - \frac{2m}{(x_\varepsilon)_\varepsilon}\right]^{1/2};\ (\tilde{c}_\varepsilon)_\varepsilon = \mp\gamma\frac{m}{[(x^2_\varepsilon)_\varepsilon]}; (\tilde{p}_{c_\varepsilon,\varepsilon})_\varepsilon = \pm(x^2_\varepsilon)_\varepsilon; \\
(\tilde{p}_{b_\varepsilon})_\varepsilon = [(x_\varepsilon)_\varepsilon]\left[1 - \frac{2m}{(x_\varepsilon)_\varepsilon}\right]^{1/2}.
\end{gathered}
\tag{1.6.32}
$$

This defines variables (1.4.5)-(1.4.6) introduced above as

$$
\begin{gathered}
(A_{x_\varepsilon,\varepsilon})_\varepsilon = (\tilde{c}_\varepsilon)_\varepsilon, (E^{x_\varepsilon}_\varepsilon)_\varepsilon = (\tilde{p}_{c_\varepsilon,\varepsilon})_\varepsilon; \gamma(K_{\phi_\varepsilon,\varepsilon})_\varepsilon = \left(\tilde{b}_\varepsilon\right)_\varepsilon, \\
\left(E^{\phi_\varepsilon}_\varepsilon\right)_\varepsilon = (\tilde{p}_{b_\varepsilon,\varepsilon})_\varepsilon;\ \eta = (2n+1)\pi, (P^{\eta_\varepsilon}_\varepsilon)_\varepsilon = 0_{\widetilde{\mathbb{R}}}
\end{gathered}
\tag{1.6.33}
$$

which constitute a 4 dimensional phase space.

Let us consider the following quantity on the point-free phase space mentioned above

$$
(\mathcal{R}_\varepsilon)_\varepsilon \equiv \frac{1}{2\pi\gamma G}\left\{[(c_\varepsilon)_\varepsilon], \left[\sqrt{[(|p_{c_\varepsilon,\varepsilon}|)_\varepsilon]}\right]\right\} = \frac{\mathbf{sign}[(p_{c_\varepsilon,\varepsilon})_\varepsilon]}{\sqrt{[(|p_{c_\varepsilon,\varepsilon}|)_\varepsilon]}} = \frac{1}{[(t_\varepsilon)_\varepsilon]}. \tag{1.6.33}
$$

Following the canonical methods presented in [44], we expand now the holonomy along x

direction of $\widetilde{\Sigma} = \widetilde{\mathbb{R}} \times \widetilde{S}^2$ with oriented generalized length $(\tau_\varepsilon)_\varepsilon$ as

$$
\left(h^{(\tau_\varepsilon)}_{x_\varepsilon,\varepsilon}\right)_\varepsilon = 1 + \epsilon\left(\int_0^{\tau_\varepsilon} dx_\varepsilon c_\varepsilon \tau_3\right)_\varepsilon + (\mathcal{O}_\varepsilon(\epsilon^2))_\varepsilon, \tag{1.6.34}
$$

and rewrite $(\mathcal{R}_\varepsilon)_\varepsilon$ as

$$
(\mathcal{R}_\varepsilon)_\varepsilon = \frac{1}{2\pi\gamma G}\mathbf{tr}\left(\tau_3\left[\left(h^{(\tau_\varepsilon)}_{x_\varepsilon,\varepsilon}\right)_\varepsilon\right]\left\{\left[\left(h^{(\tau_\varepsilon)-1}_{x_\varepsilon,\varepsilon}\right)_\varepsilon\right], \sqrt{[(|p_{c_\varepsilon,\varepsilon}|)_\varepsilon]}\right\}\right). \tag{1.6.35}
$$

Now, quantization would be straightforward:

$$
\begin{aligned}
(\hat{\mathcal{R}}_\varepsilon)_\varepsilon &= \frac{1}{2\pi\gamma\ell^2_{Pl}}\mathbf{tr}\left(\tau_3\left[\left(\hat{h}^{(\tau_\varepsilon)}_{x_\varepsilon,\varepsilon}\right)_\varepsilon\right]\left\{\left[\left(\hat{h}^{(\tau_\varepsilon)-1}_{x_\varepsilon,\varepsilon}\right)_\varepsilon\right], \sqrt{\left[\left(|\hat{p}_{c_\varepsilon,\varepsilon}|\right)_\varepsilon\right]}\right\}\right) \\
&= \frac{1}{2\pi\gamma\ell^2_{Pl}}\left(\left[\cos\left(\frac{\tau_\varepsilon c_\varepsilon}{2}\right)_\varepsilon\right]\sqrt{\left[\left(|\hat{p}_{c_\varepsilon,\varepsilon}|\right)_\varepsilon\right]}\left[\sin\left(\frac{\tau_\varepsilon c_\varepsilon}{2}\right)_\varepsilon\right] - \right. \\
&\left.\left[\sin\left(\frac{\tau_\varepsilon c_\varepsilon}{2}\right)_\varepsilon\right]\sqrt{\left[\left(|\hat{p}_{c_\varepsilon,\varepsilon}|\right)_\varepsilon\right]}\left[\cos\left(\frac{\tau_\varepsilon c_\varepsilon}{2}\right)_\varepsilon\right]\right).
\end{aligned}
\tag{1.6.36}
$$

Its action on $|(\tau_\varepsilon)_\varepsilon, (\mu_\varepsilon)_\varepsilon\rangle$ which are the simplified version of the spin network states in this reduced model (with $(\mu_\varepsilon)_\varepsilon$ being the oriented length along the equator of $\widetilde{S}^2$), then becomes:

$$
(\hat{\mathcal{R}}_\varepsilon|\tau_\varepsilon, \mu_\varepsilon\rangle)_\varepsilon = \frac{1}{2\pi\sqrt{\gamma}\,\ell_{Pl}}\left(\sqrt{|(\tau_\varepsilon)_\varepsilon + 1|} - \sqrt{|(\tau_\varepsilon)_\varepsilon - 1|}\right)|(\tau_\varepsilon)_\varepsilon, (\mu_\varepsilon)_\varepsilon\rangle. \tag{1.6.37}
$$

Such operator $(\hat{\mathcal{R}}_\varepsilon|\tau_\varepsilon, \mu_\varepsilon\rangle)_\varepsilon$ has a bounded spectrum with maximum value of $\left(\sqrt{2}\,\pi\sqrt{\gamma}\,\ell_{Pl}\right)^{-1}$.

Remark 1.6.3. Thus the Colombeau generalized Kretschmann scalar curvature $\left(\hat{R}_{\varepsilon}^{\rho\sigma\mu\nu}(t_{\varepsilon})\hat{R}_{\rho\sigma\mu\nu,\varepsilon}(t_{\varepsilon})\right)_{\varepsilon}$, which is classically has infinite large point value $\sim \mathbf{cl}[(\varepsilon^{-6})_{\varepsilon}] \in \widetilde{\mathbb{R}}$ (see Eq.(1.1.18)), at quantum level has a maximum value of :

$$\mathbf{cl}\left[\left(\hat{R}_{\varepsilon}^{\rho\sigma\mu\nu}(t_{\varepsilon})\hat{R}_{\rho\sigma\mu\nu,\varepsilon}(t_{\varepsilon})\right)_{\varepsilon}\right]\Big|_{\max} \sim \frac{M^2}{\mathbf{cl}[(t_{\varepsilon}^6)_{\varepsilon}]}\Bigg|_{\max} \leq_{\widetilde{\mathbb{R}}} \mathbf{st}\left(\frac{M^2}{\mathbf{cl}[(t_{\varepsilon}^6)_{\varepsilon}]}\Bigg|_{\max}\right) \leq \frac{M^2}{\gamma^3\pi^6 l_{Pl}^6}. \tag{1.6.38}$$

Remark 1.6.4. Note that the Colombeau generalized curvature scalar $\left(\hat{R}_{\varepsilon}(t_{\varepsilon})\right)_{\varepsilon}$ obtained at quantum level by point-free LQG by using similarly calculation as it has been applied above, has nonzero maximum value

$$\left(\left|\hat{R}_{\varepsilon}(t_{\varepsilon})\right|\right)_{\varepsilon}\Big|_{\max} \sim \frac{M}{l_{Pl}^3}\Bigg|_{\max}. \tag{1.6.39}$$

Remark 1.6.5. We emphasize that in contrast with trivial (zero valued) result obtained at quantum level for Colombeau generalized curvature scalar $\left(\hat{R}_{\varepsilon}(t_{\varepsilon})\right)_{\varepsilon}$ by using canonical LQG, see Remark 1.6.2, Colombeau generalized curvature scalar $\left(\hat{R}_{\varepsilon}(t_{\varepsilon})\right)_{\varepsilon}$ obtained at quantum level by point-free LQG has nonzero maximum value given by Eq.(1.6.39).

.

1.7.Generalized Stokes' theorem.

1.7.1.The Colombeau generalized curvilinear coordinates.

Let us consider now the Colombeau generalized transformation from one generalized coordinate system, $(x_{\varepsilon}^0)_{\varepsilon},(x_{\varepsilon}^1)_{\varepsilon},(x_{\varepsilon}^2)_{\varepsilon},(x_{\varepsilon}^3)_{\varepsilon}$, to another generalized coordinate system $(x_{\varepsilon}'^0)_{\varepsilon},(x_{\varepsilon}'^1)_{\varepsilon},(x_{\varepsilon}'^2)_{\varepsilon},(x_{\varepsilon}'^3)_{\varepsilon}$: transform according to the relation

$$(x_{\varepsilon}^i)_{\varepsilon} = \left(f_{\varepsilon}^i(x_{\varepsilon}'^0,x_{\varepsilon}'^1,x_{\varepsilon}'^2,x_{\varepsilon}'^3)\right)_{\varepsilon}, \tag{1.7.1}$$

where the $\left(f_{\varepsilon}^i\right)_{\varepsilon}$ are certain Colombeau generalized functions and where $(\mathbf{J}_{\varepsilon}(x'^0,x'^1,x'^2,x'^3))_{\varepsilon}$

$$(\mathbf{J}_{\varepsilon}(x_{\varepsilon}'^0,x_{\varepsilon}'^1,x_{\varepsilon}'^2,x_{\varepsilon}'^3))_{\varepsilon} = \left(\frac{\partial(x_{\varepsilon}^0,x_{\varepsilon}^1,x_{\varepsilon}^2,x_{\varepsilon}^3)}{\partial(x_{\varepsilon}'^0,x_{\varepsilon}'^1,x_{\varepsilon}'^2,x_{\varepsilon}'^3)}\right)_{\varepsilon} \neq 0_{\widetilde{\mathbb{R}}} \tag{1.7.2}$$

is the Jacobian of the Colombeau generalized transformation (1.7.1).

Remark 1.7.1. When we transform the coordinates, their Colombeau differentials $(dx_{\varepsilon}^i)_{\varepsilon}$ transform according to the relation

$$(dx_{\varepsilon}^i)_{\varepsilon} = \left(\frac{\partial x_{\varepsilon}^i}{\partial x_{\varepsilon}'^k}dx_{\varepsilon}'^k\right)_{\varepsilon} = \left[\left(\frac{\partial x_{\varepsilon}^i}{\partial x_{\varepsilon}'^k}\right)_{\varepsilon}\right](dx_{\varepsilon}'^k)_{\varepsilon}. \tag{1.7.3}$$

Definition 1.7.1. Every tuple of four Colombeau quantities $(A_{\varepsilon}^i)_{\varepsilon}, i = 0,1,2,3$, which under a transformation (1.7.1) of coordinates, transform like the Colombeau coordinate differentials (1.7.2), is called Colombeau contravariant four-vector:

$$(A^i_\varepsilon)_\varepsilon = \left(\frac{\partial x^i_\varepsilon}{\partial x'^k_\varepsilon} A'^k_\varepsilon\right)_\varepsilon = \left[\left(\frac{\partial x^i_\varepsilon}{\partial x'^k_\varepsilon}\right)_\varepsilon\right](A'^k_\varepsilon)_\varepsilon. \quad (1.7.4)$$

Let $(\varphi_\varepsilon)_\varepsilon$ be the Colombeau scalar. Under a coordinate transformation (1.7.1), the four Colombeau quantities $\left(\frac{\partial \varphi_\varepsilon}{\partial x^i_\varepsilon}\right)_\varepsilon, i = 0,1,2,3$ transform according to the formula

$$\left(\frac{\partial \varphi_\varepsilon}{\partial x^i_\varepsilon}\right)_\varepsilon = \left(\frac{\partial \varphi_\varepsilon}{\partial x'^k_\varepsilon}\frac{\partial x'^k_\varepsilon}{\partial x^i_\varepsilon}\right)_\varepsilon = \left(\frac{\partial \varphi_\varepsilon}{\partial x'^k_\varepsilon}\right)_\varepsilon\left(\frac{\partial x'^k_\varepsilon}{\partial x^i_\varepsilon}\right)_\varepsilon. \quad (1.7.5)$$

Definition 1.7.2. Every tuple of four Colombeau generalized functions $(A_{i,\varepsilon})_\varepsilon$ which, under
a coordinate transformation (1.7.1), transform like the Colombeau derivatives of a scalar,
is called Colombeau generalized covariant four-vector

$$(A_{i,\varepsilon})_\varepsilon = \left(\frac{\partial x'^k_\varepsilon}{\partial x^i_\varepsilon} A'_{k,\varepsilon}\right)_\varepsilon = \left(\frac{\partial x'^k_\varepsilon}{\partial x^i_\varepsilon}\right)_\varepsilon (A'_{k,\varepsilon})_\varepsilon. \quad (1.7.6)$$

Definition 1.7.3. We call the Colombeau generalized contravariant tensor of the second
rank, $(A^{ik}_\varepsilon)_\varepsilon$,any tuple of sixteen Colombeau generalized functions which transform like the
products of the components of two Colombeau generalized contravariant vectors, i.e. according to the law

$$(A^{ik}_\varepsilon)_\varepsilon = \left(\frac{\partial x'^k_\varepsilon}{\partial x^i_\varepsilon}\frac{\partial x'^m_\varepsilon}{\partial x^i_\varepsilon} A'_{im,\varepsilon}\right)_\varepsilon = \left(\frac{\partial x'^k_\varepsilon}{\partial x^i_\varepsilon}\frac{\partial x'^m_\varepsilon}{\partial x^i_\varepsilon}\right)_\varepsilon (A'_{im,\varepsilon})_\varepsilon \quad (1.7.7)$$

and a mixed Colombeau generalized tensor transforms as follows

$$(A^i_{k,\varepsilon})_\varepsilon = \left(\frac{\partial x^i_\varepsilon}{\partial x'^l_\varepsilon}\frac{\partial x'^m_\varepsilon}{\partial x^k_\varepsilon} A'^l_{m,\varepsilon}\right)_\varepsilon = \left(\frac{\partial x^i_\varepsilon}{\partial x'^l_\varepsilon}\frac{\partial x'^m_\varepsilon}{\partial x^k_\varepsilon}\right)_\varepsilon (A'^l_{m,\varepsilon})_\varepsilon. \quad (1.7.8)$$

Remark 1.7.2. Note that the scalar product of two four-vectors $(A^i_\varepsilon B_{i,\varepsilon})$ is invariant since

$$(A^i_\varepsilon B_{i,\varepsilon})_\varepsilon = \left(\frac{\partial x^i_\varepsilon}{\partial x'^l_\varepsilon}\frac{\partial x'^m_\varepsilon}{\partial x^i_\varepsilon} A'^l_\varepsilon B'_{m,\varepsilon}\right)_\varepsilon = \left(\frac{\partial x'^m_\varepsilon}{\partial x'^l_\varepsilon} A'^l_\varepsilon B'_{m,\varepsilon}\right)_\varepsilon = (A'^l_\varepsilon B'_{l,\varepsilon})_\varepsilon. \quad (1.7.9)$$

The unit four-tensor δ^i_k is defined the same as in classical case: $\delta^i_k = 0$ for $i \neq k$ and
$\delta^i_k = 1$ for $i = k$. If $(A^k_\varepsilon)_\varepsilon$is a Colombeau generalized four-vector,then multiplying by δ^i_k we
obtain

$$(A^k_\varepsilon \delta^i_k)_\varepsilon = (A^i_\varepsilon)_\varepsilon, \quad (1.7.10)$$

i.e. again Colombeau generalized four-vector; this proves that δ^i_k is a tensor.

Remark 1.7.3. The square of the Colombeau generalized line element $(ds^2_\varepsilon)_\varepsilon$ in curvilinear
coordinates is a quadratic form in the differentials $dx^i, i = 0,1,2,3$:

$$(ds^2_\varepsilon)_\varepsilon = (g_{ik,\varepsilon} dx^i dx^k)_\varepsilon = [(g_{ik,\varepsilon})_\varepsilon] dx^i dx^k. \quad (1.7.11)$$

where the $(g_{ik,\varepsilon})_\varepsilon$ are Colombeau generalized functions of the coordinates; $(g_{ik,\varepsilon})_\varepsilon$ is symmetric in the indices i and k :

$$(g_{ik,\varepsilon})_\varepsilon = (g_{ki,\varepsilon})_\varepsilon. \tag{1.7.12}$$

Definition 1.7.4. Since the (contracted) product of $(g_{ik,\varepsilon})_\varepsilon$ and the contravariant tensor $dx^i dx^k$ is a scalar, the $(g_{ik,\varepsilon})_\varepsilon$ form a covariant tensor; it is called the Colombeau generalized metric tensor.

Definition 1.7.5. Two tensors $(A_{ik,\varepsilon})_\varepsilon$ and $(B^{ik}_\varepsilon)_\varepsilon$ are said to be reciprocal to each other if

$$(A_{ik,\varepsilon}B^{ik}_\varepsilon)_\varepsilon = [(A_{ik,\varepsilon})_\varepsilon]\left[(B^{ik}_\varepsilon)_\varepsilon\right] = \delta^i_k. \tag{1.7.13}$$

In particular the contravariant metric tensor is the tensor $(g_{ik,\varepsilon})_\varepsilon$ reciprocal to the tensor $(g^{ik}_\varepsilon)_\varepsilon$, that is,

$$\{(g_{ik,\varepsilon})_\varepsilon\}\left\{(g^{ik}_\varepsilon)_\varepsilon\right\} = \delta^i_k. \tag{1.7.14}$$

The same physical quantity can be represented in contravariant or covariant components.

It is obvious that the only quantities that can determine the connection between the different forms are the components of the metric tensor. This connection is given by the formulas:

$$(A^i_\varepsilon)_\varepsilon = (g^{ik}_\varepsilon A_{k,\varepsilon})_\varepsilon, (A_{i,\varepsilon})_\varepsilon = (g_{ik,\varepsilon}A^k_\varepsilon)_\varepsilon. \tag{1.7.15}$$

These remarks also apply to Colombeau generalized tensors. The transition between the different forms of a given physical generalized tensor is accomplished by using the metric tensor according to the formulas:

$$(A^i_{k,\varepsilon})_\varepsilon = (g^{il}_\varepsilon A_{lk,\varepsilon})_\varepsilon, (A^{ik}_\varepsilon)_\varepsilon = (g^{il}_\varepsilon g^{km}_\varepsilon A_{lm,\varepsilon})_\varepsilon, \text{etc.} \tag{1.7.16}$$

The completely antisymmetric unit pseudotensor in galilean coordinates we denote by e^{iklm}. Let us transform it to an arbitrary system of Colombeau generalized coordinates, and now denote it by $(E^{iklm}_\varepsilon)_\varepsilon$. We keep the notation e^{iklm} for the quantities defined as before by $e^{0123} = 1$ (or $e_{0123} = -1$). Let the $x'^i, i = 0,1,2,3$ be galilean, and the $(x^i_\varepsilon)_\varepsilon, i = 0,1,2,3$ be arbitrary Colombeau generalized curvilinear coordinates. According to the general rules for transformation of Colombeau generalized tensors, we have

$$(E^{iklm}_\varepsilon)_\varepsilon = \left[\left(\frac{\partial x^i_\varepsilon}{\partial x'^p}\frac{\partial x^k_\varepsilon}{\partial x'^r}\frac{\partial x^l_\varepsilon}{\partial x'^s}\frac{\partial x^m_\varepsilon}{\partial x'^t}\right)_\varepsilon\right]e^{prst}, \tag{1.7.17}$$

or

$$(E^{iklm}_\varepsilon)_\varepsilon = \{(\mathbf{J}_\varepsilon(x'^0,x'^1,x'^2,x'^3))_\varepsilon\}e^{prst}, \tag{1.7.18}$$

where $(\mathbf{J}_\varepsilon(x'^0,x'^1,x'^2,x'^3))_\varepsilon \neq 0_{\widetilde{\mathbb{R}}}$ is the determinant formed from the derivatives $\partial x^i/\partial x'^p$, i.e. it is just the Colombeau generalized Jacobian of the Colombeau generalized transformation from the galilean to the Colombeau generalized curvilinear coordinates:

$$(\mathbf{J}_\varepsilon(x'^0,x'^1,x'^2,x'^3))_\varepsilon = \left(\frac{\partial(x^0_\varepsilon,x^1_\varepsilon,x^2_\varepsilon,x^3_\varepsilon)}{\partial(x'^0,x'^1,x'^2,x'^3)}\right)_\varepsilon. \tag{1.7.19}$$

This Colombeau generalized Jacobian can be expressed in terms of the determinant of the Colombeau generalized metric tensor $(g_{ik,\varepsilon})_\varepsilon$ (in the system $(x^i_\varepsilon)_\varepsilon$). To do this we write the formula for the transformation of the metric tensor:

$$(g^{ik}_\varepsilon)_\varepsilon = \left[\left(\frac{\partial x^i_\varepsilon}{\partial x'^i}\frac{\partial x^k_\varepsilon}{\partial x'^m}\right)_\varepsilon\right] g^{(0)im}, \tag{1.7.20}$$

where

$$g^{(0)im} = g^{(0)}_{im} = \left\{\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{matrix}\right\}, \tag{1.7.21}$$

and equate the determinants of the two sides of this equation. The determinant of the reciprocal tensor $\det\left|(g^{ik}_\varepsilon)_\varepsilon\right| = 1/(g_\varepsilon)_\varepsilon$. The determinant $\det|g^{(0)im}| = -1$. Thus we have $1/(g_\varepsilon)_\varepsilon = -(\mathbf{J}^2_\varepsilon(x'^0,x'^1,x'^2,x'^3))_\varepsilon$, and so

$$(\mathbf{J}^2_\varepsilon(x'^0,x'^1,x'^2,x'^3))_\varepsilon = 1/\sqrt{(g_\varepsilon)_\varepsilon}. \tag{1.7.22}$$

Thus, in curvilinear coordinates the antisymmetric unit tensor of rank four must bedefined as

$$(E^{iklm}_\varepsilon)_\varepsilon = \frac{1}{\sqrt{-(g_\varepsilon)_\varepsilon}} e^{iklm} \tag{1.7.23}$$

and its covariant components are

$$(E_{iklm,\varepsilon})_\varepsilon = \sqrt{-(g_\varepsilon)_\varepsilon}\, e_{iklm}. \tag{1.7.24}$$

In a galilean coordinate system $x'^i, i = 0,1,2,3$ the integral of a scalar with respect to $d\Omega' = dx'^0 dx'^1 dx'^2 dx'^3$ is also a scalar, i.e. the element $d\Omega'$ behaves like a scalar in the integration. On transforming to Colombeau generalized curvilinear coordinates $(x^i_\varepsilon)_\varepsilon, i = 0,1,2,3,$ the element of integration $d\Omega'$ goes over into

$$d\Omega' := \left\{(\mathbf{J}^{-1}_\varepsilon)_\varepsilon\right\} d\Omega = \sqrt{-(g_\varepsilon)_\varepsilon}\,(d\Omega_\varepsilon)_\varepsilon, \tag{1.7.25}$$

where $(d\Omega_\varepsilon)_\varepsilon = \{(dx^0_\varepsilon)_\varepsilon\}\{(dx^1_\varepsilon)_\varepsilon\}\{(dx^2_\varepsilon)_\varepsilon\}\{(dx^3_\varepsilon)_\varepsilon\}$.

Thus, in Colombeau generalized curvilinear coordinates, when integrating over a four-volume the quantity $\sqrt{-(g_\varepsilon)_\varepsilon}\,(d\Omega_\varepsilon)_\varepsilon$ behaves like an invariant.

Remark 1.7.4. The element of "area" of the Colombeau generalized hypersurface spanned
by three infinitesimal Colombeau generalized displacements is the contravariant antisymmetric Colombeau generalized tensor $(dS^{ikl}_\varepsilon)_\varepsilon$; the vector dual to it is gotten by multiplying by the tensor $\sqrt{-(g_\varepsilon)_\varepsilon}\, e_{iklm}$, so it is equal to

$$\sqrt{-(g_\varepsilon)_\varepsilon}\,(dS_{\varepsilon,i})_\varepsilon = -\tfrac{1}{6}\sqrt{-(g_\varepsilon)_\varepsilon}\, e_{iklm}(dS^{kim}_\varepsilon)_\varepsilon. \tag{1.7.26}$$

Remark 1.7.5. Let $(df^{ik}_\varepsilon)_\varepsilon$be the element of two-dimensional Colombeau generalized surface spanned by two infinitesimal Colombeau generalized displacements, the dual Colombeau generalized tensor is defined as

$$\sqrt{-(g_\varepsilon)_\varepsilon}\left(df^*_{ik,\varepsilon}\right)_\varepsilon = \frac{1}{2}\sqrt{-(g_\varepsilon)_\varepsilon}\, e_{iklm}\left(df^{lm}_\varepsilon\right)_\varepsilon. \qquad (1.7.27)$$

We will be use the designations $(dS_{\varepsilon,i})$ and $\left(df^*_{ki,\varepsilon}\right)_\varepsilon$ for $e_{iklm}(dS^{kim}_\varepsilon)_\varepsilon$ and $e_{iklm}\left(df^{lm}_\varepsilon\right)_\varepsilon$ (and not for their products by $\sqrt{-(g_\varepsilon)_\varepsilon}$).

1.7.2.Generalized Stokes' theorem.

Remark 1.7.6. Note that the canonical rules for transforming the various integrals into one another remain the same, since their derivation was formal in character and not related to the tensor properties of the different quantities. Of particular importance is the rule for transforming the integral over a hypersurface into an integral over a four-volume (Gauss' theorem), which is accomplished by the substitution

$$(dS_{i,\varepsilon})_\varepsilon := [(d\Omega_\varepsilon)_\varepsilon]\left(\frac{\partial}{\partial x^i_\varepsilon}\right)_\varepsilon. \qquad (1.7.28)$$

Remark 1.7.7. (Generalized Stokes' theorem) Note that for the integral of Colombeau generalized vector $(A^i_\varepsilon)_\varepsilon$ we have

$$\left(\oint A^i_\varepsilon dS_{i,\varepsilon}\right)_\varepsilon = \left(\int \frac{\partial A^i_\varepsilon}{\partial x^i_\varepsilon} d\Omega_\varepsilon\right)_\varepsilon = \int\left[\left(\frac{\partial A^i_\varepsilon}{\partial x^i_\varepsilon}\right)_\varepsilon\right][(d\Omega_\varepsilon)_\varepsilon]. \qquad (1.7.29)$$

which is the generalization of Stokes' theorem.

Note that in galilean coordinates the Colombeau generalized differentials $(dA_{i,\varepsilon})_\varepsilon$ of a vector $(A_{i,\varepsilon})_\varepsilon$ form the Colombeau generalized vector, and the derivatives $(\partial A_{i,\varepsilon}/\partial x^k_\varepsilon)_\varepsilon$ of the components of a vector with respect to the coordinates form the Colombeau generalized tensor. In Colombeau generalized curvilinear coordinates this is not so; $(dA_{i,\varepsilon})_\varepsilon$ is not a vector, and $(\partial A_{i,\varepsilon}/\partial x^k_\varepsilon)_\varepsilon$ is not the Colombeau generalized tensor.This is due to the fact that $(dA_{i,\varepsilon})_\varepsilon$ is the difference of vectors located at different (infinitesimally separated) points of space; at different points in space vectors transform differently, since the coefficients in the transformation formulas (1.7.3), (1.7.4) are Colombeau generalized functions of the generalized coordinates.Thus in order to compare two infmitesimally separated generalized vectors we must subject one of them to a parallel translation to the point where the second is located. Let us consider an arbitrary generalized contravariant vector ; if its value at the point x^i is $(A^i_\varepsilon)_\varepsilon$, then at the neighboring point $x^i + dx^i$ it is equal to $(A^i_\varepsilon)_\varepsilon + (dA^i_\varepsilon)_\varepsilon = (A^i_\varepsilon + dA^i_\varepsilon)_\varepsilon$. We subject the vector $(A^i_\varepsilon)_\varepsilon$ to an infinitesimal parallel displacement to the point $x^i + dx^i$; the change in the vector which results from this we denote by $(\delta A^i_\varepsilon)_\varepsilon$ Then the difference $(DA^i_\varepsilon)_\varepsilon$ between the two Colombeau generalized vectors which are now located at the same point is

$$(DA^i_\varepsilon)_\varepsilon = (dA^i_\varepsilon)_\varepsilon - (\delta A^i_\varepsilon)_\varepsilon. \qquad (1.7.30)$$

The change $(\delta A^i_\varepsilon)_\varepsilon$in the components of Colombeau generalized vector under an infinitesimal parallel displacement depends on the values of the components themselves, where the dependence must clearly be linear. This follows directly from the fact that the sum of two Colombeau generalized vectors must transform according to the same law as each of the constituents. Thus $(\delta A^i_\varepsilon)_\varepsilon$ has the form

$$(\delta A^i_\varepsilon)_\varepsilon = -(\Gamma^i_{kl,\varepsilon} A^k_\varepsilon dx^l)_\varepsilon, \qquad (1.7.31)$$

where $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ the are certain Colombeau generalized functions of the coordinates. Their form depends, of course, on the coordinate system; for a galilean coordinate system $(\Gamma^i_{kl,\varepsilon})_\varepsilon = 0_{\widetilde{\mathbb{R}}}$. From this it is already clear that the quantities $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ do not form Colombeau generalized tensor, since a tensor which is equal to zero in one coordinate system is equal to zero in every other one. In a curvilinear space it is, of course, impossible to make all the $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ vanish over all of space. But we can choose a coordinate system for which the $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ become $0_{\widetilde{\mathbb{R}}}$ over a given infinitesimal region. The quantities $(\Gamma^i_{kl,\varepsilon})_\varepsilon$, are called generalized Christoffel symbols. In addition to the quantities $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ we shall later also use Colombeau generalized quantities $(\Gamma_{i,kl,\varepsilon})_\varepsilon$ defined as follows

$$(\Gamma_{i,kl,\varepsilon})_\varepsilon = (g_{im,\varepsilon} \Gamma^m_{km,\varepsilon})_\varepsilon. \qquad (1.7.32)$$

Conversely,

$$(\Gamma^i_{kl,\varepsilon})_\varepsilon = (g^{im}_\varepsilon \Gamma_{m,kl,\varepsilon})_\varepsilon. \qquad (1.7.33)$$

It is also easy to relate the change in the components of a covariant vector under a parallel displacement to the Christoffel symbols. To do this we note that under a parallel displacement, a scalar is unchanged. In particular, the scalar product of two vectors does not change under a parallel displacement.Let $(A_{i,\varepsilon})_\varepsilon$ and $(B^i_\varepsilon)_\varepsilon$ be any covariant and contravariant vectors. Then from $\delta(A_{i,\varepsilon} B^i_\varepsilon)_\varepsilon = 0_{\widetilde{\mathbb{R}}}$, we have

$$(B^i_\varepsilon \delta A_{i,\varepsilon}) = -(A_{i,\varepsilon} \delta B^i_\varepsilon)_\varepsilon = (\Gamma^i_{kl,\varepsilon} B^k_\varepsilon A_{i,\varepsilon} dx^l)_\varepsilon \qquad (1.7.34)$$

or, changing the indices,

$$(B^i_\varepsilon \delta A_{i,\varepsilon}) = (\Gamma^k_{il,\varepsilon} B^i_\varepsilon A_{k,\varepsilon} dx^l)_\varepsilon \qquad (1.7.35)$$

From this, by the arbitrariness of the $(B^i_\varepsilon)_\varepsilon$ one obtains

$$(\delta A_{i,\varepsilon})_\varepsilon = \left((\Gamma^k_{il,\varepsilon} A_{k,\varepsilon})_\varepsilon\right) dx^l \qquad (1.7.36)$$

which determines the change in a covariant vector under a parallel displacement. Substituting (1.7.31) and $(dA^i_\varepsilon)_\varepsilon = ((\partial A^i_\varepsilon / \partial x^l)_\varepsilon) dx^l$ in (1.7.30), we obtain

$$(DA^i_\varepsilon)_\varepsilon = \left[\left(\frac{\partial A^i_\varepsilon}{\partial x^l} \right)_\varepsilon + (\Gamma^i_{kl,\varepsilon} A^k_\varepsilon)_\varepsilon \right] dx^l. \qquad (1.7.37)$$

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1.8.The Colombeau Generalized Curvature Tensor.

Remark 1.8.1. (i) Note that the notion of the classical Rimannian curvature comes from

the study of parallel transport on a classical Rimannian manifold (M,g),see Fig.1.8.1. For

instance, if a vector is moved around a loop Γ on the surface $\Sigma_\Gamma \subset M$ of a sphere keeping

parallel throughout the motion,then the final position of the vector may not be the same as
the initial position of the vector. This phenomenon is known as holonomy.
(ii) Various generalizations capture in an abstract form this idea of curvature as a measure of holonomy.

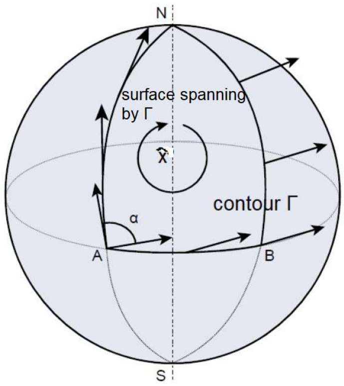

Fig.1.8.1.Parallel transporting a vector from $A \to N \to B \to A$ yields a different vector. This failure to return to the initial vector is measured by the classical holonomy of the surface Σ_Γ spanning by Γ.

Remark 1.8.2. Let (M,g) be a semirimannian manifold.Let $\Gamma_{\hat{x}^0}$ be infinitesimal closed contour and let $\Sigma_\Gamma \subset M$ be the corresponding surface spanning by Γ, see Fig.1.8.1. We assume now that christoffel symbols $\Gamma^i_{kl}(\hat{x})$ are smooth on $\Sigma_\Gamma \cup \Gamma$. The classical formula for the change in a smooth vector $A_i(\hat{x})$ after parallel displacement around infinitesimal closed contour Γ reads [4]:

$$\Delta A_k(\Gamma) = \oint_\Gamma \delta A_k = \oint_\Gamma \Gamma^i_{kl}(\hat{x}) A_k dx^l. \qquad (1.8.1)$$

We remind now the classical Stokes' theorem.

Theorem.1.8.1.(Stokes' theorem) If ω is a **smooth** $(n-1)$-form with compact support on smooth n-dimensional manifold Ω with-boundary $\partial\Omega$ of Ω given the induced orientation, and $i : \partial\Omega \hookrightarrow \Omega$ is the inclusion map, then

$$\int_\Omega d\omega = \int_{\partial\Omega} i^*\omega. \qquad (1.8.2)$$

Remark 1.8.2. Conventionally, $\int_{\partial\Omega} i^*\omega$ is abbreviated as $\int_{\partial\Omega} \omega$, since the pullback of a differential form by the inclusion map is simply its restriction to its domain $i^*\omega = \omega|\partial\Omega$

Here d is the exterior derivative, which is defined using the manifold structure only. The right-hand side is sometimes written as $\oint_{\partial\Omega} \omega$ to stress the fact that the $(n-1)$-manifold $\partial\Omega$ has no boundary.

For the further transformation of the integral (1.8.1), we must note the following. The values of the vector A_i at points inside the contour are not unique; they depend on the path along which we approach the particular point. However, as we shall see from the result obtained below, this non-uniqueness is related to terms of second order. We may therefore, with the first-order accuracy which is sufficient for the transformation, regard the components of the vector A_i at points inside the infinitesimal contour Γ as being uniquely determined by their values on the contour itself by the formulas

$$\delta A_i(x) = \Gamma^n_{il}(x) A_n(x) dx^l, \tag{1.8.3}$$

i.e.,by the derivatives

$$\frac{\partial A_i(x)}{\partial x^l} = (\Gamma^n_{il}(x) A_n(x)). \tag{1.8.4}$$

Now applying classical Stokes' theorem (see Theorem 1.8.1) to the integral (1.8.1) and considering that the area enclosed by the contour has the infinitesimal value (Δf^{lm}), we get:

$$\begin{aligned}\Delta A_k &= \frac{1}{2}\left[\frac{\partial(\Gamma^i_{km}(x)A_i(x))}{\partial x^l} - \frac{\partial(\Gamma^i_{kl}(x)A_i(x))}{\partial x^m}\right]\Delta f^{lm} = \\ &= \frac{1}{2}\left[A_i(x)\left(\frac{\partial(\Gamma^i_{km}(x))}{\partial x^l}\right)_\varepsilon - A_i(x)\frac{\partial(\Gamma^i_{kl}(x))}{\partial x^m} + \right. \\ &\left.\left(\frac{\partial A_i(x)}{\partial x^l}\right)\Gamma^i_{km}(x) - \left(\frac{\partial A_i(x)}{\partial x^m}\right)\Gamma^i_{kl}(x)\right]\Delta f^{lm}.\end{aligned} \tag{1.8.5}$$

Remark 1.8.3. Note that:(i) the regularity condition in Stokes' theorem,i.e. ω is a **smooth** $(n-1)$-form, essentially important and without this condition the **Stokes' theorem** *is no longer holds*. However in physical literature the regularity condition usually missing in formulation of the Stokes' theorem, see for example [4].
(ii) Obviously without Stokes' theorem impossible to derive the Eq.(1.8.5) and therefore
the expression in the RHS of the Eq.(1.8.5) wihout the regularity condition of the functions
$\Gamma^n_{km}(x)$, does not make any rigorous mathematical sense, i.e. the Eq.(1.8.5) becomes to
absurdum.
Substituting now the values of the derivatives (1.8.4) into Eq.(1.8.5), we get

$$\Delta A_k = \frac{1}{2} R^i_{klm}(x) A_i(x) \Delta f^{lm}, \tag{1.8.6}$$

where $R^i_{klm}(x)$ is a tensor field of the fourth rank:

$$R^i_{klm}(x) = \frac{\partial(\Gamma^i_{km}(x))}{\partial x^l} - \frac{\partial(\Gamma^i_{kl}(x))}{\partial x^m} + \Gamma^i_{nl}(x)\Gamma^n_{km}(x) - \Gamma^i_{nm}(x)\Gamma^n_{kl}(x). \tag{1.8.7}$$

Definition 1.8.1. The tensor field $R^l_{kim}(x)$ is called the classical curvature tensor or the classical Riemann tensor.
The classical Riemann tensor that is a tensorial measure of the classical holonomy.

Definition 1.8.2. Let (M,g) be a nonclassical semirimannian manifold,i.e.the manifold endowed on the tangent bundle with a symmetric bilinear form which is allowed to become degenerate (singular).Let $\Sigma_\Gamma \subset M$ be the surface spanning by Γ and let $\hat{x}^0$ be a

point such that $\hat{x}^0 \in \Sigma_\Gamma, \hat{x}^0 \notin \Gamma$. Assume that the classical Levi-Civitá connection is available on $\Sigma_\Gamma \backslash \{\hat{x}^0\}$.

(i) We will say that $\hat{x}^0$ is a singular point if the classical Levi-Civitá connection is not available on $\{\hat{x}^0\}$, see Fig.1.8.2.

(ii) We will say that a surface Σ_Γ is a singular surface if Σ_Γ contains at least one singular

point, see Fig.1.8.2.

(iii) We will say that the surface Σ_Γ admit the classical tensorial measure of holonomy (or admit the classical Riemann tensor) iff the Eq.(1.8.6) and Eq.(1.8.7) holds.

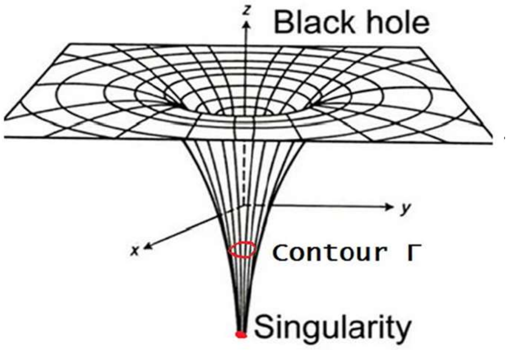

Fig.1.8.2.Infinitesimal closed contour Γ

and corresponding singular surface $\Sigma_\Gamma \ni \hat{x}^0$

spanning by Γ.

Due to the degeneracy of the metric (1.10.12)

at $r = 2m, r = 0$,

the classical Levi-Civitá connection $\Gamma^{+l}_{kj}(\{\}) =$

$$= \frac{1}{2}[g^{lm}(\{\})][(g_{mk,j}(\{\}) + g_{mj,k}(\{\}) - g_{kj,m}(\{\})]$$

is available only on $\mathbb{R}^3_+ \backslash \{r = 2m, r = 0\}$.

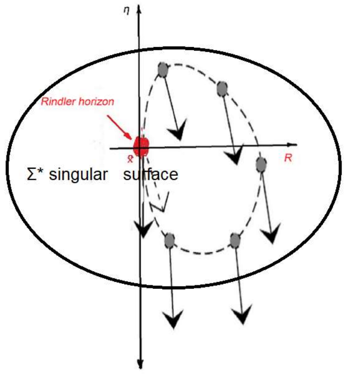

Fig.1.8.3.Infinitesimal closed contour Γ
and corresponding singular surface $\Sigma^{*}=\Sigma_{\Gamma} \ni \hat{x}^{0}$
with singular point $\hat{x}^{0}$ (Rindler horizon)
in Rindler space-time:
$ds^{2} = -\alpha R^{2} d\eta^{2} + dR^{2}$.

Remark 1.8.4. Obviously the surface Σ_{Γ} admit the classical tensorial measure of holonomy if there is no singular points $\hat{x}^{0} \in \Sigma_{\Gamma}$. Thus in order to avoid difficultness which arises from singular points one needs to replaces the definition of holonomy (1.8.1) by the definition aproporiate for the case of the singular surfaces

Remark 1.8.5. Let M be a separable, smooth orientable Hausdorff manifold of dimension n endoved with Colombeau generalized metric tensor $((g_{i,j,\varepsilon}(\hat{x}))_{ij})_{\varepsilon} \in \mathcal{G}_{0}^{2}(M)$ whose determinant $(\det(g_{ij,\varepsilon}))_{\varepsilon}$ is invertible in $\mathcal{G}(M)$, see subsection 2.1 and [15],[16],[17],[35].

Remark 1.8.6.(i) Let $(M, g_{i,j,0}(\hat{x}))$ be a semirimannian manifold endoved with $\mathbb{R}$-valued metric tensor defined by the formula: $g_{i,j,0}(\hat{x}) = ((g_{i,j,\varepsilon}(\hat{x}))_{ij})_{\varepsilon}\big|_{\varepsilon=0}$.

(ii) We assume now that $(M, g_{i,j,0}(\hat{x}))$ is a nonclassical semirimannian manifold,i.e.the manifold endowed on the tangent bundle with a symmetric bilinear form which is allowed to become degenerate (singular).

Example 1.8.1. For instance the christoffel symbols $\Gamma^{i}_{kl,0}(\hat{x},\hat{x}^{0})$ corresponding to the metric tensor $g_{i,j,0}(\hat{x})$ become infinite at some singular point $\hat{x}^{0}$ by formulae

$$\begin{cases} \Gamma^{i}_{kl,0}(\hat{x},\hat{x}^{0}) \asymp \Xi_{kl}(\hat{x})(x_{i}-x_{i}^{0})^{-\delta}, \delta \geq 1 \\ \Xi_{kl}(\hat{x}) \in C^{\infty}\left(\Sigma_{\Gamma_{\hat{x}^{0}}}\right). \end{cases} \quad (1.8.8)$$

It follows from Eq.(1.8.8) the Levi-Civitá connection is not available at point $\hat{x}^{0}$.

Let $\Gamma_{\hat{x}^{0}} \ni \hat{x}^{0}$ be infinitesimal closed contour and let $\Sigma_{\Gamma_{\hat{x}^{0}}} \subset M$ be the corresponding surface spanning by $\Gamma_{\hat{x}^{0}}$, see Fig.1.8.4.

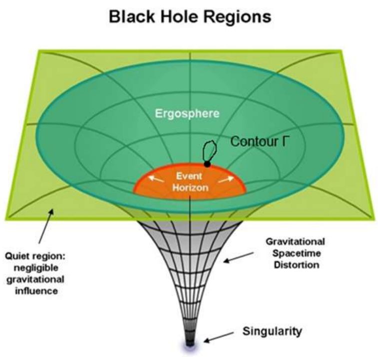

Fig.1.8.4. Singular point at BH horizon
$\{r = 2m\}$ and corresponding singular countur.

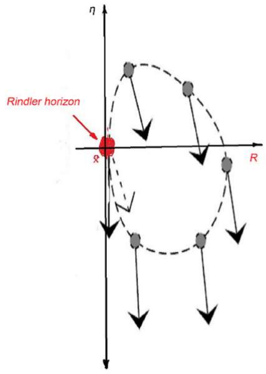

Fig.1.8.4. Singular point at Rindler horizon
$ds^2 = -\alpha R^2 d\eta^2 + dR^2.$

Remark 1.8.7. The classical formula (1.8.1) for the change in a smooth vector $A_i(\hat{x})$ after
parallel displacement around infinitesimal closed contour $\Gamma_{\hat{x}^0}$ (see Fig.1.8.2) reads:

$$\Delta A_k(\Gamma_{\hat{x}^0}) = \oint_{\Gamma_{\hat{x}^0}} \delta A_k = \oint_{\Gamma_{\hat{x}^0}} \Gamma^i_{kl}(\hat{x},\hat{x}^0)A_k dx^l. \quad (1.8.9)$$

Obviously the differential form $\Gamma^i_{kl}(\hat{x})A_k dx^l$ is not locally integrable in neighborhood of the point $\hat{x}^0 \in \Sigma_\Gamma$ and therefore $\Delta A_k(\Gamma_{\hat{x}^0}) = \infty$.

Remark 1.8.8. In order to avoid these difficultness with divergence $\Delta A_k(\Gamma_{\hat{x}^0}) = \infty$, etc. we
consider the canonical imbedding $(M, g_{i,j,0}(\hat{x})) \hookrightarrow (M, (g_{i,j,\varepsilon}(\hat{x}))_\varepsilon)$, and we extend now the
classical formula (1.8.1) from a nonclassical semirimannian manifold $(M, g_{i,j,0}(\hat{x}))$ up to Colombeau manifold $(M, (\det(g_{ij,\varepsilon}))_\varepsilon)$ in natural way and obtain the formula for the Colombeau generalized change in a vector after parallel displacement around any infinitesimal closed contour Γ. This generalized change $(\Delta A_{k,\varepsilon})_\varepsilon \in \widetilde{\mathbb{R}}$ can clearly be

written
in the following form

$$(\Delta A_{k,\varepsilon})_\varepsilon = \left(\oint_{\Gamma_{\hat{x}^0}} \delta A_{k,\varepsilon}\right)_\varepsilon, \qquad (1.8.10)$$

where the Colombeau integral is taken over the given contour $\Gamma_{\hat{x}^0}$.

Definition 1.8.3.(i) Let $(M,(g_\varepsilon)_\varepsilon)$ be the Colombeau generalized semirimannian manifold,
and let $\Gamma_{\hat{x}^0}$ be infinitesimal closed contour such that $\hat{x}^0 \in \Gamma_{\hat{x}^0} \subset M$. Let $\Theta(\hat{x}^0)$ be a closed
infinitesimal neighborhood of $\hat{x}^0$, then we we abbreviate $\delta_{\hat{x}^0} \triangleq \Gamma_{\hat{x}^0} \cap \Theta(\hat{x}^0)$. We will be say
that a point $\hat{x}^0 \in \Gamma_{\hat{x}^0}$ is a singular pont of the Colombeau generalized manifold $(M,(g_\varepsilon)_\varepsilon)$ if
$(\Delta A_{k,\varepsilon}(\Gamma_{\hat{x}^0}\backslash\{\hat{x}^0\}))_\varepsilon \in \widetilde{\mathbb{R}}_{\mathbf{inf}} = \widetilde{\mathbb{R}}\backslash\widetilde{\mathbb{R}}_{\mathbf{fin}}$ and $\Delta A_k(\Gamma_{\hat{x}^0}\backslash\delta_{\hat{x}^0}) \in \mathbb{R}$, i.e. the quantity $(\Delta A_{k,\varepsilon}(\Gamma_{\hat{x}^0}\backslash\{\hat{x}^0\}))_\varepsilon$
is infinite large Colombeau generalized number and the quantity $\Delta A_k(\Gamma_{\hat{x}^0}\backslash\delta_{\hat{x}^0})$ is finite.
(ii) We will be say that a closed contour is a singular contour $\Gamma_{\hat{x}^0}$ if it contains at least one
singular pont $\hat{x}^0 \in \Gamma_{\hat{x}^0}$, see Fig.1.8.2.-Fig.1.8.3.

Definition 1.8.4.(i) We will be say that a semirimannian manifold is a singular manifold
if there exists at least one singular (isolated) pont $\hat{x}^0 \in M$.
Substituting now in place of $(\delta A_{k,\varepsilon})_\varepsilon$the expression (1.7.36), we get

$$(\Delta A_{k,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon = \left(\oint_\Gamma \Gamma^i_{kl,\varepsilon}(\hat{x},\hat{x}^0)A_i(\hat{x})dx^l\right)_\varepsilon \in \widetilde{\mathbb{R}}, \qquad (1.8.11)$$

where for any $i,k,l = 0,1,2,3 : (\Gamma^i_{kl,\varepsilon}(\hat{x},,\hat{x}^0))_\varepsilon \in \mathcal{G}(\mathbb{R}^4), \hat{x} = x = (x^0,x^1,x^2,x^3),\ A_i(x) \in \mathcal{D}(G)$ and where $\hat{x}^0 \notin \Gamma,\ \hat{x}^0 \in \Sigma_\Gamma \subset G \subset \mathbb{R}^4$. Note that the vector A_i which appears in the integrand obviously changes as we move along the contour Γ.

Definition 1.8.5. We will be say that generalized change $(\Delta A_{k,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon$ exists in the sense
of the Schwartz distributions with compact support if for any $A_i(\hat{x}) \in \mathcal{D}(G)$ the limit:
$\lim_{\varepsilon\to 0}\Delta A_{k,\varepsilon}(\hat{x},\hat{x}^0)$ exists in $\mathcal{D}'(G)$, i.e. for any $g(\hat{x}) \in \mathcal{D}(G)$, where $\hat{x} \in G$ the following limit
exists

$$\lim_{\varepsilon\to 0}\Delta A_{k,\varepsilon}(\hat{x},\hat{x}^0) = \lim_{\varepsilon\to 0} d^4y\left(\oint_\Gamma \Gamma^i_{kl,\varepsilon}(\hat{x},\hat{x}^0)A_i(\hat{x})g(\hat{x})dx^l\right) \qquad (1.8.12)$$

Of course in this case obviously $\mathbf{cl}\left[(\Gamma^i_{kl,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon\right] \in \iota(\mathcal{D}'(G)) \cap \mathcal{G}(\mathbb{R}^4)$ where ι is an imbedding the Schwartz distributions $\mathcal{D}'(G)$ into the full Colombeau algebra $\mathcal{G}(\mathbb{R}^4)$:
$\iota : \mathcal{D}'(G) \hookrightarrow \mathcal{G}(\mathbb{R}^4)$.

For the further transformation of this Colombeau integral (1.8.11), we must note the following. The values of the vector A_i at points inside the contour are not unique; they depend on the path along which we approach the particular point. However, as we shall see from the result obtained below, this non-uniqueness is related to terms of second order. We may therefore, with the first-order accuracy which is sufficient for the transformation, regard the components of the vector $(A_{i,\varepsilon})_\varepsilon$ at points inside the infinitesimal contour Γ as being uniquely determined by their values on the contour itself by the formulas

$$(\delta A_{i,\varepsilon}(\hat{x}))_\varepsilon = (\Gamma^n_{il,\varepsilon}(\hat{x},\hat{x}^0)A_{n,\varepsilon}(\hat{x})dx^l)_\varepsilon, \tag{1.8.13}$$

i.e., by the Colombeau derivatives

$$\left(\frac{\partial A_{i,\varepsilon}(\hat{x})}{\partial x^l}\right)_\varepsilon = (\Gamma^n_{il,\varepsilon}(\hat{x},\hat{x}^0)A_{n,\varepsilon}(\hat{x}))_\varepsilon. \tag{1.8.14}$$

Now applying generalized Stokes' theorem (see Theorem 1.8.2 below) to the Colombeau integral (1.8.11) and considering that the area enclosed by the contour has the infinitesimal value $(\Delta f^{im}_\varepsilon)_\varepsilon$, we get:

$$\begin{gathered}(\Delta A_{k,\varepsilon})_\varepsilon = \\ \frac{1}{2}\left[\left(\frac{\partial(\Gamma^i_{km,\varepsilon}(\hat{x},\hat{x}^0)A_i(\hat{x}))}{\partial x^l}\right)_\varepsilon - \left(\frac{\partial(\Gamma^i_{kl,\varepsilon}(\hat{x},\hat{x}^0)A_i(\hat{x}))}{\partial x^m}\right)_\varepsilon\right](\Delta f^{im}_\varepsilon)_\varepsilon \\ = \frac{1}{2}\left[A_i(\hat{x})\left(\frac{\partial(\Gamma^i_{km,\varepsilon}(\hat{x},\hat{x}^0))}{\partial x^l}\right)_\varepsilon - A_i(\hat{x})\left(\frac{\partial(\Gamma^i_{kl,\varepsilon}(\hat{x},\hat{x}^0))}{\partial x^m}\right)_\varepsilon + \right. \\ \left.\left(\frac{\partial A_i(\hat{x})}{\partial x^l}\right)(\Gamma^i_{km,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon - \left(\frac{\partial A_i(\hat{x})}{\partial x^m}\right)(\Gamma^i_{kl,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon\right](\Delta f^{im}_\varepsilon)_\varepsilon.\end{gathered} \tag{1.8.15}$$

Definition 1.8.6. Colombeau generalized k-form $(\omega_\varepsilon)_\varepsilon$ on a differentiable manifold M is a

smooth section of the bundle of alternating Colombeau generalized k-tensors on M. Equivalently, $(\omega_\varepsilon)_\varepsilon$ associates to each $x \in M$ an alternating Colombeau generalized k-tensor $(\omega_{x,\varepsilon})_\varepsilon$, in such a way that in any chart for M, the coefficients$(\omega_{i_1\ldots i_k,\varepsilon})_\varepsilon$ are Colombeau generalized functions.

Theorem 1.8.2.(Generalized Stokes' Theorem) Let $(\omega_\varepsilon)_\varepsilon$ be Colombeau generalized differential form.Then the Colombeau integral of a differential form $(\omega_\varepsilon)_\varepsilon$ over the boundary of some orientable manifold $\Sigma \subset M$ is equal to the integral of its exterior Colombeau derivative $(d\omega_\varepsilon)_\varepsilon$ over the whole of Σ, i.e.,

$$\int_{\partial\Sigma}(\omega_\varepsilon)_\varepsilon = \left(\int_{\partial\Sigma}\omega_\varepsilon\right)_\varepsilon = \left(\int_\Sigma d\omega_\varepsilon\right)_\varepsilon = \int_\Sigma(d\omega_\varepsilon)_\varepsilon. \tag{1.8.16}$$

Proof. Immediately from the classical Stokes' Theorem and definitions.

Example 1.8.2. For example, for the integral of Colombeau generalized vector $(A_{i,\varepsilon}(x))_\varepsilon$

we have

$$\begin{gathered}\left(\oint_\Gamma A_{i,\varepsilon}dx^i\right)_\varepsilon = \left(\int_\Sigma df^{ki}\frac{\partial A_{i,\varepsilon}}{\partial x^k}\right)_\varepsilon = \frac{1}{2}\left(\int\left[\left(df^{ki}_\varepsilon\right)_\varepsilon\right]\left(\frac{\partial A_{k,\varepsilon}}{\partial x^i} - \frac{\partial A_{i,\varepsilon}}{\partial x^k}\right)\right)_\varepsilon = \\ \frac{1}{2}\int\left[\left(df^{ki}_\varepsilon\right)_\varepsilon\right]\left(\frac{\partial A_{k,\varepsilon}}{\partial x^i} - \frac{\partial A_{i,\varepsilon}}{\partial x^k}\right)_\varepsilon = \frac{1}{2}\int\left[\left(df^{ki}_\varepsilon\right)_\varepsilon\right]\left[\left(\frac{\partial A_{k,\varepsilon}}{\partial x^i}\right)_\varepsilon - \left(\frac{\partial A_{i,\varepsilon}}{\partial x^k}\right)_\varepsilon\right],\end{gathered} \tag{1.8.17}$$

where $\Gamma = \partial\Sigma$ and $\left(df^{ki}_\varepsilon\right)_\varepsilon = (dx^i_\varepsilon dx'^k_\varepsilon)_\varepsilon - (dx^k_\varepsilon dx'^i_\varepsilon)_\varepsilon$ is the infinitesimal element of surface which is given by the antisymmetric Colombeau generalized tensor of second rank $\left(df_\varepsilon^{ki}\right)_\varepsilon$.

Substituting the values of the derivatives (1.4.3) into Eq.(1.4.4), we get

$$(\Delta A_{k,\varepsilon})_\varepsilon = \frac{1}{2}(R^i_{klm,\varepsilon}(x)A_{i,\varepsilon}(x)\Delta f_\varepsilon^{lm})_\varepsilon, \tag{1.8.18}$$

where $(R^i_{klm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon$ is a Colombeau generalized tensor field of the fourth rank:

$$(R^i_{klm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon = \left(\frac{\partial(\Gamma^i_{km,\varepsilon}(\hat{x},\hat{x}^0))}{\partial x^l}\right)_\varepsilon - \left(\frac{\partial(\Gamma^i_{kl,\varepsilon}(\hat{x},\hat{x}^0))}{\partial x^m}\right)_\varepsilon + \\ (\Gamma^i_{ni,\varepsilon}(\hat{x},\hat{x}^0)\Gamma^n_{km,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon - (\Gamma^i_{nm,\varepsilon}(\hat{x},\hat{x}^0)\Gamma^n_{kl,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon. \tag{1.8.19}$$

Definition 1.8.7. The tensor field $(R^l_{kim,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon$ is called the distributional curvature tensor

or the distributional Riemann tensor.

Remark 1.8.9. Note that in general case for any $i,k,l = 0,1,2,3$:
$\mathbf{cl}\left[(R^i_{klm,\varepsilon}(x))_\varepsilon\right] \in \mathcal{G}(\mathbb{R}^4)$.

Definition 1.8.8. We will say that the distributional Riemann tensor $(R^i_{klm,\varepsilon}(x,\hat{x}^0))_\varepsilon$exists in

the sense of the Schwartz distributions if for any $i,k,l = 0,1,2,3$ and for any $A_i(x) \in \mathcal{D}(G)$

the limit exists

$$\lim_{\varepsilon\to 0}\int_G R^i_{klm,\varepsilon}(\hat{x},\hat{x}^0)A_i(x)d^4x. \tag{1.8.20}$$

Definition 1.8.9. We will say that the distributional Riemann tensor $(R^i_{klm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon$exists in

the classical sense at point $\breve{x} \in \mathbb{R}^4$ if there exists standard part of point value of Colombeau generalized function $(R^i_{klm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon$ at point $\breve{x} \in \mathbb{R}^4$, i.e.
$\mathbf{st}\left(\mathbf{cl}\left[(R^i_{klm,\varepsilon}(\breve{x},\hat{x}^0))_\varepsilon\right]\right) \in \mathbb{R}$.

From the expression (1.8.19) it follows directly that $\forall x \in \mathbb{R}$ the distributional curvature tensor is antisymmetric in the indices l and m :

$$(R^i_{klm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} -(R^i_{kml,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon \tag{1.8.21}$$

and therefore for any Colombeau generalized vector $(\hat{x}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}^4_{\mathbf{fin}}$ the following identity holds

$$(R^i_{klm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} -(R^i_{kml,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon. \tag{1.8.22}$$

Obviously the following identity holds

$$(R^l_{kim,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon + (R^i_{mkl,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon + (R^i_{lmk,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} 0_{\widetilde{\mathbb{R}}} \tag{1.8.23}$$

and therefore for any Colombeau generalized vector $(\hat{x}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}^4_{\mathbf{fin}}$ the following identity holds

$$(R^l_{kim,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon + (R^i_{mkl,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon + (R^i_{lmk,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} 0_{\widetilde{\mathbb{R}}} \tag{1.8.24}$$

In addition to the mixed distributional curvature tensor $(R^i_{klm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon$, one also uses the covariant distributional curvature tensor

$$(R_{iklm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon = (g_{in,\varepsilon}(\hat{x},\hat{x}^0)R^n_{klm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} ((g_{in,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon)\left((R^n_{klm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon\right) \tag{1.8.25}$$

and therefore for any Colombeau generalized vector $(\hat{x}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}^4_{\mathbf{fin}}$ the following identity holds

$$(R_{iklm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon = (g_{in,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0)R^n_{klm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} ((g_{in,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon)\left((R^n_{klm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon\right) \tag{1.8.26}$$

Obviously by means of simple calculation the following expressions for $(R_{iklm,\varepsilon}((x,\hat{x}^0)))_\varepsilon$ holds

$$\begin{gathered}(R_{iklm,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} \\ \frac{1}{2}\left(\left(\frac{\partial^2 g_{im,\varepsilon}(\hat{x},\hat{x}^0)}{\partial x^k \partial x^l}\right)_\varepsilon + \left(\frac{\partial^2 g_{kl,\varepsilon}(\hat{x},\hat{x}^0)}{\partial x^i \partial x^m}\right)_\varepsilon - \left(\frac{\partial^2 g_{il,\varepsilon}(\hat{x},\hat{x}^0)}{\partial x^k \partial x^m}\right)_\varepsilon - \right. \\ \left. -\left(\frac{\partial^2 g_{km,\varepsilon}(\hat{x},\hat{x}^0)}{\partial x^i \partial x^l}\right)_\varepsilon + \right. \\ +((g_{np,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon)\left[(\Gamma^n_{kl,\varepsilon}(\hat{x},\hat{x}^0)\Gamma^p_{im,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon - (\Gamma^n_{km,\varepsilon}(\hat{x},\hat{x}^0)\Gamma^p_{il,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon\right]\end{gathered} \tag{1.8.27}$$

and therefore for any Colombeau generalized vector $(\hat{x}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}^4_{\mathbf{fin}}$ the following identity holds

$$\begin{gathered}(R_{iklm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} \\ \frac{1}{2}\left(\left(\frac{\partial^2 g_{im,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0)}{\partial x^k \partial x^l}\right)_\varepsilon + \left(\frac{\partial^2 g_{kl,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0)}{\partial x^i \partial x^m}\right)_\varepsilon - \left(\frac{\partial^2 g_{il,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0)}{\partial x^k \partial x^m}\right)_\varepsilon \right. \\ \left. -\left(\frac{\partial^2 g_{km,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0)}{\partial x^i \partial x^l}\right)_\varepsilon \right) + \\ ((g_{np,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon)\left[(\Gamma^n_{kl,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0)\Gamma^p_{im,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon - (\Gamma^n_{km,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0)\Gamma^p_{il,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon\right]\end{gathered} \tag{1.8.28}$$

From this expressions (1.8.28) it follows

$$\begin{gathered}(R_{iklm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} -(R_{kilm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} -(R_{ikml,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon, \\ (R_{iklm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} (R_{lmik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon.\end{gathered} \tag{1.8.28}$$

For $(R_{iklm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon$ and for any Colombeau generalized vector $(\hat{x}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}^4_{\mathbf{fin}}$ the following identities holds

$$(R_{iklm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} (R_{lmik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon. \tag{1.8.29}$$

For $(R_{iklm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon$ the following identities holds

$$(R_{iklm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon + (R_{imkl,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon + (R_{ilmk,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} 0_{\widetilde{\mathbb{R}}} \tag{1.8.30}$$

The generalized Bianchi identity holds

$$(R^n_{ikl;m,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon + (R^n_{imk;l,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon + (R_{ilm;k,\varepsilon}(\hat{x},\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} 0_{\widetilde{\mathbb{R}}} \tag{1.8.31}$$

and for any Colombeau generalized vector $(\hat{x}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}^4_{\mathbf{fin}}$ the following identities holds

$$(R^n_{ikl;m,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon + (R^n_{imk;l,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon + (R_{ilm;k,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon =_{\widetilde{\mathbb{R}}} 0_{\widetilde{\mathbb{R}}}. \tag{1.8.32}$$

From the Colombeau generalized curvature tensor we can, by contraction, construct Colombeau generalized tensor of the second rank. This contraction can be carried out in only one way: contraction of the Colombeau generalized tensor$(R_{iklm,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon$ on the indices i and k or l and m gives zero because of the antisymmetry in these indices, while contraction on any other pair always gives the same result, except for sign. We define the Colombeau generalized tensor $(R_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon$ (the generalized Ricci tensor) as

$$(R_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon = (g_\varepsilon^{lm}R_{ilmk,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon = (R^l_{ilk,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon. \quad (1.8.33)$$

From Eq.(1.8.19) we get

$$(R_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon = \left(\frac{\partial(\Gamma^l_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))}{\partial x^l}\right)_\varepsilon - \left(\frac{\partial(\Gamma^l_{il,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))}{\partial x^m}\right)_\varepsilon + (\Gamma^l_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0)\Gamma^m_{lm,\varepsilon}(x,\hat{x}^0))_\varepsilon - (\Gamma^m_{il,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0)\Gamma^l_{km,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon. \quad (1.8.33)$$

This Colombeau generalized tensor $(R_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon$ is symmetric:

$$(R_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon = (R_{ki,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon. \quad (1.8.34)$$

Contracting $(R_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon$,we obtain the invariant

$$(R(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon = (g_\varepsilon^{ik}(\hat{x}_\varepsilon,\hat{x}^0)R_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon \quad (1.8.35)$$

which is called the Colombeau generalized scalar curvature.Finally, contracting $(R_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon$,we obtain the invariant

$$(R(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon = (g_\varepsilon^{ik}(\hat{x}_\varepsilon,\hat{x}^0)R_{ik,\varepsilon}(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon \quad (1.8.36)$$

which is the point value at point $(\hat{x}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}^4_{\mathbf{fin}}$ of the Colombeau generalized scalar curvature $(R(\hat{x},\hat{x}^0))_\varepsilon$. We remind that the point value at point $(\hat{x}_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}^4_{\mathbf{fin}}$ of the Colombeau generalized scalar curvature $(R(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon$ is $\mathbf{cl}[(R(\hat{x}_\varepsilon,\hat{x}^0))_\varepsilon] \in \widetilde{\mathbb{R}}$.

1.9.Generalized Einstein's field equations

The action functional for the gravitational field reads [37]:

$$\left(\int R_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon. \quad (1.9.1)$$

The invariant Colombeau integral (1.9.1) can be transformed by means of Gauss'theorem to the integral of an expression not containing the second derivatives. Thus Colombeau integral (1.9.1) can be presented in the following form

$$\left(\int R_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon = \left(\int G_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon + \left(\int \frac{\partial\left(\sqrt{-g_\varepsilon}\,w^i_\varepsilon\right)}{\partial x^i}d\Omega\right)_\varepsilon, \quad (1.9.2)$$

where $(G_\varepsilon)_\varepsilon$ contains only the tensor $(g_{ik,\varepsilon})_\varepsilon$ and its first derivatives, and the integrand of the second integral has the form of a divergence of a certain quantity $(w^i_\varepsilon)_\varepsilon$.According to Gauss' theorem, this second integral can be transformed into an integral over a hypersurface surrounding the four-volume over which the integration is carried out in the other two integrals. When we vary the action, the variation of the second term on the right vanishes, since in the principle of least action, the variations of the field at the limits of the region of integration are zero. Consequently, we may write

$$\delta\left(\int R_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon = \left(\delta\int R_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon = \left(\delta\int G_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon. \quad (1.9.3)$$

The left side is Colombeau scalar; therefore the expression on the right is also

Colombeau scalar (the quantity $(G_\varepsilon)_\varepsilon$ itself is, of course, not Colombeau scalar). The quantity $(G_\varepsilon)_\varepsilon$ satisfies the condition imposed above, since it contains only the $(g_{ik,\varepsilon})_\varepsilon$ and its Colombeau derivatives. Thus finally we obtain

$$\delta\mathbf{S}[(g_\varepsilon)_\varepsilon] = -\frac{c^3}{16\pi\kappa}\left(\delta\int G_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon = -\frac{c^3}{16\pi k}\left(\delta\int R_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon. \quad (1.9.4)$$

The constant κ is called the gravitational constant. The dimensions of κ follow from (1.9.4). Its numerical value is $\kappa = 6.67\times 10^{-8}\text{sm}^3\times\text{gr}^{-1}\times\sec^{-2}$.

We now proceed to the derivation of the equations of the gravitational field. These equations are obtained from the principle of least action $\delta((S_{m,\varepsilon})_\varepsilon + (S_{g_\varepsilon})_\varepsilon) = 0_{\widetilde{\mathbb{R}}}$, where $(S_{m,\varepsilon})_\varepsilon$ and $(S_{g_\varepsilon})_\varepsilon$are the distributional actions of the gravitational field and matter respectively. We now subject the gravitational Colombeau metric field, that is,the quantities g_{ik}, to variation. Calculating the variation $\delta(S_{g_\varepsilon})_\varepsilon$, we get

$$\delta\left(\int R_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon = \left(\delta\int R_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon = \left(\delta\int g_\varepsilon^{ik}R_{ik,\varepsilon}\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon =$$
$$\left\{\left(\int R_{ik,\varepsilon}\sqrt{-g_\varepsilon}\,\delta g_\varepsilon^{ik}d\Omega\right)_\varepsilon + \left(\int R_{ik,\varepsilon}g_\varepsilon^{ik}\delta\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon + \left(\int g_\varepsilon^{ik}\sqrt{-g_\varepsilon}\,\delta R_{ik,\varepsilon}d\Omega\right)_\varepsilon\right\} \quad (1.9.5)$$
$$\int\left\{\left(R_{ik,\varepsilon}\sqrt{-g_\varepsilon}\,\delta g_\varepsilon^{ik}\right)_\varepsilon + \left(R_{ik,\varepsilon}g_\varepsilon^{ik}\delta\sqrt{-g_\varepsilon}\right)_\varepsilon + \left(g_\varepsilon^{ik}\sqrt{-g_\varepsilon}\,\delta R_{ik,\varepsilon}\right)_\varepsilon\right\}d\Omega.$$

Thus, the variation $\mathbf{S}[(g_\varepsilon)_\varepsilon]$ is equal to

$$\mathbf{S}[(g_\varepsilon)_\varepsilon] = -\frac{c^3}{16\pi\kappa}\left(\int\left\{R_{ik,\varepsilon} - \frac{1}{2}g_{ik,\varepsilon}R_\varepsilon\right\}\sqrt{-g_\varepsilon}\,\delta g_\varepsilon^{ik}d\Omega\right)_\varepsilon. \quad (1.9.10)$$

Remark 1.9.1. We note that if we had started from the expression

$$\delta\mathbf{S}_g[(g_\varepsilon)_\varepsilon] = -\frac{c^3}{16\pi\kappa}\left(\delta\int G_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon \quad (1.9.11)$$

for the action of the field, then we get

$$\delta\mathbf{S}[(g_\varepsilon)_\varepsilon] =$$
$$-\frac{c^3}{16\pi\kappa}\int\delta(g_\varepsilon^{ik})_\varepsilon d\Omega\left\{\left(\frac{\partial\left\{G_\varepsilon\sqrt{-g_\varepsilon}\right\}}{\partial g_\varepsilon^{ik}}\right)_\varepsilon - \left(\frac{\partial}{\partial x^l}\frac{\partial\left\{G_\varepsilon\sqrt{-g_\varepsilon}\right\}}{\partial\frac{\partial g_\varepsilon^{ik}}{\partial x^l}}\right)_\varepsilon\right\}. \quad (1.9.12)$$

Comparing Eq.(1.9.12) with Eq.(1.9.10), we get

$$(R_{ik,\varepsilon})_\varepsilon - \frac{1}{2}(g_{ik,\varepsilon}R_\varepsilon)_\varepsilon =$$
$$\left\{\left(\frac{1}{\sqrt{-g_\varepsilon}}\right)_\varepsilon\right\}\left\{\left(\frac{\partial\left\{G_\varepsilon\sqrt{-g_\varepsilon}\right\}}{\partial g_\varepsilon^{ik}}\right)_\varepsilon - \left(\frac{\partial}{\partial x^l}\frac{\partial\left\{G_\varepsilon\sqrt{-g_\varepsilon}\right\}}{\partial\frac{\partial g_\varepsilon^{ik}}{\partial x^l}}\right)_\varepsilon\right\}. \quad (1.9.13)$$

For the variation of the action of the matter we can write

$$(\delta\mathbf{S}_{m,\varepsilon})_\varepsilon = \frac{1}{2c}\left(\int T_{ik,\varepsilon}\sqrt{-g_\varepsilon}\,\delta g_\varepsilon^{ik}d\Omega\right)_\varepsilon, \quad (1.9.14)$$

where $(T_{ik,\varepsilon})_\varepsilon \in \mathcal{G}(\mathbb{R}^4)$ is the generalized energy-momentum tensor of the matter fields.

Thus, from the principle of least action

$$\delta\{\mathbf{S}[(g_\varepsilon)_\varepsilon] + (\mathbf{S}_{m,\varepsilon})_\varepsilon\} = 0_{\widetilde{\mathbb{R}}} \tag{1.9.15}$$

one obtains

$$-\frac{c^3}{16\pi\kappa}\left(\int\left\{R_{ik,\varepsilon} - \frac{1}{2}g_{ik,\varepsilon}R_\varepsilon - \frac{8\pi\kappa}{c^4}T_{ik,\varepsilon}\right\}\sqrt{-g_\varepsilon}\,\delta g_\varepsilon^{ik}d\Omega\right)_\varepsilon = 0_{\widetilde{\mathbb{R}}}. \tag{1.9.16}$$

From Eq.(1.9.16), since of the arbitrariness of the $(\delta g_\varepsilon^{ik})_\varepsilon \in \mathcal{G}(\mathbb{R}^4)$ finally we get

$$(R_{ik,\varepsilon})_\varepsilon - \frac{1}{2}(g_{ik,\varepsilon}R_\varepsilon)_\varepsilon = \frac{8\pi\kappa}{c^4}(T_{ik,\varepsilon})_\varepsilon \tag{1.9.17}$$

or, in mixed components,

$$(R_{i,\varepsilon}^k)_\varepsilon - \frac{1}{2}\delta_i^k(R_\varepsilon)_\varepsilon = \frac{8\pi\kappa}{c^4}(T_{i,\varepsilon}^k)_\varepsilon. \tag{1.9.18}$$

They are called the generalized Einstein equations.

Contracting (1.9.18) on the indices i and k, we get

$$(R_\varepsilon)_\varepsilon = -\frac{8\pi\kappa}{c^4}(T_{i,\varepsilon}^i)_\varepsilon = -\frac{8\pi\kappa}{c^4}(T_\varepsilon)_\varepsilon. \tag{1.9.19}$$

Therefore the generalized Einstein equations of the field can also be written in the form

[37]

$$(R_{ik,\varepsilon})_\varepsilon = \frac{8\pi\kappa}{c^4}\left\{(T_{ik,\varepsilon})_\varepsilon - \frac{1}{2}(g_{ik,\varepsilon}T_\varepsilon)_\varepsilon\right\}. \tag{1.9.20}$$

Note that the generalized Einstein equations of the gravitational field are nonlinear Colombeau equations.

1.10.The densitized Einstein field equations revisited.

1.10.1.Remarks on the A. Einstein and N. Rosen paper from 1935.

The densitized Einstein field equations originally considered in A. Einstein and N. Rosen paper [32], see also [46]. As an exzample of the problem which arises from degenerasy of the metric tensor g_{ik}, the metric field is considered (see [32],eq.1):

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + \alpha^2 x_1^2 dx_4^2. \tag{1.10.1}$$

The $g_{\mu\nu}$ of this field satisfy in general the equations $R_{klm}^i = 0$, and hence the equations

$$R_{kl} = R_{klm}^m = 0. \tag{1.10.2}$$

A. Einstein emphasized that: "The $g_{\mu\nu}$ corresponding to (1.10.1) are regular for all finite (i.e. nonzero) points of space-time. Nevertheless one cannot assert that Eqs.(1.10.2) are satisfied by (1.10.1) for all finite values of $x_1,\ldots,x_4$. This is due to the fact that the determinant g of the $g_{\mu\nu}$ vanishes for $x_1 = 0$. The contravariant $g^{\mu\nu}$ therefore become infinite and the tensors R_{klm}^i and R_{kl} take on the form $0/0$. From the standpoint of Eqs.(1.10.2) the hyperplane $x_1 = 0$ then represents a singularity of the field".

We now ask whether the field law of gravitation (and later on the field law of gravitation and electricity) could not be modified in a natural way without essential change so that the solution (1.10.1) would satisfy the field equations for all finite points, i.e., also for $x_1 = 0$.

W. Mayer has called our attention to the fact that one can make R^i_{klm} and R_{kl} into rational functions of the $g_{\mu\nu}$, and their first two derivatives by multiplying them by suitable powers of g. It is easy to show that in $g^2 R_{kl}$ there is no longer any denominator. If then we replace (1.10.3) by (see [32],eq.3a):

$$R^*_{kl} = g^2 R_{kl} = 0, \qquad (1.10.3)$$

this system of equations is satisfied by (1.10.1) at all finite points. This amounts to introducing in place of the $g^{\mu\nu}$ the cofactors $[g_{\mu\nu}]$ of the $g_{\mu\nu}$ in g in order to avoid the occurrence of denominators. One is therefore operating with tensor densities of a suitable weight instead of with tensors. In this way one succeeds in avoiding singularities of that special kind which is characterized by the vanishing of g".

Remark 1.10.1.Note that A. Einstein actually rejected densitized field equations by the following reason:"The solution (1) naturally has no deeper physical significance insofar as it extends into spatial infinity. It allows one to see however to what extent the regularization of the hypersurfaces $g = 0$ leads to a theoretical representation of matter, regarded from the standpoint of the original theory. Thus, in the framework of the original theory one has the gravitational equations

$$R_{ik} - \frac{1}{2} g_{ik} R = -T_{ik}, \qquad (1.10.4)$$

where T_{ik} is the tensor of mass or energy density. Nevertheless in physical literature the densitized Einstein field equations holds from A. Einstein time untill nowoday, see for example [46].

Remark 1.10.2.Note that obviously the system of equations (1.10.3) is satisfied by (1.10.1) at all finite points.Nevertheless these equations can not solved the problem since
the ancetanty $0/0$ holds again in tensors R^i_{klm} and R_{kl} on hypersurface $x_1 = 0$.

Remark 1.10.3.Note that if some components of the Riemann curvature tensor $R^i_{klm}(\hat{x})$ become ancetanty $0/0$ or infinite at point $\hat{x}^0$ one obtain the breakdown of canonical formalism of Riemann geometry in a sufficiently small neighborhood Ω of the point $\hat{x}^0 \in \Omega$,i.e. in such neighborhood Ω Riemann curvature tensor $R^i_{klm}(\hat{x})$ must be changed
by formula (1.10.7) see remark 1.10.2.

Remark 1.10.4.Note that in Möller’s paper [38] the metric (1.10.1) has been derived in fact under abnormal assumption $0/0 = 1$ without respect to Levi-Civitá connection.

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1.10.2.Remarks on Möller abnormal famous paper from 1943

Recall that the classical Cartan’s structural equations show in a compact way the relation
between a connection and its curvature, and reveals their geometric interpretation in terms of moving frames. In order to study the mathematical properties of singularities,

we need to study the geometry of manifolds endowed on the tangent bundle with a symmetric bilinear form which is allowed to become degenerate (singular). But if the fundamental tensor is allowed to be degenerate (singular), there are some obstructions in constructing the geometric objects normally associated to the fundamental tensor. Also, local orthonormal frames and co-frames no longer exist, as well as the metric connection and its curvature operator [46].

As an important example of the geometry with the fundamental tensor which is allowed to be degenerate, we consider now Möller's uniformly accelerated frame given by Möller's line element (1.10.4).Recall that Möller dealing with the following line element [38]:

$$ds^2 = -\Delta(x)dt^2 + dx^2 + dy^2 + dz^2, \quad (1.10.4)$$

where $\Delta(x) = (a+gx)^2$.

Remark 1.10.3. Of course Möller's metric (1.10.4) degenerate at Möller horizon $x_{hor} = -a/g$.

However in contrast with A.Einstein paper [32],in famous but ubnormal paper [38] Möller mistakenly argue that metric field (1.10.4) is an global vaccuum solution of the A.Einstein field equations (1.10.5), i.e. the $g_{\mu\nu}$ of this field for all values of t,x,y,z satisfy the equations

$$R_{ik} - \frac{1}{2}g_{ik}R = 0. \quad (1.10.5)$$

Remark 1.10.4. In physical literature this Möller's abnormal mistake holds from Möller's time until nowadays.

Remark 1.10.5. Note that formally corresponding to the Möller's metric (1.10.4) classical Levi-Civitá connection reads

$$\Gamma^1_{44}(x) = a + gx, \Gamma^4_{14}(x) = \Gamma^4_{41}(x) = g(a+gx)^{-1} \quad (1.10.6)$$

and therefore classical Levi-Civit'a connection (1.10.6) of course is not available at Möller horizon since at horizon formal expressions (1.10.6) becomes infinity:

$$\Gamma^4_{14}\left(-\frac{a}{g}\right) = \Gamma^4_{41}\left(-\frac{a}{g}\right) = \infty. \quad (1.10.7)$$

Remark 1.10.6.Note that Möller dealing with Einstein's field equations in the following form

$$G^k_i = R^k_i - \frac{1}{2}\delta^k_i R = 0, \quad (1.10.8)$$

where R^k_i is the contracted Riemann-Christoffel tensor, formally calculated by canonical way by using classical Levi-Civit‘a connection (1.10.6) and where $R = R^i_i$. By using the following ansatz

$$ds^2 = -\Delta(x)dt^2 + dx^2 + dy^2 + dz^2, \tag{1.10.9}$$

Möller finally obtain

$$G^2_2(x) = G^3_3(x) = -\frac{1}{2\Delta(x)}\left[\Delta''(x) - \frac{(\Delta'(x))^2}{2\Delta(x)}\right] = -\frac{(\Delta^{1/2}(x))''}{\Delta^{1/2}(x)}. \tag{1.10.10}$$

where $\Delta'(x) = d\Delta(x)/dx$.

Remark 1.10.7. From Eq.(1.10.10) Möller obtain the following ordinary differential equation

$$(\Delta^{1/2}(x))'' = 0, \tag{1.10.11}$$

since it was mistakenly assumed that $G^2_2(x)$ and $G^3_3(x)$ for all values of x satisfy the equations

$$G^2_2(x) = G^3_3(x) \equiv 0. \tag{1.10.12}$$

The equation (1.10.11) obviously has the following trivial general solution

$$\Delta(x) = (a + gx)^2. \tag{1.10.13}$$

Remark 1.10.8. Note that at Möller horizon $x_{\mathbf{hor}} = -a/g$ the functions $G^2_2(x)$ and $G^3_3(x)$ ofcourse is not zero identically but becomes uncertainty, since

$$G^2_2(-a/g) = G^3_3(-a/g) = -\frac{\left([\Delta(-a/g)]^{1/2}\right)''}{[\Delta(-a/g)]^{1/2}} = \frac{0}{0}. \tag{1.10.14}$$

Remark 1.10.9. Note that at any point $x \neq -a/g$ obviously $G^2_2(x) = G^3_3(x) \equiv 0$ since at these
points one obtains

$$G^2_2(x) = G^3_3(x) = -\frac{1}{2\Delta(x)}\left[\Delta''(x) - \frac{(\Delta'(x))^2}{2\Delta(x)}\right] =.$$
$$-\frac{1}{2(a+gx)^2}\left[2g^2 - \frac{4g^2(a+gx)^2}{2(a+gx)^2}\right] = -\frac{1}{2(a+gx)^2}[2g^2 - 2g^2] \equiv 0. \tag{1.10.15}$$

Remark 1.10.10. At point $x = -a/g$ the quantity $G^2_2(-a/g)$ and $G^3_3(-a/g)$ well defined only by
formal limit

$$G^2_2(-a/g) = G^3_3(-a/g) = \lim_{x \to -a/g} -\frac{0}{2(a+gx)^2} = 0. \tag{1.10.16}$$

Remark 1.10.11. However in the limit $x \to -a/g$ the christoffel symbols (1.10.6) becomes
infinity:

$$\lim_{x\to -a/g} \Gamma^4_{14}(x) = \lim_{x\to -a/g} g(a+gx)^{-1} = \infty,$$
$$\lim_{x\to -a/g} \Gamma^4_{41}(x) = \lim_{x\to -a/g} g(a+gx)^{-1} = \infty. \quad (1.10.17)$$

It follows from (1.10.17) at horizon $x = -a/g$ the canonical expression for the contracted Riemann-Christoffel tensor R_i^k no longer holds, due to the degeneracy of (1.10.4), the Levi-Civitá connection is not available at Möller horizon $x = -a/g$.

In the following subsection we resolve this tension rigorously using linear distributional geometry.

1.10.3.The densitized Einstein field equations revisited by using the linear distributional geometry.

In order to derive the densitized Einstein field equations rigorously we apply in this subsection the aparatus of the linear distributional geometry.Note that in linear distributional geometry one dealing exactly with Schwartz distributions with compact support but not with full algebra of Colombeau generalized functions, see for example [30].

Remark 1.10.12. (i) Note that the notion of the Rimannian curvature comes from the study of parallel transport on a Rimannian manifold,see Fig. For instance, if a vector is moved around a loop on the surface of a sphere keeping parallel throughout the motion,
then the final position of the vector may not be the same as the initial position of the vector. This phenomenon is known as holonomy.

(ii) Various generalizations capture in an abstract form this idea of curvature as a measure of holonomy.

(iii) Classical holonomy presented by the classical formula (1.10.18) for the change ΔA_k
in a smooth vector $A_i(\hat{x})$ after parallel displacement around infinitesimal closed contour Γ.

(iv) Note that in classical case the change $\Delta A_k(\Gamma)$ always finite,i.e. $\Delta A_k(\Gamma) < \infty$.

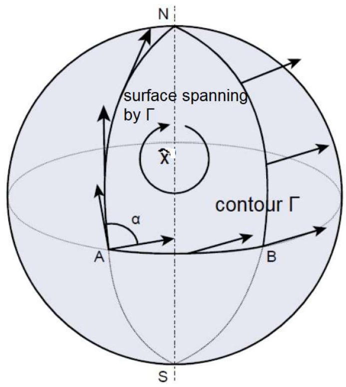

Fig.1.10.1.Parallel transporting a vector from
A → N → B → A yields a different vector.
This failure to return to the initial vector
is measured by the holonomy of the
surface spanning by Γ.

Remark 1.10.13. Let (M,g) be a semirimannian manifold.Let $\Gamma_{\hat{x}^0}$ be infinitesimal closed contour and let $\Sigma_\Gamma \subset M$ be the corresponding surface spanning by Γ, see Fig.1.10.1. We assume now that christoffel symbols $\Gamma^i_{kl}(\hat{x})$ are smooth on $\Sigma_\Gamma \cup \Gamma$. The classical formula for the change in a smooth vector $A_i(\hat{x})$ after parallel displacement around infinitesimal closed contour Γ reads [4]:

$$\Delta A_k(\Gamma) = \oint_\Gamma \delta A_k = \oint_\Gamma \Gamma^i_{kl}(\hat{x}) A_k dx^l. \tag{1.10.18}$$

We remind now the classical Stokes' theorem.

Theorem.1.10.1.(Stokes' theorem) If ω is a **smooth** $(n-1)$-form with compact support on smooth n-dimensional manifold Ω with-boundary $\partial\Omega$ of Ω given the induced orientation,

and $i : \partial\Omega \hookrightarrow \Omega$ is the inclusion map, then

$$\int_\Omega d\omega = \int_{\partial\Omega} i^*\omega. \tag{1.10.19}$$

Conventionally, $\int_{\partial\Omega} i^*\omega$ is abbreviated as $\int_{\partial\Omega} \omega$, since the pullback of a differential form by the inclusion map is simply its restriction to its domain $i^*\omega = \omega|\partial\Omega$ Here d is the exterior derivative, which is defined using the manifold structure only. The right-hand side is sometimes written as $\oint_{\partial\Omega} \omega$ to stress the fact that the $(n-1)$-manifold $\partial\Omega$ has no boundary.

For the further transformation of the integral (1.10.18), we must note the following. The values of the vector A_i at points inside the contour are not unique; they depend on the path along which we approach the particular point. However, as we shall see from the result obtained below, this non-uniqueness is related to terms of second order. We may therefore, with the first-order accuracy which is sufficient for the transformation, regard the components of the vector A_i at points inside the infinitesimal contour Γ as being uniquely determined by their values on the contour itself by the formulas

$$\delta A_i(x) = \Gamma^n_{il}(x)A_n(x)dx^l, \quad (1.10.20)$$

i.e.,by the derivatives

$$\frac{\partial A_i(x)}{\partial x^l} = (\Gamma^n_{il}(x)A_n(x)). \quad (1.10.21)$$

Now applying classical Stokes' theorem (see Theorem 1.10.1) to the integral (1.10.18) and considering that the area enclosed by the contour has the infinitesimal value (Δf^{im}), we get [4]:

$$\Delta A_k = \frac{1}{2}\left[\frac{\partial(\Gamma^i_{km}(x)A_i(x))}{\partial x^l} - \frac{\partial(\Gamma^i_{kl}(x)A_i(x))}{\partial x^m}\right]\Delta f^{im} =$$
$$= \frac{1}{2}\left[A_i(x)\left(\frac{\partial(\Gamma^i_{km}(x))}{\partial x^l}\right)_\varepsilon - A_i(x)\frac{\partial(\Gamma^i_{kl}(x))}{\partial x^m} + \right. \quad (1.10.22)$$
$$\left.\left(\frac{\partial A_i(x)}{\partial x^l}\right)\Gamma^i_{km}(x) - \left(\frac{\partial A_i(x)}{\partial x^m}\right)\Gamma^i_{kl}(x)\right]\Delta f^{im}.$$

Remark 1.10.13.Note that:(i) the regularity condition in Stokes' theorem,i.e. ω is a **smooth** $(n-1)$-form is essentially important and without this condition this theorem is no longer holds.

(ii) Obviously without Stokes' theorem impossible to derive the Eq.(1.10.22) and therefore

the expression (1.10.24) wihout the regularity condition of the functions $\Gamma^n_{km}(x)$ does not

make any sense and becomes to absurdum.

Substituting now the values of the derivatives (1.10.21) into Eq.(1.10.22), we get

$$\Delta A_k = \frac{1}{2}R^i_{klm}(x)A_i(x)\Delta f^{im}, \quad (1.10.23)$$

where $R^i_{klm}(x)$ is a tensor field of the fourth rank:

$$R^i_{klm}(x) = \frac{\partial(\Gamma^i_{km}(x))}{\partial x^l} - \frac{\partial(\Gamma^i_{kl}(x))}{\partial x^m} + \Gamma^i_{nl}(x)\Gamma^n_{km}(x) - \Gamma^i_{nm}(x)\Gamma^n_{kl}(x). \quad (1.10.24)$$

Definition 1.10.3.The tensor field $R^l_{kim}(x)$ is called the classical curvature tensor or the classical Riemann tensor.

The classical Riemann tensor that is a tensorial measure of holonomy.

Definition 1.10.4.Let (M,g) be a nonclassical semirimannian manifold.Let $\Sigma_\Gamma \subset M$ be the

surface spanning by Γ. We will say that the surface Σ_Γ admit the classical tensorial measure of holonomy (or admit the classical Riemann tensor) iff the Eq.(1.10.23) and Eq.(1.10.24) holds.

Remark 1.10.14.Let (M,g) be a nonclassical semirimannian manifold,i.e.the manifold endowed on the tangent bundle with a symmetric bilinear form which is allowed to become degenerate (singular). Let $\Gamma_{\hat{x}^0} \ni \hat{x}^0$ be infinitesimal closed contour and let $\Sigma_{\Gamma_{\hat{x}^0}} \subset M$ be the corresponding surface spanning by $\Gamma_{\hat{x}^0}$, see Fig.1.10.2.

We assume now that:

(i) christoffel symbols $\Gamma^i_{kl}(\hat{x},\hat{x}^0)$ become infinite at singular point $\hat{x}^0$ by formulae

$$\begin{cases} \Gamma^i_{kl}(\hat{x},\hat{x}^0) \asymp \Xi_{kl}(\hat{x})(x_i - x_i^0)^{-\delta}, \delta \geq 1 \\ \Xi_{kl}(\hat{x}) \in C^\infty\left(\Sigma_{\Gamma_{\hat{x}^0}}\right) \end{cases} \quad (1.10.25)$$

and (ii) $\hat{x}^0 \in \Gamma_{\hat{x}^0} \subset \Sigma_{\Gamma_{\hat{x}^0}}$. The classical formula (1.10.18) for the change in a smooth vector
$A_i(\hat{x})$ after parallel displacement around infinitesimal closed contour $\Gamma_{\hat{x}^0}$ reads:

$$\Delta A_k(\Gamma_{\hat{x}^0}) = \oint_{\Gamma_{\hat{x}^0}} \delta A_k = \oint_{\Gamma_{\hat{x}^0}} \Gamma^i_{kl}(\hat{x},\hat{x}^0) A_k dx^l. \quad (1.10.26)$$

Obviously the differential form $\Gamma^i_{kl}(\hat{x}) A_k dx^l$ does not locally integrable in neighborhood of
the point $\hat{x}^0 \in \Sigma_\Gamma$ and therefore $\Delta A_k(\Gamma_{\hat{x}^0}) = \infty$.

Remark 1.10.15. Note that under nonregularity conditions $\Delta A_k(\Gamma_{\hat{x}^0}) = \infty$ the classical formula (1.10.26) can not define correctly the holonomy of the surface $\Sigma_{\Gamma_{\hat{x}^0}}$ spanning by
$\Gamma_{\hat{x}^0}$ since the classical holonomy becomes infinity. In order to avoid this difficultness

one
needs to replace the classical formula (1.10.26) by the formula appropriate for singular case.

We started now from some definitions.

Definition 1.10.5. (i) Let (M,g) be a semirimannian manifold, and let $\Gamma_{\hat{x}^0}$ be infinitesimal
closed contour such that $\hat{x}^0 \in \Gamma_{\hat{x}^0} \subset M$. Let $\Theta(\hat{x}^0)$ be a closed infinitesimal neighborhood
of $\hat{x}^0$, then we we abbreviate $\delta_{\hat{x}^0} \triangleq \Gamma_{\hat{x}^0} \cap \Theta(\hat{x}^0)$. We will be say that a point $\hat{x}^0 \in \Gamma_{\hat{x}^0}$ is a singular pont of the manifold (M,g) if $\Delta A_k(\Gamma_{\hat{x}^0} \backslash \{\hat{x}^0\}) = \infty$ and $\Delta A_k(\Gamma_{\hat{x}^0} \backslash \delta_{\hat{x}^0}) < \infty$.

(ii) We will be say that a closed contour is a singular contour $\Gamma_{\hat{x}^0}$ if it contains at least one
singular pont $\hat{x}^0 \in \Gamma_{\hat{x}^0}$, see Fig.1.10.2-Fig.1.10.3.

Definition 1.10.6. (i) We will be say that a semirimannian manifold is a singular manifold
if there exists at least one singular (isolated) pont $\hat{x}^0 \in M$.

(ii)

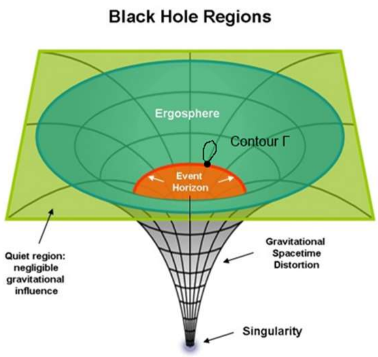

Fig.1.10.2. Singular point at BH horizon
and corresponding singular countur.

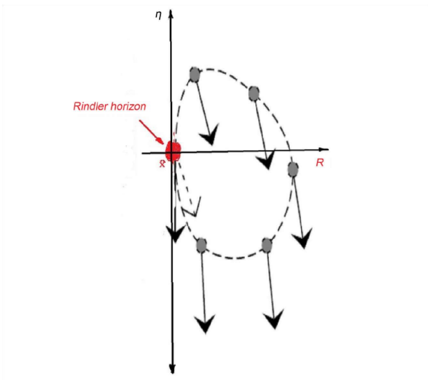

Fig.1.10.2. Singular point $(0,0)$ at Rindler horizon
$R = 0$ and corresponding singular countur.
$ds^2 = dR^2 - (\alpha R)^2 d\eta^2$.

Remark 1.10.15. Obviously the Schwarzschild singularity $r = 0$ is singular isolated pont of the Schwarzschild manifold.

Definition 1.10.3. Let (M,g) be a nonclassical semirimannian closed manifold, and let $\Gamma_{\hat{x}^0} \subset M$ be infinitesimal closed contour such that $\hat{x}^0 \in \Gamma_{\hat{x}^0} \subset M$.We will be say that a contour $\Gamma^{\#}_{\hat{x}^0} = \Gamma_{\hat{x}^0} \backslash \{\hat{x}^0\}$ with deleted point $\hat{x}^0$ (see Fig.1.10.3) is a singular truncated contour of the open manifold (M',g'),where $M' = M \backslash \partial M$ and $g' = g|_{M'}$ if $\Delta A_k(\Gamma_{\hat{x}^0} \backslash \{\hat{x}^0\}) = \infty$ and $\Delta A_k(\Gamma_{\hat{x}^0} \backslash \delta_{\hat{x}^0}) < \infty$.

Remark 1.10.16.(i) Note that the Levi-Civit'a connection $\Gamma^i_{kl}(\hat{x},\hat{x}^0)$ is available on any singular truncated contour $\Gamma^{\#}_{\hat{x}^0} = \Gamma_{\hat{x}^0} \backslash \{\hat{x}^0\}$ of the open manifold (M',g') but despite this
again $\Delta A_k(\Gamma_{\hat{x}^0} \backslash \{\hat{x}^0\}) = \infty$.(ii) Note that the semirimannian submanifold (M',g') of nonclassical semirimannian closed manifold (M,g) impossible treated classically,since the classical holonomy breaks down by divergence $\Delta A_k(\Gamma_{\hat{x}^0} \backslash \{\hat{x}^0\}) = \infty$.

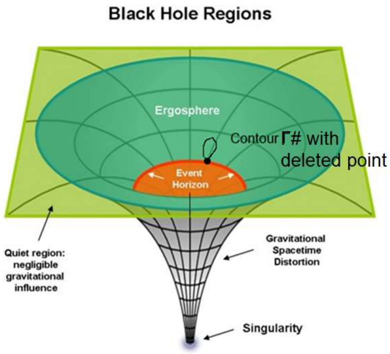

Fig.1.10.3. Singular truncated contour

$\Gamma^{\#}_{\hat{x}^0} = \Gamma_{\hat{x}^0} \backslash \{\hat{x}^0\}$

Definition 1.10.3. Let (M,g) be a semirimannian manifold, and let (M_1,g_1) be closed submanifold where $g_1 = g|_M$.

Remark 1.10.17. Let (M,g) be a nonclassical semirimannian manifold,i.e.the manifold endowed on the tangent bundle with a symmetric bilinear form which is allowed to become degenerate (singular). Let Γ be infinitesimal closed contour and let $\hat{x}^0 \in \Sigma_\Gamma \subset M, \hat{x}^0 \notin \Gamma$ be the corresponding surface spanning by Γ, see Fig.1.10.4 and Fig.1.10.5.Note that the Eq.(1.10.22) again breaks down (see Remark 1.10.13), since the

regularity condition of the functions $\Gamma^n_{km}(\hat{x},\hat{x}^0)$ are violeted at point $\hat{x}^0 \in \Sigma_\Gamma$.

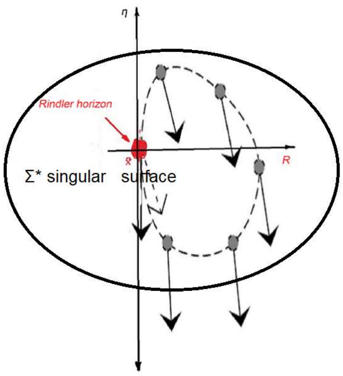

Fig.1.10.4.Infinitesimal closed contour Γ

and corresponding singular surface $\Sigma^*{=}\Sigma_\Gamma \ni \hat{x}^0$

with singular point $\hat{x}^0$ (Rindler horizon)

in Rindler space-time:

$ds^2 = -\alpha R^2 d\eta^2 + dR^2.$

Remark 1.10.18. In order to avoid the divergence mentioned above we consider the Christoffel symbols $\Gamma^i_{kl}(\hat{x},\hat{x}^0)$ as distributions on aproporiate space of the test functions.

Definition 1.10.4. Schwartz distributions with compact support (Schwartz generalized functions with compact support) are a class of linear functionals that map a space of test functions (conventional and well-behaved functions) into the set of real numbers $\mathbb{R}$.

In the simplest case, the space of test functions considered is $\mathcal{D}(\mathbb{R}^n, K)$, which is the set of functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ having two properties:

(i) φ is smooth (infinitely differentiable);

(ii) φ has compact support (is identically zero outside some compact set $K \subsetneqq \mathbb{R}^n$.

Definition 1.10.5. Schwartz distribution with compact support (Schwartz generalized functions) T is a linear mapping : $\mathcal{D}(\mathbb{R}^n, K) \to \mathbb{R}$. Instead of writing $T(\varphi)$, it is conventional to write $\langle T, \varphi \rangle$ for the value of T acting on a test function φ. A simple example of a distribution is the Dirac delta δ, defined by $\langle \delta, \varphi \rangle = \varphi(0)$, meaning that δ evaluates a test function at 0. Its physical interpretation is as the density of a point source.

Definition 1.10.6. Suppose that $f : K \subset \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function. Then a corresponding distribution $T_f \in \mathcal{D}'(\mathbb{R}^n, K)$ may be defined by

$$\langle T_f, \varphi \rangle = \int_K f(\hat{x})\varphi(\hat{x})d^n x \qquad (1.10.27)$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n, K)$.

Definition 1.10.7. We chose now the a space of test functions $\mathcal{D}(\mathbb{R}^4, K, \hat{x}^0)$:

$\mathcal{D}(\mathbb{R}^4, K, \hat{x}^0) = \{\psi(\hat{x})\} = \{\Phi(\hat{x} - \hat{x}^0)\varphi(\hat{x}) | \varphi(\hat{x}) \in \mathcal{D}(\mathbb{R}^4, K, \hat{x}^0)\}$,

where $\Phi(\hat{x} - \hat{x}^0) = \prod_{i=0}^{4}(x_i - x_i^0)^{2\delta}, \delta \geq 1, \varphi(\hat{x}) \in \mathcal{D}(\mathbb{R}^4, K), \hat{x}^0 \in K$.

Let us introduce now similarly to canonical Definition 1.8.5 the formula for the *regularized* (or generalized) change $\widetilde{\Delta A_k}(\varphi)$ in a vector $A_i(\hat{x})$ after parallel displacement around infinitesimal closed contour Γ (see Fig.1.10.3).This regularized change $\widetilde{\Delta A_k}$ can clearly be written in the form

$$\widetilde{\Delta A_k}(\varphi) = \langle \delta A_k(\hat{x}), \Phi(\hat{x} - \hat{x}^0)\varphi(\hat{x}) \rangle = \oint_\Gamma \Phi(\hat{x} - \hat{x}^0)\varphi(\hat{x})\delta A_k(\hat{x}), \qquad (1.10.28)$$

where $\Phi(\hat{x} - \hat{x}^0) = \prod_{i=0}^{4}(x_i - x_i^0)^{2\delta}, \delta \geq 1, \varphi(\hat{x}) \in \mathcal{D}(\mathbb{R}^4, K), \hat{x}^0 \in K$ and where the integral is taken over the given contour $\Gamma \subset K, \hat{x}^0 \notin \Gamma$. If $\hat{x}^0 \in \Gamma$ (see Fig.1.10.4) the regularized change $\widetilde{\Delta A_k}$ can clearly be written in the form

$$\widetilde{\Delta A_k}(\varphi) = \langle \delta A_k(\hat{x}), \Phi(\hat{x} - \hat{x}^0)\varphi(\hat{x}) \rangle = \oint_{\Gamma \setminus \{\hat{x}^0\}} \Phi(\hat{x} - \hat{x}^0)\varphi(\hat{x})\delta A_k(\hat{x}). \qquad (1.10.29)$$

Substituting in place of δA_k the canonical expression $\delta A_k = \Gamma^i_{kl}(\hat{x})A_k dx^l$ (see [4],Eq.(85.5)) we obtain

$$\begin{gathered}\widetilde{\Delta A_k}(\varphi) = \langle \Gamma^i_{kl}(\hat{x}), \Phi(\hat{x} - \hat{x}^0)\varphi(\hat{x})A_k \rangle = \\ \oint_\Gamma \Phi(\hat{x} - \hat{x}^0)\varphi(\hat{x})\delta A_k = \oint_\Gamma \Gamma^i_{kl}(\hat{x})\Phi(\hat{x} - \hat{x}^0)\varphi(\hat{x})A_k dx^l\ ,\end{gathered} \qquad (1.10.30)$$

where

$$\frac{\partial A_i}{\partial x^l} = \Gamma^i_{kl}(\hat{x})A_k. \qquad (1.10.31)$$

Remark 1.10.18.Note that: (i) Eq.(1.10.31) holds since $\hat{x}^0 \notin \Gamma$.(ii) In any neighborhood $O(\hat{x}^0,\varepsilon) = \{\hat{x} : \|\hat{x}-\hat{x}^0\| \leq \varepsilon\}, \varepsilon > 0, \varepsilon \ll 1$of the singular point $\hat{x}^0$ the functions $\Gamma^i_{kl}(\hat{x})\Phi(\hat{x}-\hat{x}^0)$ is regular.(iii) At singular point $\hat{x}^0$ the quantities $\Gamma^i_{kl}(\hat{x})\Phi(\hat{x}-\hat{x}^0)|_{\hat{x}=\hat{x}^0}$ are well defined by the limit:

$$\Gamma^i_{kl}(\hat{x})\Phi(\hat{x}-\hat{x}^0)|_{\hat{x}=\hat{x}^0} = \lim_{\hat{x}\to\hat{x}^0}\Gamma^i_{kl}(\hat{x})\Phi(\hat{x}-\hat{x}^0),$$

since for any i,k,l the limit $\lim_{\hat{x}\to\hat{x}^0}\Gamma^i_{kl}(\hat{x})\Phi(\hat{x}-\hat{x}^0)$ exists and finite by the choosing of the function $\Phi(\hat{x}-\hat{x}^0)$.

(iv) It follows from (i)-(iii) the classical Stokes' theorem (see [4],Eq.(6.19)) holds for the integral (1.10.30).

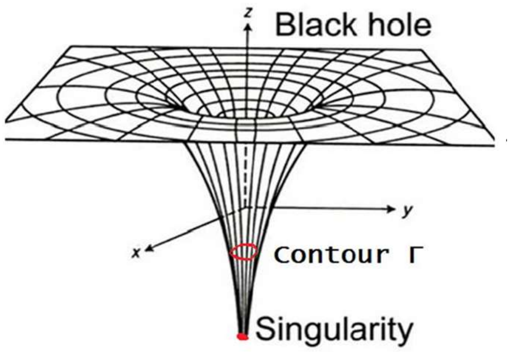

Fig.1.10.5.Infinitesimal closed contour Γ
and corresponding singular surface $\Sigma_\Gamma \ni \hat{x}^0$
spanning by Γ.
Due to the degeneracy of the
Schwarzschild metric (1.1)
at point $r = 0$,
the Levi-Civitá connection $\Gamma^{+l}_{kj}(\{\}) =$
$= \frac{1}{2}[g^{lm}(\{\})][(g_{mk,j}(\{\}) + g_{mj,k}(\{\}) - g_{kj,m}(\{\})]$
is not available on $\mathbb{R}^3_+ \cup \{0\}$.

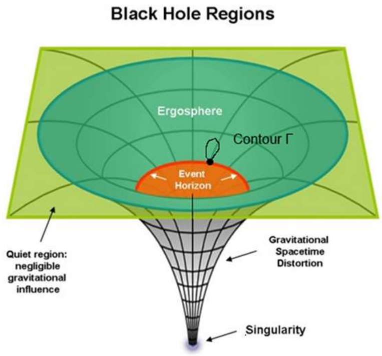

Fig.1.10.6.Infinitesimal closed contour Γ with a singularity at point $\hat{x}^0$ on Horizon and corresponding singular surface $\Sigma_\Gamma \ni \hat{x}^0$ spanning by Γ.
Due to the degeneracy of of the Schwarzschild metric field (1.1) at $r = 2m$, the classical Levi-Civitá connection $\Gamma_{kj}^{+l}(\{\}) =$
$= \frac{1}{2}[g^{lm}(\{\})][(g_{mk,j}(\{\}) + g_{mj,k}(\{\}) - g_{kj,m}(\{\})]$
is not available on $\mathbb{R}^3_+ \cup \{r = 2m\}$ in classical sense but is available on $\mathbb{R}^3_+ \cup \{r = 2m\} \cup \{r = 0\}$ in the sense of the generalized functions in $\mathcal{D}'(\mathbb{R}^4, K, \hat{x}^0)$.

Remark 1.10.19. Note that:(i) by using the Eq.(1.10.30) the classical singular Levi-Civitá connection corresponding to the degenerate and singular Schwarzschild metric field (1.6) now is avaluble on extended Schwarzschild spacetime $\overline{\mathbf{Sh}}$=$(S^2 \times \{r \geq 2m\} \cup \{0 \leq r \leq 2m\}) \times \mathbb{R}$, since the singular Christoffel symbols (1.7) in Schwarzschild coordinates are well defined as generalized functions in $\mathcal{D}'(\mathbb{R}^4, K, \hat{x}^0)$.
(ii) The same holds for the Schwarzschild metric in isotropic coordinates (1.11).
Now applying Stokes' theorem (see [4],Eq.(6.19)) to the integral (1.10.30) and considering that the area enclosed by the contour has the infinitesimal value Δf^{lm}, we get

$$\widetilde{\Delta A_k} = \oint_{\Gamma} \Phi(\hat{x}-\hat{x}^0)\ \Gamma^i_{kl}(\hat{x})A_k dx^l =$$

$$= \frac{1}{2}\int_{\Sigma_\Gamma}\left[\frac{\partial(\Gamma^i_{km}(\hat{x})A_i\Phi(\hat{x}-\hat{x}^0))}{\partial x^l} - \frac{\partial(\Gamma^i_{kl}(\hat{x})A_i\Phi(\hat{x}-\hat{x}^0))}{\partial x^m}\right]df^{lm} \approx$$

$$\approx \left[\frac{\partial(\Gamma^i_{km}(\hat{x})A_i\Phi(\hat{x}-\hat{x}^0))}{\partial x^l} - \frac{\partial(\Gamma^i_{kl}(\hat{x})A_i\Phi(\hat{x}-\hat{x}^0))}{\partial x^m}\right]\frac{\Delta f^{lm}}{2} =$$

$$\left[\Phi(\hat{x}-\hat{x}^0)\frac{\partial(\Gamma^i_{km}(\hat{x}-\hat{x}^0)A_i)}{\partial x^l} + (\Gamma^i_{km}(\hat{x})A_i)\frac{\partial\Phi(\hat{x}-\hat{x}^0)}{\partial x^l} - \right. \tag{1.10.32}$$

$$\left. -\Phi(\hat{x}-\hat{x}^0)\frac{\partial(\Gamma^i_{kl}(\hat{x})A_i)}{\partial x^m} - (\Gamma^i_{kl}(\hat{x})A_i)\frac{\partial\Phi(\hat{x}-\hat{x}^0)}{\partial x^m}\right]\frac{\Delta f^{lm}}{2} =$$

$$\left[\Phi(\hat{x}-\hat{x}^0)\frac{\partial(\Gamma^i_{km}(\hat{x})A_i)}{\partial x^l} - \Phi(\hat{x}-\hat{x}^0)\frac{\partial(\Gamma^i_{kl}(\hat{x})A_i)}{\partial x^m} - \right.$$

$$\left. A_i(\hat{x})\Phi(\hat{x}-\hat{x}^0)\frac{2\delta\Gamma^i_{km}(\hat{x})}{x_l - x^0_l} - A_i(\hat{x})\Phi(\hat{x}-\hat{x}^0)\frac{2\delta\Gamma^i_{kl}(\hat{x})}{x_m - x^0_m}\right]\frac{\Delta f^{lm}}{2}.$$

Substituting the values of the derivatives (1.10.31) into Eq.(1.10.32), we get finally:

$$\widetilde{\Delta A_k} = \widetilde{R}^i_{klm}\frac{A_i(\hat{x})\Phi(\hat{x}-\hat{x}^0)\Delta f^{lm}}{2}, \tag{1.10.33}$$

where $\widetilde{R}^i_{klm}$, is a tensor of the fourth rank

$$\widetilde{R}^i_{klm} = R^i_{klm} + 2\delta\left[\frac{\Gamma^i_{km}(\hat{x})}{x_l - x^0_l} - \frac{\Gamma^i_{kl}(\hat{x})}{x_m - x^0_m}\right]. \tag{1.10.34}$$

Here R^i_{klm} is the classical Riemann curvature tensor.That $\widetilde{R}^i_{klm}$ is a tensor is clear from the fact that in (1.10.6) the left side is a vector–the difference $\widetilde{\Delta A_k}$ between the values of vectors at one and the same point.

Definition1.10.8. The tensor $\widetilde{R}^i_{klm}$ is called the generalized curvature tensor or the generalized Riemann tensor. Note that for any i,k,l,m : $\widetilde{R}^i_{klm} \in \mathcal{D}'(\mathbb{R}^4, K, \hat{x}^0)$.

Definition1.10.9. The generalized Ricci curvature tensor $\widetilde{R}_{km}$ is defined as

$$\widetilde{R}_{km} = \widetilde{R}^i_{kim}. \tag{1.10.35}$$

Remark 1.10.20.Note that for any k,m : $\widetilde{R}_{km} \in \mathcal{D}'(\mathbb{R}^4, K, \hat{x}^0)$.

Definition1.10.10. The generalized Ricci scalar $\widetilde{R}$ is defined as

$$\widetilde{R} = g^{km}\widetilde{R}_{km}. \tag{1.10.36}$$

Remark 1.10.21.Note that the generalized Ricci scalar $\widetilde{R} \in \mathcal{D}'(\mathbb{R}^4, K, \hat{x}^0)$.

Definition1.10.11. The generalized Einstein tensor for any k,m : $\widetilde{G}_{km} \in \mathcal{D}'(\mathbb{R}^4, K, \hat{x}^0)$ is defined as

$$\widetilde{G}_{km} = \widetilde{R}_{km} - \frac{1}{2}g_{km}\widetilde{R}. \tag{1.10.37}$$

Thus the revisited densitized Einstein field equations in $\mathcal{D}'(\mathbb{R}^4, K, \hat{x}^0)$ reads:

$$\int_K \widetilde{G}_{km}(\hat{x})\Phi(\hat{x}-\hat{x}^0)df^{km} = \int_K \Big(\widetilde{R}_{km}(\hat{x}) - \frac{1}{2}g_{km}(\hat{x})\widetilde{R}(\hat{x})\Big)\Phi(\hat{x}-\hat{x}^0)df^{km} = \\ = -\int_K T_{km}(\hat{x})\Phi(\hat{x}-\hat{x}^0)df^{km}. \tag{1.10.38}$$

Remark 1.10.22. Note that Beyond any small neighborhood $O(\hat{x}^0,\varepsilon) = \{\hat{x} : \|\hat{x}-\hat{x}^0\| \leq \varepsilon\}, \quad \varepsilon > 0, \varepsilon \ll 1$of the singular point $\hat{x}^0$ the equations (1.10.38) becomes to the following form

$$\widetilde{G}_{km}(\hat{x})\Phi(\hat{x}-\hat{x}^0) = \widetilde{R}_{km}(\hat{x}) - \frac{1}{2}g_{km}\widetilde{R}(\hat{x}) = -T_{km}(\hat{x}), \tag{1.10.39}$$

where $\hat{x} \notin O(\hat{x}^0,\varepsilon)$.We rewrite now the Eq.(1.10.34) in the following form

$$\widetilde{R}^i_{klm} = R^i_{klm} + \Re^i_{klm} \\ \Re^i_{klm} = 2\delta\left[\frac{\Gamma^i_{km}(\hat{x})}{x_l - x^0_l} - \frac{\Gamma^i_{kl}(\hat{x})}{x_m - x^0_m}\right]. \tag{1.10.40}$$

Thus the revisited densitized Einstein field equations (1.10.38) reads

$$\int_K \widetilde{R}_{km}(\hat{x}) - \frac{1}{2}\int_K g_{km}(\hat{x})\widetilde{R}(\hat{x}) = \\ \int_K R_{km}\Phi(\hat{x}-\hat{x}^0)df^{km} + \int_K \Re_{km}\Phi(\hat{x}-\hat{x}^0)df^{km} - \\ \frac{1}{2}\int_K g_{km}(\hat{x})(R(\hat{x}) + \Re(\hat{x}))\Phi(\hat{x}-\hat{x}^0)df^{km} = \\ \int_K \Big(R_{km}(\hat{x}) - \frac{1}{2}g_{km}(\hat{x})R(\hat{x})\Big)\Phi(\hat{x}-\hat{x}^0)df^{km} + \\ \int_K \Big(\Re_{km}(\hat{x}) - \frac{1}{2}g_{km}(\hat{x})\Re(\hat{x})\Big)\Phi(\hat{x}-\hat{x}^0)df^{km} = -\int_K T_{km}(\hat{x})\Phi(\hat{x}-\hat{x}^0)df^{km}. \tag{1.10.41}$$

We assume now that

$$\int_K \Big(\Re_{km}(\hat{x}) - \frac{1}{2}g_{km}(\hat{x})\Re(\hat{x})\Big)\Phi(\hat{x}-\hat{x}^0)df^{km} = -\int_K T_{km}(\hat{x})\Phi(\hat{x}-\hat{x}^0)df^{km}. \tag{1.10.42}$$

From the Eqs.(1.10.41)-(1.10.42) we get the (revisited) densitized "vacuum" equations:

$$\int_K \Big(R_{km}(\hat{x}) - \frac{1}{2}g_{km}(\hat{x})R(\hat{x})\Big)\Phi(\hat{x}-\hat{x}^0)df^{km} = 0. \tag{1.10.43}$$

Remark 1.10.23. It follows that a metric field $g_{km}(\hat{x})$ has a distributional source in $\mathcal{D}'(\mathbb{R}^4, K, \hat{x}^0)$ given by Eq.(1.10.42),since the "vacuum" equations (1.10.43) follows from the densitized Einstein field equations (1.10.38) under setting given by Eq.(1.10.42).

Remark 1.10.24. Beyond any neighborhood $O(\hat{x}^0,\varepsilon) = \{\hat{x} : \|\hat{x}-\hat{x}^0\| \leq \varepsilon\}, \varepsilon > 0,$of the singular point $\hat{x}^0$ the equations (1.10.43) becomes to the following canonical "vacuum" form

$$R_{km}(\hat{x}) - \frac{1}{2}g_{km}(\hat{x})R(\hat{x}) = 0. \tag{1.10.44}$$

Remark 1.10.25. The "vacuum" equations (1.10.44) return the canonical Schwarzschild solutions (1.6) and (1.11) except any neighborhood $O(\{r-2m\} \cup \{r=0\},\varepsilon), \varepsilon > 0$ of horizon and Schwarzschild singularity.But we ephazized that scalar curvature of the Schwarzschild spacetime is given exactly by the generalized Ricci scalar $\widetilde{R} \neq R$.

Remark 1.10.26. It follows from the Eqs.(1.10.41) that the Möller metric field (1.10.9) has

a singular source T_{km}.

Beyond the neighborhood $O(x_{\mathbf{hor}},\varepsilon) = \{x : |a+gx| \leq \varepsilon\},\varepsilon > 0$ of the Möller horizon $x_{\mathbf{hor}} = -a/g$, but not on whole Möller spacetime, one obtains in accordance with classical

Möller's result,see Eqs.(1.10.15):

$$R^k_m(x) - \frac{1}{2}g^k_m(x)R(x) = 0. \tag{1.10.45}$$

Inside the submanifold $O(x_{\mathbf{hor}},\varepsilon)$ the corresponding densitized "vacuum" equations (1.10.43) reads

$$\int_{O(x_{\mathbf{hor}},\varepsilon)} \Big(R^k_m(x) - \frac{1}{2}g^k_m(x)R(x)\Big)\Delta(x)df^{km} = 0 \tag{1.10.46}$$

Therefore from the Eqs.(1.10.46) and Eqs.(1.10.10) one obtains

$$\begin{gathered}\int_{O(x_{\mathbf{hor}},\varepsilon)} G^2_2(x)\Delta(x)dx = \int_{O(x_{\mathbf{hor}},\varepsilon)} G^3_3(x)dx = \\ -\int_{O(x_{\mathbf{hor}},\varepsilon)} \frac{1}{2\Delta(x)}\left[\Delta''(x) - \frac{(\Delta'(x))^2}{2\Delta(x)}\right]\Delta(x)dx \equiv 0,\end{gathered} \tag{1.10.47}$$

since at Möller horizon $x_{\mathbf{hor}} = -a/g$ the function

$$\Xi(x) = \frac{1}{\Delta(x)}\left[\Delta''(x) - \frac{(\Delta'(x))^2}{2\Delta(x)}\right]\Delta(x) = \frac{1}{\Delta(x)}\left[2g^2 - \frac{4g^2\Delta(x)}{2\Delta(x)}\right] \tag{1.10.48}$$

is well defind by taking the limit $x \to x_{\mathbf{hor}}$ (see Remark 1.10.18) and therefore

$$\begin{gathered}\lim_{x\to x_{\mathbf{hor}}} \Xi(x) = \lim_{x\to x_{\mathbf{hor}}}\left(\frac{\Delta(x)}{\Delta(x)}\left[2g^2 - \frac{4g^2\Delta(x)}{2\Delta(x)}\right]\right) = \\ \lim_{x\to x_{\mathbf{hor}}}\left(\frac{\Delta(x)}{\Delta(x)}\right)\left[2g^2 - 2g^2\lim_{x\to x_{\mathbf{hor}}}\left(\frac{\Delta(x)}{\Delta(x)}\right)\right] = 2g^2 - 2g^2 \equiv 0.\end{gathered} \tag{1.10.49}$$

Remark 1.10.27. Note that in contrast with abnormal Möller's calculation, see Remark 1.10.10-1.10.11, Eq.(1.10.16)

Remark 1.10.28. (**I**) Note that the Schwarzschild metric field (1.6) in classical sense is well defined only for $r > 2m$,The boundary of the manifold $\{r > 2m\}$ in $\mathbb{R}^3 \times \mathbb{R}$ is the submanifold $\{r = 2m\}$ of $\mathbb{R}^3 \times \mathbb{R}$, diffeomorfic to a product $S^2 \times \mathbb{R}$.This submanifold is colled the event horizon, or simply the horizon [33],[34].

(**II**) The Schwarzschild metric (1.10.12) in canonical coordinates (x^0,r,θ,ϕ),with $m > 0$, ceases to be a smooth Lorentzian metric for $r = 2m$, because for such a value of r the coefficient g_{00} becomes zero while g_{11} becomes infinite. For $0 < r < 2m$ the metric (1.9.9) again a smooth Lorentzian metric but t is a space coordinate while r a time coordinate. Hence the metric cannot be said to be either spherically symmetric or static for $r < 2m$ [33].

(**III**) From consideration above obviously follows that on Schwarzschild spacetime

$\mathbf{Sh} = (S^2 \times \{r > 2m\} \cup \{0 < r < 2m\}) \times \mathbb{R}$ the Levi-Civita connection

$$\left\{ \Gamma^{+l}_{kj}(\{\}) == \frac{1}{2}[g^{lm}(\{\})][(g_{mk,j}(\{\}) + g_{mj,k}(\{\}) - g_{kj,m}(\{\})] \right. \qquad (1.10.50)$$

is not available in classical sense and that is well known many years from mathematical

literature, see for example [22] and Remark 1.10.19 above.

(**IV**) Note that [4] : (**i**) The determinat $\det(g_{lm}(\{\})) = -r^4 \sin^2\theta$ of the metric (1.6)

is reqular on horizon,i.e., smooth and non-vanishing for $r = 2m$.

In addition:

(**ii**) The curvature scalar $\mathbf{R}(\{\}) = g^{\mu\nu}\mathbf{R}_{\mu\nu}(\{\})$ is zero for $r = 2m$.

(**iii**) The none of higher-order scalars such as $\mathbf{R}^{\mu\nu}(\{\})\mathbf{R}_{\mu\nu}(\{\})$, etc. blows up. For example

the quadratic scalar $\mathbf{R}^{\rho\sigma\mu\nu}(\{\})\mathbf{R}_{\rho\sigma\mu\nu}(\{\}) = 48m^2/r^6$ is reqular on horizon,i.e.,smooth and

non-vanishing for $r = 2m$.

(**V**) Note that: (**i**) In physical literature (see for example [4],[33],[35],) it was wrongly assumed that a properties (**i**)-(**iii**) is enough to convince us that $r = 2m$ represent the non honest physical singularity but only coordinate singularity.

(**VI**) Such assumption based only on wrong formal extensions

$\widehat{\mathbf{R}}(\{\}), \widehat{\mathbf{R}}^{\mu\nu}(\{\})\widehat{\mathbf{R}}_{\mu\nu}(\{\}), \ldots,$

$\widehat{\mathbf{R}}^{\rho\sigma\mu\nu}(\{\})\widehat{\mathbf{R}}_{\rho\sigma\mu\nu}(\{\})$ of the curvature scalar $\mathbf{R}(\{\})$ and higher-order scalars such as $\mathbf{R}^{\mu\nu}(\{\})\mathbf{R}_{\mu\nu}(\{\}), \ldots, \mathbf{R}^{\rho\sigma\mu\nu}(\{\})\mathbf{R}_{\rho\sigma\mu\nu}(\{\})$ on horizon $r = 2m$ and on origin $r = 0$ by formulae

$$\widehat{\mathbf{R}}(r)\Big|_{r=2m} = \lim_{r\to 2m} \mathbf{R}(r) = 0, \widehat{\mathbf{R}}(r)\Big|_{r=0} = \lim_{r\to 0} \mathbf{R}(r) = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\widehat{\mathbf{R}}^{\rho\sigma\mu\nu}(r)\widehat{\mathbf{R}}_{\rho\sigma\mu\nu}(r)\Big|_{r=2m} = \lim_{r\to 2m}(\mathbf{R}^{\rho\sigma\mu\nu}(r)\mathbf{R}_{\rho\sigma\mu\nu}(r)) = \lim_{r\to 2m}\frac{48m^2}{r^6} = \frac{48m^2}{r^6}\Big|_{r=2m}, \qquad (1.10.51)$$

$$\widehat{\mathbf{R}}^{\rho\sigma\mu\nu}(r)\widehat{\mathbf{R}}_{\rho\sigma\mu\nu}(r)\Big|_{r=0} = \lim_{r\to 0}(\mathbf{R}^{\rho\sigma\mu\nu}(r)\mathbf{R}_{\rho\sigma\mu\nu}(r)) = \lim_{r\to 0}\frac{48m^2}{r^6} = \infty.$$

However in the limit $r \to 2m$ the Levi-Civitá connection $\Gamma^{+l}_{kj}(\{\})$ becomes infinite [4]:

$$\Gamma^1_{00}(r)|_{r=2m} = \lim_{r\to 2m}\frac{m(r-2m)}{r^3} = 0, \Gamma^1_{11}(r)|_{r=2m} = \lim_{r\to 2m}\frac{-m}{r(r-2m)} = \infty,$$

$$\Gamma^0_{01}(r)|_{r=2m} = \lim_{r\to 2m}\frac{m}{r(r-2m)} = \infty,$$

$$\Gamma^2_{12}(r)|_{r=2m} = \lim_{r\to 2m}\frac{1}{r} = 2^{-1}m^{-1}, \Gamma^1_{22}|_{r=2m} = -\lim_{r\to 2m}(r-2m) = 0,$$

$$\Gamma^3_{13}|_{r=2m} = \lim_{r\to 2m}\frac{1}{r} = 2^{-1}m^{-1}, \Gamma^1_{33}|_{r=2m} = -\lim_{r\to 2m}(r-2m)\sin^2\theta = 0, \qquad (1.10.52)$$

$$\Gamma^1_{00}(r)|_{r=0} = \lim_{r\to 0}\frac{m(r-2m)}{r^3} = \infty\Gamma^1_{11}(r)|_{r=0} = \lim_{r\to 0}\frac{-m}{r(r-2m)} = \infty,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta, \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}.$$

Thus obviously by consideration above (see Remark 1.1-Remark 1.10.2) this extension given by Eq.(1.10.15) has no any sense in respect of the canonical Riemannian geometry.

(**VII**) From consideration above (see Remark 1.10.1-Remark 1.10.2) obviously follows that the scalars such as $\widehat{\mathbf{R}}(\{\}), \widehat{\mathbf{R}}^{\mu\nu}(\{\})\widetilde{\mathbf{R}}_{\mu\nu}(\{\}), \ldots, \widehat{\mathbf{R}}^{\rho\sigma\mu\nu}(\{\})\widehat{\mathbf{R}}_{\rho\sigma\mu\nu}(\{\})$ has no any rigorous sense in respect to the canonical Levi-Civitá connection (1.10.11) and therefore cannot be said to be either honest physical singularity or only coordinate singularity in respect of the canonical Riemannian geometry.

Remark 1.10.29. Note that in physical literature the spacetime singularity usually is defined as location where the quantities that are used to measure the gravitational field become infinite in a way that does not depend on the coordinate system. These quantities are the classical scalar invariant curvatures of singular spacetime, which includes a measure of the density of matter.

Remark 1.10.30. In general relativity, many investigations have been derived with regard to singular exact *vacuum solutions* of the Einstein equation and the singularity structure of space-time. Such solutions have been formally derived under condition $\mathbf{T}^{\nu}_{\mu}(x) = 0,$ where $\mathbf{T}^{\nu}_{\mu}(x)$ represent the energy-momentum densities of the gravity source. This for example is the case for the well-known Schwarzschild solution, which is given by, in the Schwarzschild coordinates (x^0, r, θ, ϕ),

$$ds^2 = -h(r)(dx^0)^2 + h^{-1}(r)(dr)^2 + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], h(r) = 1 - \frac{r_s}{r}, \quad (1.10.52)$$

where, r_s is the Schwarzschild radius $r_s = 2GM/c^2$ with G, M and c being the Newton gravitational constant, mass of the source, and the light velocity in vacuum Minkowski space-time, respectively. The metric (1.10.44) describe the gravitational field produced by a point-like particle located at $r = 0$, see [30].

Remark 1.10.31. Note that when we say, on the basis of the canonical expression of the curvature square

$$\mathbf{R}^{\rho\sigma\mu\nu}(r)\mathbf{R}_{\rho\sigma\mu\nu}(r) = \frac{12r_s^2}{r^6} \quad (1.10.53)$$

formally obtained from the metric (1.10.44), that $r = 0$ is a singularity of the Schwarzschild space-time, the source is considered to be point-like and this metric is regarded as *meaningful everywhere* in space-time.

Remark 1.10.32. From the metric (1.10.44), the calculation of the canonical Einstein tensor proceeds in a straighforward manner gives for $r \neq 0$

$$\begin{aligned} G^t_t(r) = G^r_r(r) &= -\frac{h'(r)}{\hat{r}} - \frac{1 + h(r)}{\hat{r}^2} \equiv 0 \ , \\ G^\theta_\theta(r) = G^\varphi_\varphi(r) &= -\frac{h''(r)}{2} - \frac{h(r)}{\hat{r}^2} \equiv 0, \end{aligned} \quad (1.10.54)$$

where $h(r) = -1 + r_s/r.$ Using Eq.(1.10.54) one formally obtains a boundary conditions

$$\begin{cases} G^t_t(0) \triangleq \lim\limits_{r\to 0} G^t_t(r) = 0, G^r_r(0) \triangleq \lim\limits_{r\to 0} G^r_r(r) = 0, \\ G^\theta_\theta(0) \triangleq \lim\limits_{r\to 0} G^\theta_\theta(r) = 0, G^\varphi_\varphi(0) \triangleq \lim\limits_{r\to 0} G^\varphi_\varphi(r) = 0. \end{cases} \quad (1.10.55)$$

However as pointed out above the canonical expression of the Einstein tensor in a sufficiently small neighborhood Ω of the point $r = 0$ and must be replaced by the generalized Einstein tensor $\widetilde{G}_{km}$ (1.10.37). By simple calculation easy to see that

$$\begin{cases} \widetilde{G}_t^t(0) \triangleq \lim_{r\to 0} \widetilde{G}_t^t(r) = -\infty, \widetilde{G}_r^r(0) \triangleq \lim_{r\to 0} \widetilde{G}_r^r(r) = -\infty, \\ \widetilde{G}_\theta^\theta(0) \triangleq \lim_{r\to 0} \widetilde{G}_\theta^\theta(r) = -\infty, \widetilde{G}_\varphi^\varphi(0) \triangleq \lim_{r\to 0} \widetilde{G}_\varphi^\varphi(r) = -\infty. \end{cases} \quad (1.10.56)$$

and therefore the boundary conditions (1.10.56) are completely wrong. But other hand as pointed out by many authors [5]-[17] that the canonical representation of the Einstein tensor, valid only in a weak (distributional) sense,i.e. [12]:

$$G_b^a(\hat{x}) = -8\pi m \delta_0^a \delta_b^0 \delta^3(\hat{x}) \quad (1.10.57)$$

and therefore again we obtain $G_b^a(0) = -\infty \times (\delta_0^a \delta_b^0)$. Thus canonical property fo the Einstein tensor: $G_b^a(\hat{x}) \equiv 0$, is breakdown in rigorous mathematical sense for the Schwarzschild solution at origin $r = 0$.

1.11.The nonlinear distributional Möller geometry using the full algebra of the Colombeau generalized functions. A.Einstein as the Copernicus of nonlinear distributional geometry.

Note that the ε-regularization related to Colombeau geometry originally considered in A. Einstein and N. Rosen paper [32]. A. Einstein emphasized that:

"The solution (1.10.1) (see [32],eq.1) naturally has no deeper physical significance insofar as it extends into spatial infinity. It allows one to see however to what extent the regularization of the hypersurfaces $g = 0$ leads to a theoretical representation of matter, regarded from the standpoint of the original theory. Thus, in the framework of the original theory one has the gravitational equations (see [32],eq.4)

$$R_{ik} - \frac{1}{2} g_{ik} R = -T_{ik}, \quad (1.11.1)$$

where T_{ik} is the tensor of mass or energy density. To interpret (1.11.1) in the framework of this theory we must approximate the line element by a slightly different one which avoids the singularity $g = 0$. Accordingly we introduce a small constant σ and let (see [32],eq.1a)

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + (\alpha^2 x_1^2 + \sigma) dx_4^2 \quad (1.11.2)$$

the smaller σ (> 0) is chosen, the nearer does this gravitational field come to that of (1.11.2). If one calculates from this the (fictitious) (see Remark 1.11.1) energy tensor T_{ik} one obtains as nonvanishing components

$$T_{22} = T_{23} = \frac{\alpha^2 \sigma}{(\alpha^2 x_1^2 + \sigma)^2}. \quad (1.11.3)$$

We see then that the smaller one takes σ the more is the tensor concentrated in the neighborhood of the hypersurface $x_1 = 0$. From the standpoint of the original theory the solution (1.11.1) contains a singularity which corresponds to an energy or mass concentrated in the surface $x_1 = 0$; from the standpoint of the modified theory, however, (1.11.2) is a solution of (1.10.3) (see [32],eq.3a), free from singularities, which describes the "field-producing mass, " without requiring for this the introduction of any new field quantities".

Remark 1.11.1. Note that the energy tensor T_{ik} (1.11.3) A. Einstein mistakenly has been considered as "fictitious". It is clear that A. Einstein thought that by using any fixed semi Riemannian metric and thus in particular by the fixed metric field mentioned in his paper [32]:

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + \alpha^2 x_1^2 dx_4^2 \qquad (1.11.4)$$

one obtains full and unique semi Riemannian structure corresponding to uniformly accelerated reference system,etc. This principle mistake holds in physical community from A. Einstein time until nowadays. However even in A. Einstein time it were well known that for non classical semi Riemannian metric fields this opinion completely wrong and one needs something more than only simply formula (1.11.4),see [4]-[6] in Refferences B.

Actually as we know the imbedding $ds^2 \hookrightarrow (ds_\sigma^2)_\sigma$ in Colombeau generalized object $(ds_\sigma^2)_\sigma$ is necessary.

Remark 1.11.2.Note that the regularization mentioned in A. Einstein and N. Rosen paper [32]. by introducing ε-regularization $\varepsilon = \sqrt{\sigma}, \varepsilon \in (0,\delta], \delta \ll 1$ in contemporary Colombeau notations reads

$$(ds_\varepsilon^2)_\varepsilon = -dx_1^2 - dx_2^2 - dx_3^2 + \left((\alpha^2 x_1^2 + \varepsilon^2)_\varepsilon\right) dx_4^2 \qquad (1.11.5)$$

Thus in accordance with the Colombeau framework the classical metric field (1.1.1) embedded in Colombeau object $(ds_\varepsilon^2)_\varepsilon$.

Remark 1.11.3.(i) Note that in contemporary Colombeau notations A. Einstein "fictitious" energy tensor $T_{ik}(\sigma)$ obviously reads [5]:

$$(T_{22}(\varepsilon))_\varepsilon = T_{23}(\varepsilon) = \alpha^2 \left(\frac{\varepsilon^2}{(\alpha^2 x_1^2 + \varepsilon^2)^2} \right)_\varepsilon, \qquad (1.11.6)$$

where $\varepsilon = \sqrt{\sigma}, \varepsilon \in (0,\delta], \delta \ll 1.$ The generalized Einstein field equations (1.9.17) corresponding to distributional energy tensor (1.11.5.) reads

$$(R_{ik,\varepsilon})_\varepsilon - \frac{1}{2} g_{ik,\varepsilon} (R_\varepsilon)_\varepsilon = -(T_{ik}(\varepsilon))_\varepsilon. \qquad (1.11.7)$$

(ii) Note that in any classical point (x_1,x_2,x_3,x_4) with $x_1 \in \mathbb{R} \backslash \{0\}$ the energy tensor (1.10.) becomes to infinite small values, i.e. $(T_{ik}(\varepsilon))_\varepsilon \approx_{\widetilde{\mathbb{R}}} 0$

(iii) Note that in any classical point (x_1,x_2,x_3,x_4) with $x_1 = 0$ the energy tensor (1.11.6) has well defined point value $\in \widetilde{\mathbb{R}}_{\mathbf{inf}} \subset \widetilde{\mathbb{R}}$.

(iv) Note that in accordance with result obtained above in subsection 1.10.3 the

generalized Einstein field equations meant that metric field (1.11.1) has a non trivial

Colombeau distributional source

$$\left(\frac{\varepsilon^2}{\left(\alpha^2 x_1^2+\varepsilon^2\right)^2}\right)_\varepsilon \in \mathcal{G}(\mathbb{R}). \tag{1.11.8}$$

(v) Note that it follows from Eq.() the Colombeau curvature scalar $(R_\varepsilon)_\varepsilon$
in any classical point (x_1,x_2,x_3,x_4) with $x_1 \in \mathbb{R}\backslash\{0\}$
We consider now the Möller's metric (1.10.4)

$$ds^2 = -\Delta(x)dt^2 + dx^2 + dy^2 + dz^2, \tag{1.11.9}$$

where $\Delta(x) = (a+gx)^2$.

In order to avoid difficultness with degeneracy of the classical metric (1.11.9) mentioned
above in subsection 1.10.2, we replace now Möller's line element (1.11.8) by corresponding Colombeau line element

$$(d_\varepsilon s^2)_\varepsilon = -[(\Delta_\varepsilon(x))_\varepsilon]dt^2 + dx^2 + dy^2 + dz^2, \tag{1.11.10}$$

where $\Delta_\varepsilon(x) = \left[(a+gx)^2+\varepsilon^2\right], \varepsilon \in (0,1]$.

Remark 1.11.4. Note that in contrast with a wrong formal expression (1.10.6) the distributional Levi-Cività connection corresponding Colombeau line element (1.11.10) reads

$$(\Gamma^1_{44,\varepsilon}(x))_\varepsilon = ((a+gx))_\varepsilon, (\Gamma^4_{14,\varepsilon}(x)) = \Gamma^4_{41,\varepsilon}(x) = \frac{a+gx}{\left((a+gx)^2+\varepsilon^2\right)_\varepsilon} \tag{1.11.11}$$

Notice that the distributional Levi-Cività connection (1.11.11) is well defined and even regular on whole Möller's space-time.

Let $(G^k_{i,\varepsilon}(x))_\varepsilon$ be the distributional Einstein tensor $(G^k_i(x))_\varepsilon \triangleq (R^k_i(x))_\varepsilon - \frac{1}{2}\delta^k_i(R_\varepsilon(x))_\varepsilon$,
where $(R^k_{i,\varepsilon}(x))_\varepsilon$ is the contracted distributional Riemann-Christoffel tensor calculated by
using distributional Levi-Civit'a connection (1.11.1) corresponding to Colombeau line element (1.11.10) [5],[35] and $(R_\varepsilon(x))_\varepsilon = (R^i_i(\varepsilon))_\varepsilon$. Therefore for the case of the Colombeau line element (1.11.10) we get

$$\begin{gathered}(G^2_2(x;\varepsilon))_\varepsilon = (G^3_3(x;\varepsilon))_\varepsilon = \\ -\frac{1}{2(\Delta_\varepsilon(x))_\varepsilon}\left\{(\Delta''_\varepsilon(x))_\varepsilon - \frac{[(\Delta'_\varepsilon(x))_\varepsilon]^2}{2(\Delta_\varepsilon(x))_\varepsilon}\right\} = -\left(\frac{g^2\varepsilon^2}{\Delta^2_\varepsilon(x)}\right)_\varepsilon.\end{gathered} \tag{1.11.12}$$

Notice that $(\Im(x;\varepsilon))_\varepsilon \triangleq -\left(\frac{g^2\varepsilon^2}{\Delta^2_\varepsilon(x)}\right)_\varepsilon, \varepsilon \in (0,1]$ is Colombeau generalized function such that $\mathbf{cl}[(\Im(x;\varepsilon))_\varepsilon] \in \mathbf{G}(\mathbb{R})$ and $\mathbf{cl}\left[\left(\Im\left(-(a/g)^{-1};\varepsilon\right)\right)_\varepsilon\right] = \mathbf{cl}[(\varepsilon^{-2})_\varepsilon] \in \widetilde{\mathbb{R}}_{\mathbf{inf}} \subset \widetilde{\mathbb{R}}$.

Thus Colombeau generalized fundamental tensor $(g_{ik}(\varepsilon))_\varepsilon$ corresponding to Colombeau
metric (1.11.10) that is non vacuum Colombeau solution of the of the generalized Einstein's field equations (1.11.7) with Colombeau generalized source.For Colombeau scalars $(R_\varepsilon(x))_\varepsilon$ and $(R^{\mu\nu}(x,\varepsilon)R_{\mu\nu}(x,\varepsilon))_\varepsilon$ we get [5]:

$$(R_\varepsilon(x))_\varepsilon = \left(\frac{g^2\varepsilon^2}{\Delta^2_\varepsilon(x)}\right)_\varepsilon \tag{1.11.13}$$

and

$$(R^{\mu\nu}(x,\varepsilon)R_{\mu\nu}(x,\varepsilon))_\varepsilon = g^4\varepsilon^4\left[(a+gx)^2+\varepsilon^2\right]^{-4} \quad (1.11.14)$$

correspondingly.

1.12.The distributional Schwarzschild geometry by using the linear L.Schwartz distributions and by using the full algebra of the Colombeau generalized functions.

General relativity as a physical theory is governed by particular physical equations; the focus of interest is the breakdown of physics which need not coincide with the breakdown of geometry. It has been suggested to describe singularity at the origin as *internal* point of the Schwarzschild spacetime, where the Einstein field equations are satisfied in a weak sense [5]-[22].

1.12.1.The distributional Schwarzschild geometry at the origin.The smooth regularization of the singularity at the origin.

The two singular functions we will work with throughout this paper (namely the singular components of the Schwarzschild metric) are $\frac{1}{r}$ and $\frac{1}{r-r_s}$, $r_s \geq 0$. Since $\frac{1}{r} \in L^1_{loc}(\mathbb{R}^3)$, it obviously gives the regular distribution $\frac{1}{r} \in D'(\mathbb{R}^3)$. By convolution with a mollifier $\rho(x)$ (adapted to the symmetry of the spacetime, i.e. chosen radially symmetric) we embed it into the Colombeau algebra $\mathcal{G}(\mathbb{R}^3)$ [22]:

$$\frac{1}{r} \xrightarrow{\iota} \iota\left(\frac{1}{r}\right) \triangleq \left(\frac{1}{r}\right) * \rho_\varepsilon \triangleq \left(\frac{1}{r}\right)_\varepsilon, \rho_\varepsilon = \frac{1}{\varepsilon^3}\rho\left(\frac{r}{\varepsilon}\right), \varepsilon \in (0,1]. \quad (1.12.1)$$

Inserting (1.12.1) into Schwarzschild metric (1.10.44) we obtain a generalized Colombeau object modeling the singular Schwarzschild spacetime [22]:

$$\begin{cases} (ds^2_\varepsilon)_\varepsilon = (h_\varepsilon(r)(dt)^2)_\varepsilon - (h^{-1}_\varepsilon(r)(dr)^2)_\varepsilon + r^2[(d\theta)^2+\sin^2\theta(d\phi)^2], \\ h_\varepsilon(r) = -1 + r_s\left(\frac{1}{r}\right)_\varepsilon, \varepsilon \in (0,1]. \end{cases} \quad (1.12.2)$$

Remark 1.12.1. Note that under regularization (1.12.1) for any $\varepsilon \in (0,1]$ the metric

$$ds^2_\varepsilon = h_\varepsilon(r)(dt)^2 - h^{-1}_\varepsilon(r)(dr)^2 + r^2[(d\theta)^2+\sin^2\theta(d\phi)^2]$$

obviously is a classical Riemannian object and there no exist an the breakdown of canonical formalism of Riemannian geometry for these metrics, even at origin $r=0$. It has been suggested by many authors to describe singularity at the origin as an *internal point*, where the Einstein field equations are satisfied in a distributional sense [5]-[22]. From the Colombeau metric (1.12.2) one obtains in a distributional sense [22]:

$$\begin{aligned} (R^2_2(r,\varepsilon))_\varepsilon = (R^3_3(r,\varepsilon))_\varepsilon &= \left(\frac{h'_\varepsilon(r)}{r} + \frac{1+h_\varepsilon(r)}{r^2}\right)_\varepsilon = 8\pi m\frac{\delta(r)}{r^2}, \\ (R^0_0(r,\varepsilon))_\varepsilon = (R^1_1(r,\varepsilon))_\varepsilon &= \frac{1}{2}\left(\frac{h''_\varepsilon(r)}{2} + \frac{h'_\varepsilon(r)}{r}\right)_\varepsilon = -4\pi m\delta\frac{\delta(r)}{r^2}. \end{aligned} \quad (1.12.3)$$

Hence, the Colombeau distributional Ricci tensor and the Colombeau distributional curvature scalar $(R_\varepsilon(r))_\varepsilon$ are of δ-type, i.e. $(R_\varepsilon(r))_\varepsilon = \pi m \frac{\delta(r)}{r^2} \in \mathcal{D}(\mathbb{R}^3)$.

Remark 1.12.2. Note that the formulae (1.12.3) should be contrasted with what is the expected result $G^a_b(x) = -8\pi m \delta^a_0 \delta^0_b \delta^3(x)$ given by Eq.(1.10.49). However the equations (1.12.3) are obviously given in spherical coordinates and therefore strictly speaking this is not correct, because the basis fields $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \theta}\right\}$ are not globally defined. Representing distributions concentrated at the origin requires a basis regular at the origin. Transforming the formulae for $(R_{ij}(\varepsilon))_\varepsilon$ into Cartesian coordinates associated with the spherical ones, i.e., $\{r,\theta,\varphi\} \leftrightarrow \{x^i\}$, we obtain, e.g., for the Einstein tensor the expected result $G^a_b(x) = -8\pi m \delta^a_0 \delta^0_b \delta^3(x)$ given by Eq.(1.10.49), see [22].

1.12.2.The nonsmooth regularization of the singularity at the origin.

The nonsmooth regularization of the Schwarzschild singularity at the origin $r = 0$ is considered by N. R. Pantoja and H. Rago in paper [12]. Pantoja non smooth regularization regularization of the Schwarzschild singularity reads

$$(h_\varepsilon(r))_\varepsilon = -1 + \left(\frac{r_s}{r}\Theta(r-\varepsilon)\right)_\varepsilon, \varepsilon \in (0,1], r < r_s. \tag{1.12.4}$$

Here $\Theta(u)$ is the Heaviside function and the limit $\varepsilon \to 0$ is understood in a distributional sense.Equation (1.12.2) with h_ε as given in (1.12.4) can be considered as an regularized version of the Schwarzschild line element in curvature coordinates. From equation (1.12.4), the calculation of the distributional Einstein tensor proceeds in a straighforward manner. By simple calculation it gives [12]:

$$\begin{cases} (G^t_t(r,\varepsilon))_\varepsilon = (G^r_r(r,\varepsilon))_\varepsilon = -\left(\frac{h'_\varepsilon(r)}{r}\right)_\varepsilon - \left(\frac{1+h_\varepsilon(r)}{r^2}\right)_\varepsilon = \\ \qquad = -r_s\left(\frac{\delta(r-\varepsilon)}{r^2}\right)_\varepsilon = -r_s\frac{\delta(r)}{r^2} \end{cases} \tag{1.12.5}$$

and

$$\begin{cases} (G^\theta_\theta(r,\varepsilon))_\varepsilon = (G^\varphi_\varphi(r,\varepsilon))_\varepsilon = -\left(\frac{h''_\varepsilon(r)}{2}\right)_\varepsilon - \left(\frac{h_\varepsilon(r)}{r^2}\right)_\varepsilon = \\ r_s\left(\frac{\delta(r-\varepsilon)}{r^2}\right)_\varepsilon - r_s\left(\frac{\varepsilon}{r^2}\frac{d}{dr}\delta(r-\varepsilon)\right)_\varepsilon = -r_s\frac{\delta(r)}{r^2}. \end{cases} \tag{1.12.6}$$

which is exactly the result obtained in Ref. [9] using smoothed versions of the Heaviside function $\Theta(r-\varepsilon)$. Transforming now the formulae for $(G^a_b(r,\varepsilon))_\varepsilon$ into Cartesian coordinates associated with the spherical ones, i.e., $\{r,\theta,\varphi\} \leftrightarrow \{x^i\}$, we obtain for the generalized Einstein tensor the expected result given by Eq.(1.10.49)

$$G^a_b(x) = -8\pi m \delta^a_0 \delta^0_b \delta^3(x), \tag{1.12.7}$$

see Remark 1.12.2.

1.12.3.The smooth regularization via Horizon in

Schwarzschild coordinates and the smooth regularization at horizon in isotropic coordinates.

1.12.3.1.The smooth regularization via Horizon in Schwarzschild coordinates

The smooth regularization via Horizon is considered by J.M.Heinzle and R.Steinbauer in paper [22]. Note that $\frac{1}{r-r_s} \notin L^1_{loc}(\mathbb{R}^3)$. An canonical regularization is the principal value $\mathbf{vp}\left(\frac{1}{r-r_s}\right) \in D'(\mathbb{R}^3)$ which can be embedded into $\mathcal{G}(\mathbb{R}^3)$ [22]:

$$\frac{1}{r-r_s} \overset{\mathbf{vp}}{\to} \mathbf{vp}\left(\frac{1}{r-r_s}\right) \overset{\iota}{\to} \iota\left[\rho_\varepsilon * \mathbf{vp}\left(\frac{1}{r-r_s}\right)\right] \triangleq \left(\frac{1}{r-r_s}\right)_\varepsilon \in \mathcal{G}(\mathbb{R}^3). \qquad (1.12.8)$$

Inserting now(1.12.8) into Schwarzschild metric (1.10.44) we obtain a generalized Colombeau object modeling the singular Schwarzschild space-time [22]:

$$(ds^2_\varepsilon)_\varepsilon = (h(r)(dt)^2)_\varepsilon - (h^{-1}_\varepsilon(r)(dr)^2)_\varepsilon + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], \qquad (1.12.9)$$

where

$$h(r) = -1 + \frac{r_s}{r}, h^{-1}_\varepsilon(r) = -1 - r_s\left(\frac{1}{r-r_s}\right)_\varepsilon, \varepsilon \in (0,1]. \qquad (1.12.10)$$

Remark 1.12.3. Note that obviously Colombeau object (1.12.9) is degenerate at $r = r_s$, because $h(r)$ is zero at the horizon. However, this does not come as a surprise. Both $h(r)$ and $h^{-1}(r)$ are positive outside of the black hole and negative in the interior. As a consequence any *smooth* regularization of $h(r)$ (or h^{-1}) must pass through zero somewhere and, additionally, this zero must converge to $r = r_s$ as the regularization parameter goes to zero.

Remark 1.12.4. Note that due to the degeneracy of Colombeau object (1.10.26), even the
distributional Levi-Civitá connection obviously is not available by using the smooth regularization via horizon [22].

Remark 1.12.5. Note that the smooth regularization (1.12.8) doesn't make any sense, since Colombeau object (1.12.9) again is degenerate at $r = r_s$. However in isotropic coordinates (t,ρ,θ,ϕ) the smooth regularization at horizon obviously possible.

1.12.3.2.The smooth regularization at horizon in isotropic coordinates.

The Schwarzschild metric in isotropic coordinates (t,ρ,θ,ϕ) reads [3]:

$$ds^2 = -\frac{(4\rho - r_s)^2}{4\rho + r_s}dt^2 + \left(1 + \frac{r_s}{4\rho}\right)^4 d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1.12.11)$$

where we let $c = 1$. The canonical simple regularization at horizon reads

$$(4\rho - r_s)^2 \to (4\rho - r_s)^2 + \varepsilon^2. \qquad (1.12.12)$$

Inserting now(1.12.12) into Schwarzschild metric (1.12.11) we obtain a generalized Colombeau object modeling the degenerate Schwarzschild space-time in isotropic coordinates

$$(ds^2_\varepsilon)_\varepsilon = -\frac{\left((4\rho - r_s)^2 + \varepsilon^2\right)_\varepsilon}{4\rho + r_s}dt^2 + \left(1 + \frac{r_s}{4\rho}\right)^4 d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (1.12.13)$$

Corresponding to (1.12.13) Colombeau distributional geometry considered in sect. 2.4.

1.12.4.The nonsmooth regularization via Gorizon

In this paper we leave the neighborhood of the singularity at the origin and turn to the singularity at the horizon. The question we are aiming at is the following: using distributional geometry (thus without leaving Schwarzschild coordinates), is it possible to show that the horizon singularity of the Schwarzschild metric is not merely a coordinate singularity. In order to investigate this issue we calculate the distributional curvature at the horizon in Schwarzschild coordinates.

The main focus of this work is a (nonlinear) *superdistributional* description of the Schwarzschild spacetime. Although the nature of the Schwarzschild singularity is much "worse" than the quasi-regular conical singularity, there are several distributional treatments in the literature [8]-[29], mainly motivated by the following considerations: the physical interpretation of the Schwarzschild metric is clear as long as we consider it merely as an exterior (vacuum) solution of an extended (sufficiently large) massive spherically symmetric body. Together with the interior solution it describes the entire spacetime. The concept of point particles—well understood in the context of linear field theories—suggests a mathematical idealization of the underlying physics: one would like to view the Schwarzschild solution as defined on the entire spacetime and regard it as generated by a point mass located at the origin and acting as the gravitational source.

This of course amounts to the question of whether one can reasonably ascribe distributional curvature quantities to the Schwarzschild singularity at the horizon.

The emphasis of the present work lies on mathematical rigor. We derive the "physically expected" result for the distributional energy momentum tensor of the Schwarzschild geometry, i.e., $T_0^0 = 8\pi m\delta^{(3)}(\vec{x})$, in a conceptually satisfactory way. Additionally, we set up a unified language to comment on the respective merits of some of the approaches taken so far. In particular, we discuss questions of differentiable structure as well as smoothness and degeneracy problems of the regularized metrics, and present possible refinements and workarounds.These aims are accomplished using the framework of nonlinear supergeneralized functions (supergeneralized Colombeau algebras $\widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$).Examining the Schwarzschild metric (1.12) in a neighborhood of the horizon, we see that, whereas $h(r)$ is smooth, $h^{-1}(r)$ is not even L^1_{loc} (note that the origin is now always excluded from our considerations; the space we are working on is $\mathbb{R}^3\backslash\{0\}$). Thus, regularizing the Schwarzschild metric amounts to embedding h^{-1} into $\widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$ (as done in (3.2)).Obviously, (3.1) is degenerate at $r = 2m$, because $h(r)$ is zero at the horizon. However, this does not come as a surprise. Both $h(r)$ and $h^{-1}(r)$ are positive outside of the black hole and negative in the interior. As a consequence any (smooth) regularization $h^+_\epsilon(r)$ ($h^-_\epsilon(r)$) [above (below) horizon] of $h(r)$ must pass through small enough vicinity $O^+_\epsilon(2m) = \{\vec{x} \in \mathbb{R}^3 | \|\vec{x}\| > 2m, \|\vec{x} - 2m\| \leqslant \epsilon\}$ ($O^-_\epsilon(2m) = \{\vec{x} \in \mathbb{R}^3 | \|\vec{x}\| < 2m, \|\vec{x} - 2m\| \leqslant \epsilon\}$) of zeros set $O_0(2m) = \{\vec{y} \in \mathbb{R}^3 | \|\vec{y}\| = 2m\}$ somewhere and, additionally, this vicinity $O^+_\epsilon(2m)$ ($O^-_\epsilon(2m)$) must converge to $O_0(2m)$ as the regularization parameter ϵ goes to zero.Due to the degeneracy of the Schwarzschild metric (1.10.44) , the Levi-Cività connection is not available. By apriporiate nonsmooth regularization (see sect. 3) we obtain an Colombeau generalized object modeling the singular Schwarzschild metric above and below horizon, i.e.,

$$\begin{cases} (ds_\epsilon^{+2})_\epsilon = (h_\epsilon^+(r)dt^2)_\epsilon - \left([h_\epsilon^+(r)]^{-1}dr^2\right)_\epsilon + r^2 d\Omega^2, \\ (ds_\epsilon^{-2})_\epsilon = (h_\epsilon^-(r)dt^2)_\epsilon - \left([h_\epsilon^-(r)]^{-1}dr^2\right)_\epsilon + r^2 d\Omega^2, \\ \epsilon \in (0,1]. \end{cases} \tag{1.12.14}$$

Consider corresponding distributional connections $(\Gamma_{kj}^{+l}(\epsilon))_\epsilon = (\Gamma_{kj}^{+l}[h_\epsilon^+])_\epsilon \in \widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$ and $(\Gamma_{kj}^{-l}(\epsilon))_\epsilon = (\Gamma_{kj}^{-l}[h_\epsilon^+])_\epsilon \in \widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$:

$$\begin{aligned} (\Gamma_{kj}^{+l}(\epsilon))_\epsilon &= \frac{1}{2}((g_\epsilon^{+lm})[(g_\epsilon^+)_{mk,j} + (g_\epsilon^+)_{mj,k} - (g_\epsilon^+)_{kj,m}])_\epsilon, \\ (\Gamma_{kj}^{-l}(\epsilon))_\epsilon &= \frac{1}{2}((g_\epsilon^{-lm})[(g_\epsilon^-)_{mk,j} + (g_\epsilon^-)_{mj,k} - (g_\epsilon^-)_{kj,m}])_\epsilon. \end{aligned} \tag{1.12.15}$$

Obviously $(\Gamma_{kj}^{+l}[h_\epsilon^+])_\epsilon, (\Gamma_{kj}^{-l}[h_\epsilon^+])_\epsilon$ coincides with the corresponding Levi-Cività connection on $\mathbb{R}^3 \backslash \{(r=0) \cup (r=2m)\}$, as $(h_\epsilon^+)_\epsilon = h_0^+, (h_\epsilon^-)_\epsilon = h_0^-$, and $(g_\epsilon^{+lm})_\epsilon = g_0^{+lm}, (g_\epsilon^{-lm})_\epsilon = g_0^{-lm}$ there. Clearly, connections $\Gamma_{kj}^{+l}(\epsilon), \Gamma_{kj}^{-l}(\epsilon), \epsilon \in (0,1]$ in respect the regularized metric $g_\epsilon^\pm, \epsilon \in (0,1]$, i.e., $(g_\epsilon^\pm)_{ij;k} = 0$. Proceeding in this manner, we obtain the nonstandard result

$$\begin{cases} \left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon = \left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon \approx -m\tilde{\Phi}(2m), \\ \left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon = \left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon \simeq m\tilde{\Phi}(2m). \end{cases} \tag{1.12.16}$$

Investigating the weak limit of the angular components of the generalized Ricci tensor using the abbreviation

$$\tilde{\Phi}(r) = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \Phi(x)$$

and let $\Phi(x)$ be the function $\Phi(x) \in \mathcal{S}_{2m}^+(\mathbb{R}^3)$ ($\Phi(x) \in \mathcal{S}_{2m}^-(\mathbb{R}^3)$), where by $\mathcal{S}_{2m}^+(\mathbb{R}^3)$ ($\mathcal{S}_{2m}^-(\mathbb{R}^3)$) we denote the class of all functions $\Phi(x)$ with compact support such that
(i) **supp**$(\Phi(x)) \subset \{x | \|x\| \geq 2m\}$ (**supp**$(\Phi(x)) \subset \{x | \|x\| \leq 2m\}$) (ii) $\tilde{\Phi}(r) \in C^\infty(\mathbb{R})$. Then for any function $\Phi(x) \in \mathcal{S}_{2m}^\pm(\mathbb{R}^3)$ we get:

$$\begin{cases} w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^+]_1^1 = w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^+]_0^0 = m\langle\tilde{\delta}|\Phi\rangle = -m\tilde{\Phi}(2m), \\ w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^-]_1^1 = w\text{-}\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^-]_0^0 = m\langle\tilde{\delta}|\Phi\rangle = m\tilde{\Phi}(2m), \end{cases} \tag{1.12.17}$$

i.e., the Schwarzschild spacetime is weakly Ricci-nonflat (the origin was excluded from our considerations). Furthermore,the Tolman formula [3],[4] for the total energy of a static and asymptotically flat spacetime with g the determinant of the four dimensional

metric and d^3x the coordinate volume element, gives

$$E_T = \int \left(\mathbf{T}_r^r + \mathbf{T}_\theta^\theta + \mathbf{T}_\phi^\phi + \mathbf{T}_t^t\right)\sqrt{-g}\, d^3x = m, \tag{1.12.18}$$

as it should be.

The paper is organized in the following way: in section II we discuss the conceptual as well as the mathematical prerequisites. In particular we comment on geometrical matters (differentiable structure, coordinate invariance) and recall the basic facts of nonlinear superdistributional geometry in the context of algebras $\widetilde{\mathcal{G}}(M,\Sigma)$ of supergeneralized functions. Moreover, we derive sensible nonsmooth regularizations of the singular functions to be used throughout the paper. Section III is devoted to these approach to the problem. We present a new conceptually satisfactory method to derive the main result. In these final section III we investigate the horizon and describe its distributional curvature. Using nonlinear superdistributional geometry and supergeneralized functions it seems possible to show that the horizon singularity is not only a coordinate singularity without leaving Schwarzschild coordinates.

1.12.5. Distributional Eddington-Finkelstein spacetime.

In physical literature many years exist belief that Schwarzschild spacetime $(S^2 \times \{r > 2m\}) \times \mathbb{R}$ is extendible, in the sense that it can be immersed in a larger spacetime whose manifold is not covered by the canonical Schwarzschild coordinate with $r > 2m$. In physical literature [4],[33], [34],[35] one considers the formal change of coordinates obtained by replacing the canonical Schwarzschild time by "retarded time" above horizon v^+ given when $r > 2m$ by

$$v^+ = t + r + 2m \ln\left(\frac{r}{2m} - 1\right). \tag{1.12.19}$$

From (1.12.19) it follows for $r > 2m$

$$dt = -\frac{dr}{1 - \frac{2m}{r}} + dv^+. \tag{1.12.20}$$

The Schwarzschild metric (1.10.44) above horizon ds^{+2} (see section 3) in this coordinate obviously takes the form

$$ds^{+2} = -\left(1 - \frac{2m}{r}\right)dv^{+2} + 2drdv^- + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \tag{1.12.21}$$

When $r < 2m$ we replace (1.12.19) below horizon by

$$v^- = t + r + 2m \ln\left(1 - \frac{r}{2m}\right) \tag{1.12.22}$$

From (1.12.22) it follows for $r < 2m$

$$dt = \frac{dr}{\frac{2m}{r} - 1} + dv^-. \tag{1.12.23}$$

The Schwarzschild metric (1.10.44) below horizon ds_-^2 (see section 3) in this coordinate obviously takes the form

$$ds^{-2} = \left(\frac{2m}{r} - 1\right)dv^{-2} - 2drdv^- + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \tag{1.12.24}$$

Remark 1.12.6.(i) Note that the metric (1.12.21) is well defined on the manifold $S^2 \times (r > 0) \times \mathbb{R}$ and obviously it is regular Lorentzian metric: its coefficients are smooth.

(ii) The term $2drdv$ ensures its non-degeneracy for $r = 2m$.

(iii) Due to the nondegeneracy of the metric (1.12.24) the Levi-Civita connection

$$\left\{ \Gamma_{kj}^{+l}(\{\}) == \frac{1}{2}[g^{lm}(\{\})][(g_{mk,j}(\{\}) + g_{mj,k}(\{\}) - g_{kj,m}(\{\})] \right. \tag{1.12.25}$$

obviously now available and therefore nonsingular on horizon in contrast with Schwarzschild metric one obtains [3]:

$$\Gamma^{v}_{vv} = \frac{r_s}{2r^2}, \Gamma^{r}_{vv} = \frac{r_s(r-r_s)}{2r^3}, \Gamma^{r}_{vr} = -\frac{r_s}{2r^2}, \Gamma^{\theta}_{r\theta} = \frac{1}{r},$$
$$\Gamma^{\varphi}_{r\varphi} = \frac{1}{r}, \Gamma^{v}_{\theta\theta} = -r, \Gamma^{r}_{\theta\theta} = -r(r-r_s), \Gamma^{\varphi}_{\theta\varphi} = \cot\theta, \quad (1.12.26)$$
$$\Gamma^{v}_{\varphi\varphi} = -r\sin^2\theta, \Gamma^{r}_{\varphi\varphi} = -r(r-r_s)\sin^2\theta, \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta.$$

(**iv**) In physical literature [3],[4] by using properties (**i**)-(**iii**) this spacetime wrongly convicted as an rigorous mathematical extension of the Schwarzschild spacetime.

Remark 1.12.7. Let us consider now the coordinates: (i) $v^{+}, r' = r, \theta' = \theta, \varphi' = \varphi$ and (ii) $v^{-}, r' = r, \theta' = \theta,\ \varphi' = \varphi$. Obviously both transformations given by Eq.(1.12.20) and Eq.(1.12.23) are singular because the both Jacobian of these transformations is singular at $r = 2m$:

$$\begin{pmatrix} \frac{\partial v^{+}}{\partial t} & \frac{\partial v^{+}}{\partial r} \\ \frac{\partial r'}{\partial t} & \frac{\partial r'}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & \frac{r}{r-2m} \\ 0 & 1 \end{pmatrix} \quad (1.12.27)$$

and

$$\begin{pmatrix} \frac{\partial v^{-}}{\partial t} & \frac{\partial v^{-}}{\partial r} \\ \frac{\partial r'}{\partial t} & \frac{\partial r'}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{r}{2m-r} \\ 0 & 1 \end{pmatrix}. \quad (1.12.28)$$

Remark 1.12.8. Note first (i) such singular transformations not allowed in conventional Lorentzian geometry and second (ii) both Eddington-Finkelstein metrics given by Eq.(1.12.21) and by Eq.(1.12.24) well defined in rigorous mathematical sence at $r = 2m.$

Remark 1.12.9. (**I**) From consideration above follows that Schwarzschild spacetime $(S^2 \times \{r > 2m\}) \times \mathbb{R}$ is not extendible, in the sense that it can be immersed in a larger spacetime whose manifold is not covered by the canonical Schwarzschild coordinate with $r > 2m.$ Thus Eddington-Finkelstein spacetime cannot be considered as an extension of the Schwarzschild spacetime in natural way in respect with conventional Lorentzian geometry. Such "extension" are the extension by abnormal definition and nothing more. (**II**) However distributional Eddington-Finkelstein spacetime (1.10.53) is equivalent of the distributional Schwarzschild spacetime in natural way.

Remark 1.12.10. From consideration above follows that it is necessary an regularization of the Eq.(1.12.20) and Eq.(1.12.23) on horizon. However obviously only nonsmooth regularization via horizon $r = 2m$ possible. Under nonsmooth regularization (see section 3) Eq.(1.12.20) and Eq.(1.12.23) takes the form

$$dt = -\frac{dr}{\frac{1}{r}\sqrt{(r-2m)^2 + \epsilon^2}} + dv^{+}_{\epsilon}, \quad (1.12.29)$$
$$\epsilon \in (0,1]$$

and

$$dt = \frac{dr}{\frac{1}{r}\sqrt{(2m-r)^2 + \epsilon^2}} + dv^{-}_{\epsilon}, \quad (1.12.30)$$
$$\epsilon \in (0,1]$$

correspondingly. Therefore Eq.(1.12.27)-Eq.(1.12.28) takes the form

$$\begin{pmatrix} \frac{\partial v_\epsilon^+}{\partial t} & \frac{\partial v_\epsilon^+}{\partial r} \\ \frac{\partial r'}{\partial t} & \frac{\partial r'}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & \frac{r}{\sqrt{(r-2m)^2+\epsilon^2}} \\ 0 & 1 \end{pmatrix} \tag{1.12.32}$$

and

$$\begin{pmatrix} \frac{\partial v_\epsilon^-}{\partial t} & \frac{\partial v_\epsilon^-}{\partial r} \\ \frac{\partial r'}{\partial t} & \frac{\partial r'}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{r}{\sqrt{(2m-r)^2+\epsilon^2}} \\ 0 & 1 \end{pmatrix}. \tag{1.12.33}$$

From Eq.(1.12.29)-Eq.(1.12.30) one obtain generalized Eddington-Finkelstein transformatios

$$dt = -\frac{rdr}{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon} + (dv_\epsilon^+)_\epsilon, \quad \epsilon \in (0,1] \tag{1.12.34}$$

and

$$dt = \frac{rdr}{\left(\sqrt{(2m-r)^2+\epsilon^2}\right)_\epsilon} + (dv_\epsilon^-)_\epsilon, \quad \epsilon \in (0,1]. \tag{1.12.35}$$

Therefore Eq.(1.12.32)-Eq.(1.12.33) takes the form

$$\begin{pmatrix} \left(\frac{\partial v_\epsilon^+}{\partial t}\right)_\epsilon & \left(\frac{\partial v_\epsilon^+}{\partial r}\right)_\epsilon \\ \frac{\partial r'}{\partial t} & \frac{\partial r'}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & \frac{r}{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon} \\ 0 & 1 \end{pmatrix} \tag{1.12.36}$$

and

$$\begin{pmatrix} \left(\frac{\partial v_\epsilon^-}{\partial t}\right)_\epsilon & \left(\frac{\partial v_\epsilon^-}{\partial r}\right)_\epsilon \\ \frac{\partial r'}{\partial t} & \frac{\partial r'}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{r}{\left(\sqrt{(2m-r)^2+\epsilon^2}\right)_\epsilon} \\ 0 & 1 \end{pmatrix}. \tag{1.12.37}$$

At point $r = 2m$ one obtain

$$\left.\begin{pmatrix} \left(\frac{\partial v_\epsilon^+}{\partial t}\right)_\epsilon & \left(\frac{\partial v_\epsilon^+}{\partial r}\right)_\epsilon \\ \frac{\partial r'}{\partial t} & \frac{\partial r'}{\partial r} \end{pmatrix}\right|_{r=2m} = \begin{pmatrix} 1 & r(\epsilon^{-1})_\epsilon \\ 0 & 1 \end{pmatrix} \tag{1.12.38}$$

and

$$\left.\begin{pmatrix} \left(\frac{\partial v_\epsilon^-}{\partial t}\right)_\epsilon & \left(\frac{\partial v_\epsilon^-}{\partial r}\right)_\epsilon \\ \frac{\partial r'}{\partial t} & \frac{\partial r'}{\partial r} \end{pmatrix}\right|_{r=2m} = \begin{pmatrix} 1 & -r(\epsilon^{-1})_\epsilon \\ 0 & 1 \end{pmatrix}, \tag{1.12.39}$$

where $(\epsilon^{-1})_\epsilon \in \widetilde{\mathbb{R}}$. Thus generalized Eddington-Finkelstein transformations (1.12.34)-(1.12.35) well defined in the sense of Colombeau generalized functions. From the

Eq.(1.12.34) one obtains

$$dt^2 = \left[-\frac{rdr}{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon} + (dv_\epsilon^+) \right]^2 = \frac{r^2dr^2}{\left((r-2m)^2+\epsilon^2\right)_\epsilon} -$$
$$-\frac{2rdr(dv_\epsilon^+)}{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon} + (dv_\epsilon^{+2})_\epsilon,$$
$$dt^2(h_\epsilon^+(r))_\epsilon = -\frac{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon}{r}dt^2 = \tag{1.12.40}$$
$$-\frac{rdr^2}{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon} + 2dr(dv_\epsilon^+) - \frac{1}{r}\sqrt{(r-2m)^2+\epsilon^2}\,(dv_\epsilon^{+2})_\epsilon.$$

From the Eq.(1.12.35) one obtains

$$dt^2 = \left[\frac{rdr}{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon} + (dv_\epsilon^-) \right]^2 = \frac{r^2dr^2}{\left((r-2m)^2+\epsilon^2\right)_\epsilon} +$$
$$+\frac{2rdr(dv_\epsilon^-)}{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon} + (dv_\epsilon^{-2})_\epsilon,$$
$$dt^2(h_\epsilon^-(r))_\epsilon = \frac{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon}{r}dt^2 = \tag{1.12.41}$$
$$\frac{rdr^2}{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon} + 2dr(dv_\epsilon^-) + \frac{1}{r}\sqrt{(r-2m)^2+\epsilon^2}\,(dv_\epsilon^{-2})_\epsilon.$$

Substituting Eqs.(1.12.40)-(1.12.41) into distributional Schwarzschild metric (1.12.42) above (below) gorizon (see subsect.3.1)

$$(ds_\epsilon^{+2})_\epsilon = (h_\epsilon^+(r)dt^2)_\epsilon + \left([h_\epsilon^+(r)]^{-1}dr^2\right)_\epsilon + r^2d\Omega^2,$$
$$(ds_\epsilon^{-2})_\epsilon = (h_\epsilon^-(r)dt^2)_\epsilon - \left([h_\epsilon^-(r)]^{-1}dr^2\right)_\epsilon + r^2d\Omega^2, \tag{1.12.42}$$
$$(h_\epsilon^+(r))_\epsilon = -\frac{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon}{r}, (h_\epsilon^-(r))_\epsilon = \frac{\left(\sqrt{(r-2m)^2+\epsilon^2}\right)_\epsilon}{r}$$

one obtains

$$(ds_\epsilon^{+2})_\epsilon = -\frac{1}{r}\sqrt{(r-2m)^2+\epsilon^2}\,dv_\epsilon^{+2} + 2drdv_\epsilon^+ + r^2[(d\theta)^2+\sin^2\theta(d\phi)^2].$$
$$(ds_\epsilon^{-2})_\epsilon = \frac{1}{r}\sqrt{(2m-r)^2+\epsilon^2}\,dv_\epsilon^{-2} + 2drdv_\epsilon^- + r^2[(d\theta)^2+\sin^2\theta(d\phi)^2]. \tag{1.12.43}$$

Therefore Colombeau generalized object modeling the classical Eddington-Finkelstein metric given Eq.(1.12.21) and Eq.(1.10.24) above and below gorizon takes the form

$$(ds^{+2}_{\epsilon})_{\epsilon} = h^{+}_{\epsilon}(r)dv^{+2}_{\epsilon} + 2drdv^{+}_{\epsilon} + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2],$$
$$(ds^{-2}_{\epsilon})_{\epsilon} = h^{-}_{\epsilon}(r)dv^{-2}_{\epsilon} + 2drdv^{-}_{\epsilon} + r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \quad (1.12.44)$$

It easily to verify by using formula A.2 (see appendix) that the distributional curvature scalar $(\mathbf{R}(\epsilon))_{\epsilon}$ again singular at $r = 2m$ as in the case of the distributional Schwarzschild spacetime given by Eq.(1.12.42). However this is not surprising because the classical Eddington- Finkelstein spacetime and generalized Eddington-Finkelstein specetime given by Eq.(1.12.44) that is essentially different geometrical objects.

2. Generalized Colombeau Calculus

2.1.Notation and basic notions from standard Colombeau theory.

We use [1],[2],[7] as standard references for the foundations and various applications of standard Colombeau theory. We briefly recall the basic Colombeau construction. Throughout the paper Ω will denote an open subset of $\mathbb{R}^n$. Stanfard Colombeau generalized functions on Ω are defined as equivalence classes $u = [(u_\varepsilon)_\varepsilon]$ of nets of smooth functions $u_\varepsilon \in C^\infty(\Omega)$ (regularizations) subjected to asymptotic norm conditions with respect to $\varepsilon \in (0,1]$ for their derivatives on compact sets.

The basic idea of *classical Colombeau's theory of nonlinear generalized functions* [1],[2] is regularization by sequences (nets) of smooth functions and the use of asymptotic estimates in terms of a regularization parameter ε. Let $(u_\varepsilon)_{\varepsilon\in(0,1]}$ with $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M)$ for all $\varepsilon \in \mathbb{R}_+$,where M a separable, smooth orientable Hausdorff manifold of dimension n.

Definition 2.1.1. The classical Colombeau's algebra of generalized functions on M is defined as the quotient:

$$\mathcal{G}(M) \triangleq \mathcal{E}_M(M)/\mathcal{N}(M) \quad (2.1.1)$$

of the space $\mathcal{E}_M(M)$ of sequences of moderate growth modulo the space $\mathcal{N}(M)$ of negligible sequences. More precisely the notions of moderateness resp. negligibility are defined by the following asymptotic estimates (where $\mathfrak{X}(M)$ denoting the space of smooth vector fields on M):

$$\begin{cases} \mathcal{E}_M(M) \triangleq \{(u_\varepsilon)_\varepsilon | \ \forall K(K \subsetneqq M)\, \forall k(k \in \mathbb{N})\, \exists N(N \in \mathbb{N}) \\ \forall \xi_1,\dots,\xi_k(\xi_1,\dots,\xi_k \in \mathfrak{X}(M)) \left[\sup_{p\in K} |L_{\xi_1}\dots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0 \right] \Big\}, \end{cases} \quad (2.1.2)$$

$$\begin{cases} \mathcal{N}(M) \triangleq \{(u_\varepsilon)_\varepsilon | \ \forall K(K \subsetneqq M),\, \forall k(k \in \mathbb{N}_0)\forall q(q \in N) \\ \forall \xi_1,\dots,\xi_k(\xi_1,\dots,\xi_k \in \mathfrak{X}(M)) \left[\sup_{p\in K} |L_{\xi_1}\dots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0 \right] \Big\}. \end{cases} \quad (2.1.3)$$

Remark 2.1.1. In the definition the Landau symbol $a_\varepsilon = O(\psi(\varepsilon))$ appears, having the following meaning: $\exists C(C > 0)\exists\varepsilon_0(\varepsilon_0 \in (0,1])\forall\varepsilon(\varepsilon < \varepsilon_0)[a_\varepsilon \leq C\psi(\varepsilon)]$.

Definition 2.1.2. Elements of $\mathcal{G}(M)$ are denoted by:

$$u = \mathbf{cl}[(u_\varepsilon)_\varepsilon] \triangleq (u_\varepsilon)_\varepsilon + \mathcal{N}(M). \quad (2.1.4)$$

Remark 2.1.2. With componentwise operations $(\cdot,\pm)$ $\mathcal{G}(M)$ is a fine sheaf of differential algebras with respect to the Lie derivative defined by $L_\xi u \triangleq \mathbf{cl}[(L_\xi u_\varepsilon)_\varepsilon]$.

The spaces of moderate resp. negligible sequences and hence the algebra itself may be

characterized locally, i.e., $u \in \mathcal{G}(M)$ iff $u \circ \psi_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))$ for all charts (V_α, ψ_α), where on

the open set $\psi_\alpha(V_\alpha) \subset \mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives.

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Remark 2.1.3. Smooth functions $f \in \mathcal{C}^\infty(M)$ are embedded into $\mathcal{G}(M)$ simply by the "constant" embedding σ, i.e., $\sigma(f) = \mathbf{cl}[(f)_\varepsilon]$, hence $\mathcal{C}^\infty(M)$ is a faithful subalgebra of $\mathcal{G}(M)$.

2.2.Point Values of a Generalized Functions on M. Generalized Numbers.

Within the classical distribution theory, distributions cannot be characterized by their point values in any way similar to classical functions. On the other hand, there is a very natural and direct way of obtaining the point values of the elements of Colombeau's algebra: points are simply inserted into representatives. The objects so obtained are sequences of numbers, and as such are not the elements in the field $\mathbb{R}$ or $\mathbb{C}$. Instead, they are the representatives of *Colombeau's generalized numbers.* We give the exact definition of these "numbers".

Definition 2.2.1. Inserting $p \in M$ into $u \in \mathcal{G}(M)$ yields a well defined element of the ring of constants (also called generalized numbers) $\mathcal{K}$ (corresponding to $\mathbf{K} = \mathbb{R}$ resp. $\mathbb{C}$), defined as the set of moderate nets of numbers ($(r_\varepsilon)_\varepsilon \in \mathbf{K}^{(0,1]}$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some N) modulo negligible nets ($|r_\varepsilon| = O(\varepsilon^m)$ for each m); componentwise insertion of points of M into elements of $\mathcal{G}(M)$ yields well-defined generalized numbers, i.e.,elements of the ring of constants:

$$\mathcal{K} = \mathcal{E}_{\mathbf{c}}(M)/\mathcal{N}_{\mathbf{c}}(M) \tag{2.2.1}$$

(with $\mathcal{K} = \widetilde{\mathbb{R}}$ or $\mathcal{K} = \widetilde{\mathbb{C}}$ for $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$), where

$$\begin{cases} \mathcal{E}_{\mathbf{c}}(M) = \left\{(r_\epsilon)_\epsilon \in \mathbf{K}^I | \exists n(n \in \mathbb{N})\left[|r_\epsilon| = O(\epsilon^{-n}) \text{ as } \varepsilon \to 0\right]\right\} \\ \mathcal{N}_{\mathbf{c}}(M) = \left\{(r_\epsilon)_\epsilon \in \mathbf{K}^I | \forall m(m \in \mathbb{N})\left[|r_\epsilon| = O(\epsilon^{m}) \text{ as } \varepsilon \to 0\right]\right\} \\ I = (0,1]. \end{cases} \tag{2.2.2}$$

Generalized functions on M are characterized by their generalized point values, i.e., by their values on points in $\tilde{M}_c$, the space of equivalence classes of compactly supported nets $(p_\varepsilon)_\varepsilon \in M^{(0,1]}$ with respect to the relation $p_\varepsilon \sim p'_\varepsilon :\Leftrightarrow d_h(p_\varepsilon, p'_\varepsilon) = O(\varepsilon^m)$ for all m, where d_h denotes the distance on M induced by any Riemannian metric.

Definition 2.2.2. For $u \in \mathcal{G}(M)$ and $x_0 \in M$, the point value of u at the point $x_0, u(x_0)$, is defined as the class of $(u_\varepsilon(x_0))_\varepsilon$ in $\mathcal{K}$.

Definition 2.2.3. We say that an element $r \in \mathcal{K}$ is *strictly nonzero* if there exists a representative $(r_\varepsilon)_\varepsilon$ and a $q \in \mathbb{N}$ such that $|r_\varepsilon| \geqslant \varepsilon^q$ for ε sufficiently small. If r is strictly nonzero, then it is also invertible with the inverse $[(1/r_\varepsilon)_\varepsilon]$. The converse is true as well.

Treating the elements of Colombeau algebras as a generalization of classical functions,
the question arises whether the definition of point values can be extended in such a way
that each element is characterized by its values. Such an extension is indeed possible.

Definition 2.2.4. Let Ω be an open subset of $\mathbb{R}^n$. On a set $\hat{\Omega}$:

$$\begin{cases} \hat{\Omega} = \{(x_\varepsilon)_\varepsilon \in \Omega^I | \exists p(p > 0)[|x_\varepsilon| = O(\varepsilon^p)]\} = \\ \left\{(x_\varepsilon)_\varepsilon \in \Omega^I | \exists p(p > 0)\exists \varepsilon_0(\varepsilon_0 > 0)\left[|x_\varepsilon| \leq \varepsilon^p, \text{ for } 0 < \varepsilon < \varepsilon_0\right]\right\} \end{cases} \tag{2.2.3}$$

we introduce an equivalence relation:

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow \forall q(q > 0)\forall \varepsilon(\varepsilon > 0)\left[|x_\varepsilon - y_\varepsilon| \leq \varepsilon^q, \text{ for } 0 < \varepsilon < \varepsilon_0\right] \tag{2.2.4}$$

and denote by $\widetilde{\Omega} = \hat{\Omega}/\sim$ the set of generalized points. The set of points with compact support is

$$\widetilde{\Omega}_{\mathbf{c}} = \left\{\widetilde{x} = \mathbf{cl}[(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega} | \exists K(K \subset \Omega)\exists \varepsilon_0(\varepsilon_0 > 0)\left[x_\varepsilon \in K \text{ for } 0 < \varepsilon < \varepsilon_0\right]\right\} \tag{2.2.5}$$

Definition 2.2.5. A generalized function $u \in \mathcal{G}(M)$ is called associated to zero, $u \approx 0$ on
$\Omega \subseteq M$ in L.Schwartz sense if one (hence any) representative $(u_\epsilon)_\epsilon$ converges to zero weakly,i.e.

$$w\text{-}\lim_{\epsilon \to 0} u_\epsilon = 0 \tag{2.2.6}$$

We shall often write:

$$u \underset{\mathbf{Sch}}{\approx} 0. \tag{2.2.7}$$

The $\mathcal{G}(M)$-module of generalized sections in vector bundles-especially the space of generalized tensor fields $\mathcal{T}_s{}^r(M)$-is defined along the same lines using analogous asymptotic estimates with respect to the norm induced by any Riemannian metric on the respective fibers. However, it is more convenient to use the following algebraic description of generalized tensor fields

$$\mathcal{G}^r_s(M) = \mathcal{G}(M) \otimes \mathcal{T}_s{}^r(M), \tag{2.2.8}$$

where $\mathcal{T}_s{}^r(M)$ denotes the space of smooth tensor fields and the tensor product is taken over the module $C^\infty(M)$. Hence generalized tensor fields are just given by classical ones with generalized coefficient functions. Many concepts of classical tensor analysis carry over to the generalized setting [1]-[2], in particular Lie derivatives with respect to both classical and generalized vector fields, Lie brackets, exterior algebra, etc. Moreover, generalized tensor fields may also be viewed as $\mathcal{G}(M)$-multilinear maps taking generalized vector and covector fields to generalized functions, i.e., as $\mathcal{G}(M)$-modules we have

$$\mathcal{G}^r_s(M) \cong L_{(M)}(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M)). \tag{2.2.9}$$

In particular a generalized metric is defined to be a symmetric, generalized $(0,2)$-tensor field $g_{ab} = [((g_\epsilon)_{ab})_\epsilon]$ (with its index independent of ε and) whose determinant $\det(g_{ab})$ is invertible in $\mathcal{G}(M)$. The latter condition is equivalent to the following notion called strictly nonzero on compact sets: for any representative $\det((g_\epsilon)_{ab})_\epsilon$ of $\det(g_{ab})$ we have $\forall K \subset M\, \exists m \in \mathbb{N}[\inf_{p\in K}|\det(g_{ab}(\epsilon))| \geq \epsilon^m]$ for all ϵ small enough. This notion captures the intuitive idea of a generalized metric to be a sequence of classical metrics approaching a singular limit in the following sense: g_{ab} is a generalized metric iff (on every relatively compact open subset V of M) there exists a representative $((g_\epsilon)_{ab})_\epsilon$ of g_{ab} such that for fixed ϵ (small enough)$(g_\epsilon)_{ab} = g_{ab}(\epsilon)$ (resp. $(g_\epsilon)_{ab}|_V$) is a classical pseudo-Riemannian metric and $\det(g_{ab})$ is invertible in the algebra of generalized functions. A generalized metric induces a $\mathcal{G}(M)$-linear isomorphism from $\mathcal{G}_0^1(M)$ to $\mathcal{G}_1^0(M)$ and the inverse metric $g^{ab} \triangleq [(g_{ab}^{-1}(\epsilon))_\epsilon]$ is a well defined element of $\mathcal{G}_0^2(M)$ (i.e., independent of the representative $((g_\epsilon)_{ab})_\epsilon$). Also the generalized Levi-Civita connection as well as the generalized Riemann-, Ricci- and Einstein tensor of a generalized metric are defined simply by the usual coordinate formulae on the level of representatives.

2.3. The nonlinear distributional Schwarzschild geometry. The truncated distributional Schwarzschild geometry. Gravitational singularity.

There exist two different types of distributional Colombeau solution of the Generalized Einstein's Field Equations (1.9.18) corresponding to classical Schwarzschild solution. That is: (i) full distributional Schwarzschild blackhole geometry, given by Colombeau generalized object $\left\{(ds_\epsilon^{+2})_\epsilon, (ds_\epsilon^{-2})_\epsilon\right\}$, given by Equations (1.10.28), see Figure 2.3.1(a) and (ii) the truncated distributional Schwarzschild space-time given by Colombeau generalized object $(ds_\epsilon^{+2})_\epsilon$,i.e. in this case distributional space-time ends just on the Schwarzschild horizon, see Figure 2.3.1(b).

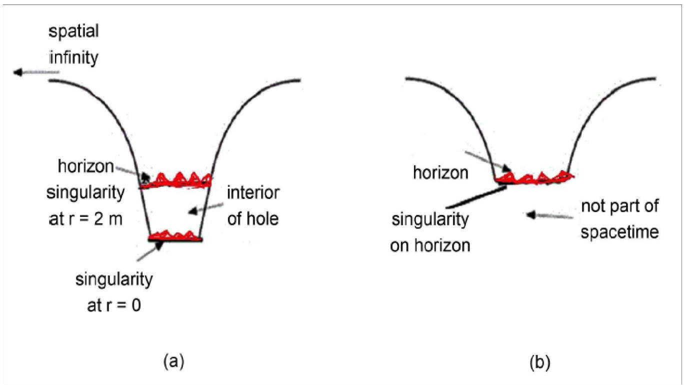

Fig.2.3.1.

Fig.2.3.1.(a) The picture of a distributional Schwarzschild blackhole, given by the full Colombeau generalized object (1.10.28). Distributional spacetime ends just on the Schwarzschild singularity. (b) The truncated Schwarzschild distributional geometry, given by Colombeau generalized object . Distributional spacetime ends just on the Schwarzschild horizon.

The Colombeau generalized Ricci tensor above horizon $[\mathbf{R}^+(r)]_\alpha^\beta$ (see Eq.3.1.4) reads

$$\begin{cases} \left([\mathbf{R}_\epsilon^+(r)]_0^0\right)_\epsilon = \left([\mathbf{R}_\epsilon^+(r)]_1^1\right)_\epsilon = \frac{1}{2}\left((h_\epsilon^{+''}(r))_\epsilon + \frac{2}{r}(h_\epsilon^{+'}(r))_\epsilon\right) \\ \left([\mathbf{R}_\epsilon^+(r)]_2^2\right)_\epsilon = \left([\mathbf{R}_\epsilon^+(r)]_3^3\right)_\epsilon = \frac{(h_\epsilon^{+'}(r))_\epsilon}{r} + \frac{1+(h_\epsilon^+(r))_\epsilon}{r^2}. \end{cases} \quad (2.3.1)$$

From Eq.(3.1.5) we obtain

$$\begin{gathered} \left(2r^2[\mathbf{R}_\epsilon^+(r)]_0^0\right)_\epsilon = \left(2r^2[\mathbf{R}_\epsilon^+(r)]_1^1\right)_\epsilon = r^2(h_\epsilon^{+''}(r))_\epsilon + 2r(h_\epsilon^{+'}(r))_\epsilon = \\ -\left(\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}\right)_\epsilon + \left(\frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}\right)_\epsilon = \\ -\frac{r}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{3/2}} \end{gathered} \quad (2.3.2)$$

and

$$\begin{gathered} \left(r^2[\mathbf{R}_\epsilon^+(r)]_2^2\right)_\epsilon = \left(r^2[\mathbf{R}_\epsilon^+(r)]_3^3\right)_\epsilon = r(h_\epsilon^{+'}(r))_\epsilon + 1 + (h_\epsilon^+(r))_\epsilon = \\ -\left(\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}\right)_\epsilon + 1 = -\frac{r-2m}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{1/2}} + 1. \end{gathered} \quad (2.3.3)$$

For any $r-2m \in \mathbb{R}_+$ from Eq.(2.3.2)-Eq.(2.3.3) one obtains

$$\begin{gathered} \left(2r^2[\mathbf{R}_\epsilon^+(r)]_0^0\right)_\epsilon = \left(2r^2[\mathbf{R}_\epsilon^+(r)]_1^1\right)_\epsilon = r^2(h_\epsilon^{+''})_\epsilon + 2r(h_\epsilon^{+'})_\epsilon = \\ -\frac{r}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{3/2}} = \\ -\frac{r}{(r-2m)\left[1+(r-2m)^{-2}(\epsilon^2)_\epsilon\right]^{1/2}} + \frac{r(r-2m)^2}{(r-2m)^3\left[1+(r-2m)^{-2}(\epsilon^2)_\epsilon\right]^{3/2}} = \\ -\frac{r}{(r-2m)} + \frac{r}{(r-2m)} + O\left((\epsilon^2)_\epsilon\right) = O\left((\epsilon^2)_\epsilon\right) \end{gathered} \quad (2.3.4)$$

and

$$\begin{gathered} \left(r^2[\mathbf{R}_\epsilon^+(r)]_2^2\right)_\epsilon = \left(r^2[\mathbf{R}_\epsilon^+(r)]_3^3\right)_\epsilon = r(h_\epsilon^{+'}(r))_\epsilon + 1 + (h_\epsilon^+(r))_\epsilon = \\ -\frac{r-2m}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{1/2}} + 1 = -\frac{r-2m}{(r-2m)\left[1+(r-2m)^{-2}(\epsilon^2)_\epsilon\right]^{1/2}} + 1 = \\ -1+1+O\left((\epsilon^2)_\epsilon\right) = O\left((\epsilon^2)_\epsilon\right), \end{gathered} \quad (2.3.5)$$

where $\epsilon \in (0,\delta], \delta \ll 1$.

Thus for any $r-2m \in \mathbb{R}_+$ the nonlinear distributional Schwarzschild geometry returns the classical result.

For any $r-2m = 0_{\widetilde{\mathbb{R}}}$ from Eq.(2.3.2)-Eq.(2.3.3) in contrast with ubnormal canonical result

one obtains

$$\left([\mathbf{R}_{\epsilon}^{+}(r)]_0^0\right)_{\epsilon}\Big|_{r=2m} = \left([\mathbf{R}_{\epsilon}^{+}(r)]_1^1\right)_{\epsilon}\Big|_{r=2m} =$$

$$-\frac{r}{4m^2\left[(r-2m)^2+(\epsilon^2)_{\epsilon}\right]^{1/2}}\Bigg|_{r=2m} + \frac{r(r-2m)^2}{4m^2\left[(r-2m)^2+(\epsilon^2)_{\epsilon}\right]^{3/2}}\Bigg|_{r=2m} = \tag{2.3.6}$$

$$= -2^{-1}m^{-1}(\epsilon^{-1})_{\epsilon}$$

and

$$\left([\mathbf{R}_{\epsilon}^{+}(r)]_2^2\right)_{\epsilon}\Big|_{r=2m} = \left([\mathbf{R}_{\epsilon}^{+}(r)]_3^3\right)_{\epsilon}\Big|_{r=2m} =$$

$$-\frac{r-2m}{4m^2\left[(r-2m)^2+(\epsilon^2)_{\epsilon}\right]^{1/2}}\Bigg|_{r=2m} + 4^{-1}m^{-2} = 4^{-1}m^{-2}. \tag{2.3.7}$$

It follows from Eq.(2.3.7)

$$\left([\mathbf{R}_{\epsilon}^{+}(r)]_2^2\right)_{\epsilon} = \left([\mathbf{R}_{\epsilon}^{+}(r)]_3^3\right)_{\epsilon} = 4^{-1}m^{-2}\psi_{+}(r,2m), \tag{2.3.8}$$

where

$$\psi_{+}(r,2m) = \begin{cases} 0 & \text{if} \quad r > 2m \\ \\ 1 & \text{if} \quad r = 2m \end{cases} \tag{2.3.9}$$

The Colombeau generalized Ricci tensor below horizon $[\mathbf{R}_{\epsilon}^{-}]_{\alpha}^{\beta} = [\mathbf{R}_{\epsilon}^{-}]_{\alpha}^{\beta}$(see Eq.3.1.21) reads

$$\begin{gathered}\left([\mathbf{R}_{\epsilon}^{-}(r)]_0^0\right)_{\epsilon} = \left([\mathbf{R}_{\epsilon}^{-}(r)]_1^1\right)_{\epsilon} = \frac{1}{2}\left((h_{\epsilon}^{-\prime\prime}(r))_{\epsilon} + \frac{2}{r}(h_{\epsilon}^{-\prime}(r))_{\epsilon}\right),\\ \left([\mathbf{R}_{\epsilon}^{-}(r)]_2^2\right)_{\epsilon} = \left([\mathbf{R}_{\epsilon}^{-}(r)]_3^3\right)_{\epsilon} = \frac{(h_{\epsilon}^{-\prime}(r))_{\epsilon}}{r} + \frac{1+(h_{\epsilon}^{-}(r))_{\epsilon}}{r^2}.\end{gathered} \tag{2.3.10}$$

From Eq.(3.1.22) we obtain

$$\begin{gathered}\left(2r^2[\mathbf{R}_{\epsilon}^{-}(r)]_0^0\right)_{\epsilon} = \left(2r^2[\mathbf{R}_{\epsilon}^{-}(r)]_1^1\right)_{\epsilon} =\\ \frac{r}{\left[(r-2m)^2+(\epsilon^2)_{\epsilon}\right]^{1/2}} - \frac{r(r-2m)^2}{\left[(r-2m)^2+(\epsilon^2)_{\epsilon}\right]^{3/2}}\end{gathered} \tag{2.3.11}$$

and

$$\left(r^2[\mathbf{R}_{\epsilon}^{-}(r)]_2^2\right)_{\epsilon} = \left(r^2[\mathbf{R}_{\epsilon}^{-}(r)]_3^3\right)_{\epsilon} = \frac{r-2m}{\left[(r-2m)^2+(\epsilon^2)_{\epsilon}\right]^{1/2}} + 1. \tag{2.3.12}$$

For any $r-2m \in \mathbb{R}_{-}$ from Eq.(2.3.11)-Eq.(2.3.12) one obtains

$$\left(2r^2[\mathbf{R}_\epsilon^-(r)]_0^0\right)_\epsilon = \left(2r^2[\mathbf{R}_\epsilon^-(r)]_1^1\right)_\epsilon =$$
$$\frac{r}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{1/2}} - \frac{r(r-2m)^2}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{3/2}} =$$
$$\frac{r}{(r-2m)\left[1+(r-2m)^{-2}(\epsilon^2)_\epsilon\right]^{1/2}} - \frac{r(r-2m)^2}{(r-2m)^3\left[1+(r-2m)^{-2}(\epsilon^2)_\epsilon\right]^{3/2}} = \quad (2.3.13)$$
$$\frac{r}{(r-2m)} - \frac{r}{(r-2m)} + O\left((\epsilon^2)_\epsilon\right) = O\left((\epsilon^2)_\epsilon\right)$$

and

$$\left(r^2[\mathbf{R}_\epsilon^-(r)]_2^2\right)_\epsilon = \left(r^2[\mathbf{R}_\epsilon^-(r)]_3^3\right)_\epsilon = \frac{r-2m}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{1/2}} + 1 = O\left((\epsilon^2)_\epsilon\right) \quad (2.3.14)$$

where $\epsilon \in (0,\delta], \delta \ll 1$.
Thus for any $r-2m \in \mathbb{R}_-$ the nonlinear distributional Schwarzschild geometry returns the classical result.
For any $r-2m = 0_{\widetilde{\mathbb{R}}}$ from Eq.(2.3.11)-Eq.(2.3.12) in contrast with ubnormal canonical result one obtains

$$\left(2r^2[\mathbf{R}_\epsilon^-(r)]_0^0\right)_\epsilon\Big|_{r=2m} = \left(2r^2[\mathbf{R}_\epsilon^-(r)]_1^1\right)_\epsilon\Big|_{r=2m} =$$
$$\frac{r}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{1/2}}\Bigg|_{r=2m} - \frac{r(r-2m)^2}{\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{3/2}}\Bigg|_{r=2m} = 2^{-1}m^{-1}(\epsilon^{-1})_\epsilon \quad (2.3.15)$$

and

$$\left([\mathbf{R}_\epsilon^-(r)]_2^2\right)_\epsilon\Big|_{r=2m} = \left([\mathbf{R}_\epsilon^-(r)]_3^3\right)_\epsilon\Big|_{r=2m} =$$
$$\frac{r-2m}{4m^2\left[(r-2m)^2+(\epsilon^2)_\epsilon\right]^{1/2}}\Bigg|_{r=2m} + 4^{-1}m^{-2} = 4^{-1}m^{-2}. \quad (2.3.14)$$

It follows from Eq.(2.3.14)

$$\left([\mathbf{R}_\epsilon^-(r)]_2^2\right)_\epsilon = \left([\mathbf{R}_\epsilon^-(r)]_3^3\right)_\epsilon = 4^{-1}m^{-2}\psi_-(r,2m), \quad (2.3.15)$$

where

$$\psi_-(r,2m) = \begin{cases} 0 & \text{if} \quad r < 2m \\ 1 & \text{if} \quad r = 2m \end{cases} \quad (2.3.16)$$

Remark 2.3.1. Note that any sequence $[\mathbf{R}_\epsilon^+(r)]_0^0, [\mathbf{R}_\epsilon^+(r)]_1^1, [\mathbf{R}_\epsilon^+(r)]_2^2, [\mathbf{R}_\epsilon^+(r)]_3^3$ and any sequence $[\mathbf{R}_\epsilon^-(r)]_0^0, [\mathbf{R}_\epsilon^-(r)]_1^1, [\mathbf{R}_\epsilon^-(r)]_2^2, [\mathbf{R}_\epsilon^-(r)]_3^3$ has a weak limit in $\mathcal{D}'(\mathbb{R}^3)$: w-$\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^+(r)]_0^0 \in \mathcal{D}'(\mathbb{R}^3), \ldots,$ w-$\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^-(r)]_3^3 \in \mathcal{D}'(\mathbb{R}^3)$, see Section 3.

2.4. The nonlinear distributional Schwarzschild geometry in isotropic coordinates.

The Schwarzschild metric in isotropic coordinates (t,ρ,θ,ϕ) reads [3]:

$$ds^2 = -\left(\frac{1 - r_s/4\rho}{1 + r_s/4\rho}\right)^2 dt^2 + \left(1 + \frac{r_s}{4\rho}\right)^4 d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2.4.1)$$

see Eq.(1.1) where we let $c = 1$, or in the equivalent form

$$ds^2 = -\frac{(4\rho - r_s)^2}{4\rho + r_s} dt^2 + \left(1 + \frac{r_s}{4\rho}\right)^4 d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2.4.2)$$

In order to obtain Colombeau solutions of the generalized Einstein field equations (1.9.18), corresponding to the nonclassical semi Riemannian metric (2.4.2) we apply the regulariation $(4\rho - r_s)^2 \to (4\rho - r_s)^2 + \varepsilon^2$ and embed (2.4.2) in the Colombeau object

$$(ds_\varepsilon^2)_\varepsilon = -\frac{\left((4\rho - r_s)^2 + \varepsilon^2\right)_\varepsilon}{4\rho + r_s} dt^2 + \left(1 + \frac{r_s}{4\rho}\right)^4 d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2.4.3)$$

We rewrite now Eq.(2.4.3) in the following form (see Appendix A,Eq.(A.1))

$$\begin{gathered}(ds_\varepsilon^2)_\varepsilon = -[(A_\varepsilon(\rho))_\varepsilon]dt^2 + [B(\rho) + C(\rho)]d\rho^2 + \\ B(\rho)\rho^2(d\theta^2 + \sin^2\theta d\phi^2),\end{gathered} \qquad (2.4.4)$$

where

$$A_\varepsilon(\rho) = \frac{(4\rho - r_s)^2 + \varepsilon^2}{4\rho + r_s}, B(\rho) = 1, B(\rho) + C(\rho) = \left(1 + \frac{r_s}{4\rho}\right)^4 = G(\rho). \qquad (2.4.5)$$

The Colombeau generalized curvature scalar $(R_\varepsilon(\rho))_\varepsilon$ reads (see Appendix A,Eq.(A.2))

$$\begin{gathered}(R_\varepsilon(\rho))_\varepsilon = G(\rho)\left(\left[\frac{2}{\rho}\left(-\frac{A'_\varepsilon(\rho)}{A_\varepsilon(\rho)} + \frac{\Delta'_\varepsilon(\rho)}{\Delta_\varepsilon(\rho)}\right) + \frac{2C(\rho)}{\rho^2} - \frac{A''_\varepsilon(\rho)}{A_\varepsilon(\rho)}\right.\right. \\ \left.\left.+\frac{1}{2}\frac{A'_\varepsilon(\rho)}{A_\varepsilon(\rho)}\frac{\Delta'_\varepsilon(\rho)}{\Delta_\varepsilon(\rho)}\right]\right)_\varepsilon,\end{gathered} \qquad (2.4.6)$$

where

$$\begin{gathered}\Delta_\varepsilon(\rho) = A_\varepsilon(\rho)G(\rho), \Delta'_\varepsilon(\rho) = A'_\varepsilon(\rho)G(\rho) + A_\varepsilon(\rho)G'(\rho) \\ \frac{\Delta'_\varepsilon(\rho)}{\Delta_\varepsilon(\rho)} = \frac{A'_\varepsilon(\rho)G(\rho) + A_\varepsilon(\rho)G'(\rho)}{A_\varepsilon(\rho)G(\rho)} = \frac{A'_\varepsilon(\rho)}{A_\varepsilon(\rho)} + \frac{G'(\rho)}{G(\rho)}.\end{gathered} \qquad (2.4.7)$$

Substituting Eq.(2.4.7) into Eq.(2.4.3) we get

$$\begin{gathered}(R_\varepsilon(\rho))_\varepsilon = G(\rho)\left(\left[\frac{2}{\rho}\left(-\frac{A'_\varepsilon(\rho)}{A_\varepsilon(\rho)} + \frac{\Delta'_\varepsilon(\rho)}{\Delta_\varepsilon(\rho)}\right) + \frac{2C(\rho)}{\rho^2} - \frac{A''_\varepsilon(\rho)}{A_\varepsilon(\rho)}\right.\right. \\ \left.\left.+\frac{1}{2}\frac{A'_\varepsilon(\rho)}{A_\varepsilon(\rho)}\frac{\Delta'_\varepsilon(\rho)}{\Delta_\varepsilon(\rho)}\right]\right)_\varepsilon = \\ G(\rho)\left(\left[\frac{2}{\rho}\frac{G'(\rho)}{G(\rho)} + \frac{2C(\rho)}{\rho^2} - \frac{A''_\varepsilon(\rho)}{A_\varepsilon(\rho)} + \frac{1}{2}\frac{A'_\varepsilon(\rho)}{A_\varepsilon(\rho)}\left(\frac{A'_\varepsilon(\rho)}{A_\varepsilon(\rho)} + \frac{G'(\rho)}{G(\rho)}\right)\right]\right)_\varepsilon,\end{gathered} \qquad (2.4.8)$$

where

$$A_\varepsilon(\rho) = \frac{(4\rho - r_s)^2 + \varepsilon^2}{4\rho + r_s}, A'_\varepsilon(\rho) = \frac{8(4\rho - r_s)}{4\rho + r_s} - \frac{(4\rho - r_s)^2 + \varepsilon^2}{(4\rho + r_s)^2},$$

$$A''_\varepsilon(\rho) = \frac{64 r_s}{(4\rho + r_s)^2} + \frac{8(\varepsilon^2 + 2r_s^2 - 8\rho r_s)}{(4\rho + r_s)^3}, \tag{2.4.9}$$

$$\frac{A''_\varepsilon(\rho)}{A_\varepsilon(\rho)} = \frac{64 r_s}{\left[(4\rho - r_s)^2 + \varepsilon^2\right](4\rho + r_s)} + \frac{8}{(4\rho + r_s)^2}$$

and

$$\frac{A'_\varepsilon(\rho)}{A_\varepsilon(\rho)} = \frac{8(4\rho - r_s)}{(4\rho - r_s)^2 + \varepsilon^2} - \frac{1}{4\rho + r_s}. \tag{2.4.10}$$

From Eq.(2.4.9)-Eq.(2.4.10) we obtain

$$\left.\frac{A'_\varepsilon(\rho)}{A_\varepsilon(\rho)}\right|_{4\rho = r_s} = \left.\frac{8(4\rho - r_s)}{(4\rho - r_s)^2 + \varepsilon^2}\right|_{4\rho = r_s} - \left.\frac{1}{4\rho + r_s}\right|_{4\rho = r_s} = -\frac{1}{2r_s}$$

$$\left.\frac{A''_\varepsilon(\rho)}{A_\varepsilon(\rho)}\right|_{4\rho = r_s} = \left.\frac{8r_s(4\rho + r_s)}{\left[(4\rho - r_s)^2 + \varepsilon^2\right]}\right|_{4\rho = r_s} + \left.\frac{8}{(4\rho + r_s)^2}\right|_{4\rho = r_s} = \tag{2.4.11}$$

$$\frac{8r_s(4\rho + r_s)}{\left[(4\rho - r_s)^2 + \varepsilon^2\right](4\rho + r_s)} = \frac{16 r_s}{\varepsilon^2} + \frac{2}{r_s^2}.$$

From Eq.(2.4.11) and Eq.(2.4.8) finally we obtain

$$(R_\varepsilon(\rho))_\varepsilon|_{4\rho = r_s} = \frac{16 r_s}{(\varepsilon^2)_\varepsilon} + const. \tag{2.4.11}$$

2.5.Generalized Colombeau Calculus.

2.5.1.Super Generalized Functions.

We briefly recall the basic generalized Colombeau construction. Colombeau supergeneralized functions on $\Omega \subseteq \mathbb{R}^n$, where $\dim(\Omega) = n$ are defined as equivalence classes $u = [(u_\varepsilon)_\varepsilon]$ of nets of functions $u_\varepsilon \in C^\infty(\Omega \backslash \Sigma), \varepsilon \in (0,\delta]$ such that any u_ε is a net of functions smooth on $\Omega \backslash \Sigma$ and has a discontinuity on a subset $\Sigma \subset \Omega$, where $\dim(\Sigma) < n$. We assume that for any $\varepsilon \in (0,\delta]$ the derivative $\frac{\partial^m u_\varepsilon}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$ exists in the sense of the theory of generalized functions and $\frac{\partial^m u_\varepsilon}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \in \mathcal{D}'(\Omega)$.

The basic idea of generalized *Colombeau's theory of nonlinear supergeneralized functions* [33] is regularization by sequences (nets) of nonsmooth functions with derivatives in $\mathcal{D}'(\Omega)$ and the use of asymptotic estimates in terms of a regularization parameter ε. Let $(u_\varepsilon)_{\varepsilon \in (0,1]}$ with u_ε such that: (i) $u_\varepsilon \in \mathcal{C}^\infty(M \backslash \Sigma)$ and (ii) $L_{\xi_1} \ldots L_{\xi_k} u_\varepsilon \in \mathcal{D}'(M)$, for all $\varepsilon \in (0,\delta]$, where M a separable, smooth orientable Hausdorff manifold of dimension n.

Definition 2.5.1. The supergeneralized Colombeau's algebra $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}(M,\Sigma)$ of supergeneralized functions on M, where $\Sigma \subset M, \dim(M) = n, \dim(\Sigma) < n$, is defined as the quotient:

$$\widetilde{\mathcal{G}}(M,\Sigma) \triangleq \mathcal{E}_M(M,\Sigma)/\mathcal{N}(M,\Sigma) \tag{2.5.1}$$

of the space $\mathcal{E}_M(M,\Sigma)$ of sequences of moderate growth modulo the space $\mathcal{N}(M,\Sigma)$ of negligible sequences. More precisely the notions of moderateness resp. negligibility are defined by the following asymptotic estimates (where $\mathfrak{X}(M\backslash\Sigma)$ denoting the space of smooth vector fields on $M\backslash\Sigma$):

$$\begin{gathered}\mathcal{E}_M(M,\Sigma) \triangleq \{(u_\varepsilon)_\varepsilon | \, \forall K(K \subsetneq M\backslash\Sigma)\, \forall k(k \in \mathbb{N})\, \exists N(N \in \mathbb{N}) \\ \forall \xi_1,\ldots,\xi_k(\xi_1,\ldots,\xi_k \in \mathfrak{X}(M\backslash\Sigma)) \left[\sup_{p\in K} |L_{\xi_1}\ldots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{-N}), \varepsilon \to 0 \right] \& \\ \forall K(K \subsetneq M)\, \forall k(k \in \mathbb{N})\, \exists N(N \in \mathbb{N}) \forall (f \in \mathcal{C}^\infty(M)) \forall \xi_1,\ldots,\xi_k(\xi_1,\ldots,\xi_k \in \mathfrak{X}(M)) \\ \left[\|L^w_{\xi_1}\ldots L^w_{\xi_k} u_\varepsilon\| = \left(\sup_{f\in\mathcal{C}^\infty(M)} \left|L^w_{\xi_1}\ldots L^w_{\xi_k} u_\varepsilon(f)\right| \right) = O(\varepsilon^{-N}), \varepsilon \to 0 \right]\},\end{gathered} \tag{2.5.2}$$

$$\begin{gathered}\mathcal{N}(M,\Sigma) \triangleq \{(u_\varepsilon)_\varepsilon | \, \forall K(K \subsetneq M\backslash\Sigma),\, \forall k(k \in \mathbb{N}_0) \forall q(q \in N) \\ \forall \xi_1,\ldots,\xi_k(\xi_1,\ldots,\xi_k \in \mathfrak{X}(M\backslash\Sigma)) \left[\sup_{p\in K} |L_{\xi_1}\ldots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{q}), \varepsilon \to 0 \right]\} \& \\ \forall K(K \subsetneq M)\, \forall k(k \in \mathbb{N})\, \exists N(N \in \mathbb{N}) \forall (f \in \mathcal{C}^\infty(M)) \forall \xi_1,\ldots,\xi_k(\xi_1,\ldots,\xi_k \in \mathfrak{X}(M)) \\ \left[\|L^w_{\xi_1}\ldots L^w_{\xi_k} u_\varepsilon\| = \left(\sup_{f\in\mathcal{C}^\infty(M)} \left|L^w_{\xi_1}\ldots L^w_{\xi_k} u_\varepsilon(f)\right| \right) = O(\varepsilon^{q}), \varepsilon \to 0 \right]\},\end{gathered} \tag{2.5.3}$$

where $L^w_{\xi_k}$ denoting the weak Lie derivative in L.Schwartz sense.In the definition the Landau symbol $a_\varepsilon = O(\psi(\varepsilon))$ appears, having the following meaning: $\exists C(C > 0)\exists\varepsilon_0(\varepsilon_0 \in (0,1])\forall\varepsilon(\varepsilon < \varepsilon_0)[a_\varepsilon \leq C\psi(\varepsilon)]$.

Definition 2.5.2. Elements of $\widetilde{\mathcal{G}}(M,\Sigma)$ are denoted by:

$$u = \mathbf{cl}[(u_\varepsilon)_\varepsilon] \triangleq (u_\varepsilon)_\varepsilon + \mathcal{N}(M,\Sigma). \tag{2.5.4}$$

Remark 2.5.1.With componentwise operations $(\cdot,\pm)$ $\widetilde{\mathcal{G}}(M,\Sigma)$ is a fine sheaf of differential algebras with respect to the Lie derivative defined by $L_\xi u \triangleq \mathbf{cl}[(L_\xi u_\varepsilon)_\varepsilon]$.

The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $u \in \widetilde{\mathcal{G}}(M,\Sigma)$ iff $u \circ \psi_\alpha \in \widetilde{\mathcal{G}}(\psi_\alpha(V_\alpha))$ for all charts (V_α,ψ_α), where on the open set $\psi_\alpha(V_\alpha) \subset \mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives.

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Remark 2.5.2. Smooth functions $f \in \mathcal{C}^\infty(M\backslash\Sigma)$ are embedded into $\widetilde{\mathcal{G}}(M,\Sigma)$ simply by the "constant" embedding σ, i.e., $\sigma(f) = \mathbf{cl}[(f)_\varepsilon]$, hence $\mathcal{C}^\infty(M\backslash\Sigma)$ is a faithful subalgebra of $\widetilde{\mathcal{G}}(M,\Sigma)$.

2.5.2.Point Values of a Supergeneralized Functions on M. Supergeneralized Numbers

Within the classical distribution theory, distributions cannot be characterized by their point values in any way similar to classical functions. On the other hand, there is a very natural and direct way of obtaining the point values of the elements of Colombeau's algebra: points are simply inserted into representatives. The objects so obtained are sequences of numbers, and as such are not the elements in the field $\mathbb{R}$ or $\mathbb{C}$. Instead, they are the representatives of *Colombeau's generalized numbers.* We give the exact definition of these "numbers".

Definition 2.5.3. Inserting $p \in M$ into $u \in \widetilde{\mathcal{G}}(M,\Sigma)$ yields a well defined element of the ring of constants (also called generalized numbers) $\widetilde{\mathcal{K}}$ (corresponding to $\mathbf{K} = \mathbb{R}$ resp. $\mathbb{C}$), defined as the set of moderate nets of numbers ($(r_\varepsilon)_\varepsilon \in \mathbf{K}^{(0,1]}$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some N) modulo negligible nets ($|r_\varepsilon| = O(\varepsilon^m)$ for each m); componentwise insertion of points of M into elements of $\widetilde{\mathcal{G}}(M,\Sigma)$ yields well-defined generalized numbers, i.e.,elements of the ring of constants:

$$\widetilde{\mathcal{K}}_\Sigma = \mathcal{E}_{\mathbf{c}}(M,\Sigma)/\mathcal{N}_{\mathbf{c}}(M,\Sigma) \tag{2.5.5}$$

(with $\widetilde{\mathcal{K}}_\Sigma = \widetilde{\mathbb{R}}_\Sigma$ or $\mathcal{K} = \widetilde{\mathbb{C}}_\Sigma$ for $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$), where

$$\begin{cases} \mathcal{E}_{\mathbf{c}}(M,\Sigma) = \left\{(r_\epsilon)_\epsilon \in \mathbf{K}^I | \exists n (n \in \mathbb{N})\left[|r_\varepsilon| = O(\varepsilon^{-n}) \text{ as } \varepsilon \to 0\right]\right\}, \\ \mathcal{N}_{\mathbf{c}}(M,\Sigma) = \left\{(r_\epsilon)_\epsilon \in \mathbf{K}^I | \forall m (m \in \mathbb{N})\left[|r_\varepsilon| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\right]\right\} \\ I = (0,1]. \end{cases} \tag{2.5.6}$$

Supergeneralized functions on M are characterized by their generalized point values, i.e., by their values on points in $\tilde{M}_c$, the space of equivalence classes of compactly supported nets $(p_\varepsilon)_\varepsilon \in (M \backslash \Sigma)^{(0,1]}$ with respect to the relation
$p_\varepsilon \sim p'_\varepsilon :\Leftrightarrow d_h(p_\varepsilon, p'_\varepsilon) = O(\varepsilon^m)$ for all m, where d_h denotes the distance on $M \backslash \Sigma$ induced by any Riemannian metric.

Definition 2.5.4. For $u \in \widetilde{\mathcal{G}}(M,\Sigma)$ and $x_0 \in M$, the point value of u at the point $x_0, u(x_0)$, is
defined as the class of $(u_\varepsilon(x_0))_\varepsilon$ in $\widetilde{\mathcal{K}}$.

Definition 2.5.5. We say that an element $r \in \widetilde{\mathcal{K}}$ is *strictly nonzero* if there exists a representative $(r_\varepsilon)_\varepsilon$ and a $q \in \mathbb{N}$ such that $|r_\varepsilon| \geqslant \varepsilon^q$ for ε sufficiently small. If r is strictly nonzero, then it is also invertible with the inverse $[(1/r_\varepsilon)_\varepsilon]$. The converse is true as well.

Treating the elements of Colombeau algebras as a generalization of classical functions, the question arises whether the definition of point values can be extended in such a way that each element is characterized by its values. Such an extension is indeed possible.

Definition 2.5.6. Let Ω be an open subset of $\mathbb{R}^n \backslash \Sigma$. On a set $\hat{\Omega}_\Sigma$:

$$\begin{cases} \hat{\Omega}_\Sigma = \left\{(x_\varepsilon)_\varepsilon \in (\Omega \backslash \Sigma)^I | \exists p (p > 0)[|x_\varepsilon| = O(\varepsilon^p)]\right\} = \\ \left\{(x_\varepsilon)_\varepsilon \in (\Omega \backslash \Sigma)^I | \exists p (p > 0) \exists \varepsilon_0 (\varepsilon_0 > 0)\left[|x_\varepsilon| \leq \varepsilon^p, \text{ for } 0 < \varepsilon < \varepsilon_0\right]\right\} \end{cases} \tag{2.5.7}$$

we introduce an equivalence relation:

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow \forall q (q > 0) \forall \varepsilon (\varepsilon > 0)\left[|x_\varepsilon - y_\varepsilon| \leq \varepsilon^q, \text{ for } 0 < \varepsilon < \varepsilon_0\right] \tag{2.5.8}$$

and denote by $\widetilde{\Omega}_\Sigma = \hat{\Omega}_\Sigma / \sim$ the set of supergeneralized points. The set of points with compact support is

$$\widetilde{\Omega}_{\Sigma,\mathbf{c}} = \left\{\widetilde{x} = \mathbf{cl}[(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_\Sigma | \exists K(K \subset \Omega \backslash \Sigma)\exists \varepsilon_0(\varepsilon_0 > 0)\left[x_\varepsilon \in K \text{ for } 0 < \varepsilon < \varepsilon_0\right]\right\} \quad (2.5.9)$$

Definition 2.5.7. A supergeneralized function $u \in \widetilde{\mathcal{G}}(M,\Sigma)$ is called associated to zero, $u \approx 0$ on $\Omega \subseteq M$ in *L. Schwartz's sense* if one (hence any) representative $(u_\varepsilon)_\varepsilon$ converges to zero weakly,i.e.

$$w\text{-}\lim_{\varepsilon\to 0} u_\varepsilon = 0 \quad (2.5.10)$$

We shall often write:

$$u \underset{\mathbf{Sch}}{\approx} 0. \quad (2.5.11)$$

Definition 2.5.8. The $\widetilde{\mathcal{G}}(M,\Sigma)$-module of supergeneralized sections in vector bundles-especially the space of generalized tensor fields $\mathcal{T}_s{}^r(M\backslash\Sigma)$-is defined along the same lines using analogous asymptotic estimates with respect to the norm induced by any Riemannian metric on the respective fibers. However, it is more convenient to use the following algebraic description of generalized tensor fields

$$\widetilde{\mathcal{G}}^r_s(M,\Sigma) = \widetilde{\mathcal{G}}(M,\Sigma) \otimes \mathcal{T}_s{}^r(M\backslash\Sigma), \quad (2.5.12)$$

where $\mathcal{T}_s{}^r(M\backslash\Sigma)$ denotes the space of smooth tensor fields and the tensor product is taken over the module $C^\infty(M\backslash\Sigma)$. Hence super generalized tensor fields are just given by classical ones with generalized coefficient functions. Many concepts of classical tensor analysis carry over to the generalized setting, in particular Lie derivatives with respect to both classical and generalized vector fields, Lie brackets, exterior algebra, etc. Moreover, generalized tensor fields may also be viewed as $\widetilde{\mathcal{G}}(M,\Sigma)$-multilinear maps taking generalized vector and covector fields to generalized functions, i.e., as $\widetilde{\mathcal{G}}(M,\Sigma)$-modules we have

$$\widetilde{\mathcal{G}}^r_s(M,\Sigma) \cong L_{(M)}(\widetilde{\mathcal{G}}^0_1(M,\Sigma)^r, \widetilde{\mathcal{G}}^1_0(M,\Sigma)^s; \widetilde{\mathcal{G}}(M,\Sigma)). \quad (2.5.13)$$

In particular a supergeneralized metric is defined to be a symmetric, supergeneralized $(0,2)$-tensor field $g_{ab} = [((g_\varepsilon)_{ab})_\varepsilon]$ (with its index independent of ε and) whose determinant $\det(g_{ab})$ is invertible in $\widetilde{\mathcal{G}}(M,\Sigma)$. The latter condition is equivalent to the following notion called strictly nonzero on compact sets: for any representative $\det((g_\varepsilon)_{ab})_\varepsilon$ of $\det(g_{ab})$ we have $\forall K \subset M\backslash\Sigma \; \exists m \in \mathbb{N}[\inf_{p\in K}|\det(g_{ab}(\varepsilon))| \geq \varepsilon^q]$ for all ε small enough. This notion captures the intuitive idea of a generalized metric to be a sequence of classical metrics approaching a singular limit in the following sense: g_{ab} is a generalized metric iff (on every relatively compact open subset V of M) there exists a representative $((g_\varepsilon)_{ab})_\varepsilon$ of g_{ab} such that for fixed ε (small enough)$(g_\varepsilon)_{ab} = g_{ab}(\varepsilon)$ (resp. $(g_\varepsilon)_{ab}|_V$) is a classical pseudo-Riemannian metric and $\det(g_{ab})$ is invertible in the algebra of generalized functions. A generalized metric induces a $\widetilde{\mathcal{G}}(M,\Sigma)$-linear isomorphism from $\widetilde{\mathcal{G}}^1_0(M,\Sigma)$ to $\widetilde{\mathcal{G}}^0_1(M,\Sigma)$ and the inverse metric $g^{ab} \triangleq [(g^{-1}_{ab}(\varepsilon))_\varepsilon]$ is a well defined element of $\widetilde{\mathcal{G}}^2_0(M,\Sigma)$ (i.e., independent of the representative $((g_\varepsilon)_{ab})_\varepsilon$). Also the supergeneralized Levi-Civita connection as well as the supergeneralized Riemann, Ricci and Einstein tensor of a supergeneralized metric are defined simply by the usual coordinate formulae on the level of representatives.

2.5.3.Remarks on distributional derivatives in $\mathcal{D}'(\mathbb{R})$

Let $f(x)$ be a function of a single variable $x \in \mathbb{R}$ that has a jump discontiuity at $x = x_0$ of magnitude $[f]_{x_0} = f(x_0+0) - f(x_0-0)$ but has a continuous derivative everywhere else except $x = x_0$ (see Fig.2.5.1) i.e.,

$$f(x) = \begin{cases} f_1(x) & x < x_0 \\ f_2(x) & x \geq x_0 \end{cases}$$

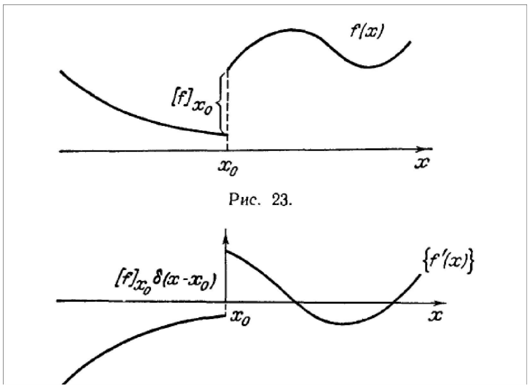

Fig.2.5.1.

Let the derivative in the interval $x \leq x_0$ and $x > x_0$ be denoted $\{f'(x)\}$, where $\{f'(x)\} = f'(x)$ iff $x \neq x_0$. This derivative $\{f'(x)\}$ is undefined at $x = x_0$. With the help of generalized functions, however, the distributional derivative $\bar{f}'(x)$ is obtained by setting:

$$g(x) = f(x) - [f]_{x_0} H(x - x_0), \tag{2.5.14}$$

where $H(x-x_0)$ is the Heaviside function. The function $g(x)$ is continuous at $x = x_0$. Its derivative coincides with that of $f(x)$ on both sides of x_0. Accordingly, we differentiate both sides of Eq.(2.5.14) and therefore one obtains

$$\{f'(x)\} = \bar{f}'(x) - [f]_{x_0} \delta(x - x_0). \tag{2.5.15}$$

Thus finally we get

$$\bar{f}'(x) = \{f'(x)\} + [f]_{x_0} \delta(x - x_0). \tag{2.5.16}$$

or

$$\bar{f}'(x) = \begin{cases} f_1'(x) + [f]_{x_0} \delta(x - x_0) & x < x_0 \\ f_2'(x) + [f]_{x_0} \delta(x - x_0) & x \geq x_0 \end{cases} \tag{2.5.17}$$

Equation (2.5.17) is easily generalized to a function $f(x)$ that has jumps of magnitude $[f]_{x_0}, [f]_{x_1}, \ldots, [f]_{x_k}$ at $x_0, x_1, \ldots, x_k$. The result is

$$\bar{f}'(x) = \{f'(x)\} + \textstyle\sum_{i=0}^{k} [f]_{x_i} \delta(x - x_i). \tag{2.5.18}$$

Let us now consider a function $f(x)$ that admits derivatives up to the second order on both sides of the point x_0, that has a jump discontinuity of strength $[f]_{x_0}$, and whose derivative has a jump discontinuity of strength $[f']_{x_0}$ at this point. To obtain $\bar{f}''(x)$, one substitutes $f'(x)$ for $f(x)$ in (2.5.14) and immediately one obtains

$$\bar{f}''(x) = \left\{f''(x)\right\} + [f]_{x_0}\delta'(x-x_0) + \left[f'\right]_{x_0}\delta(x-x_0), \qquad (2.5.19)$$

where $\left\{f''(x)\right\} = f''(x)$ iff $x \neq x_0$, or

$$\bar{f}''(x) = \begin{cases} f_1'(x) + [f]_{x_0}\delta'(x-x_0) + \left[f'\right]_{x_0}\delta(x-x_0) & x < x_0 \\ f_2'(x) + [f]_{x_0}\delta(x-x_0) + \left[f'\right]_{x_0}\delta(x-x_0) & x \geq x_0 \end{cases} \qquad (2.5.20)$$

This process can be continued for higher derivatives and for singularities at several points.Thus, a function $f(x)$ that admits continuous derivatives up to the m-th order in each of the intervals $(x_{j-1},x_j), j = 1,2,\ldots,l$ has m-th order distributional derivative $\bar{f}^{(m)}(x)$:

$$\bar{f}^{(m)}(x) = \left\{f^{(m)}(x)\right\} + \sum_{j=1}^{l}\left[a_j\delta^{(m-1)}(x-x_j) + b_j\delta^{(m-2)}(x-x_j) + \ldots + f_j\delta(x-x_j)\right], \quad (2.5.21)$$

where $a_j = [f(x)]_{x_j}, b_j = \left[f'(x)\right]_{x_j}, \ldots, f_j = \left[f^{(m-1)}(x)\right]_{x_j}$ and $[\bullet]_{x_j}$ stands for the jump in the quantity across the point x_j.

2.5.4.The super distributional geometry of the Schwarzschild space-time.

Note that the Colombeau generalized object $\left\{(ds_\epsilon^{+2})_\epsilon, (ds_\epsilon^{-2})_\epsilon\right\}$, given by Equations (1.10.28) that is a natural way is a supergeneralized $(0,2)$-tensor field $g_{ab} = [((g_\varepsilon)_{ab})_\varepsilon]$ whose determinant $\det(g_{ab})$ is invertible in $\widetilde{\mathcal{G}}(\mathbb{R}^4, \{r = 2m\})$ and the inverse metric $g^{ab} \triangleq [(g_{ab}^{-1}(\varepsilon))_\varepsilon]$ is a well defined element of $\widetilde{\mathcal{G}}_0^2(\mathbb{R}^4, \{r = 2m\})$. Thus the full super generalized Ricci tensor reads

$$\begin{gathered} \left([\mathbf{R}_\epsilon(r)]_0^0\right)_\epsilon = \left([\mathbf{R}_\epsilon(r)]_1^1\right)_\epsilon = \frac{1}{2}\left((\bar{h}_\epsilon''(r))_\epsilon + \frac{2}{r}(\bar{h}_\epsilon'(r))_\epsilon\right), \\ \left([\mathbf{R}_\epsilon(r)]_2^2\right)_\epsilon = \left([\mathbf{R}_\epsilon(r)]_3^3\right)_\epsilon = \frac{\left(\bar{h}_\epsilon'(r)\right)_\epsilon}{r} + \frac{1 + (h_\epsilon(r))_\epsilon}{r^2}, \end{gathered} \qquad (2.5.22)$$

where $(h_\epsilon)_\epsilon = \{(h_\epsilon^-)_\epsilon, (h_\epsilon^+)_\epsilon\}$ and the istributional derrivatives $\bar{h}_\epsilon'(r)$ and $\bar{h}_\epsilon''(r)$ is defined by Eq.(2.5.17) and by Eq.(2.5.20) correspondingly.

From Eq.(3.1.2) we obtain

$$\begin{gathered} ([h_\varepsilon]_{2m})_\varepsilon = (h_\varepsilon^+(2m+0))_\varepsilon - (h_\varepsilon^-(2m-0))_\varepsilon = (h_\varepsilon^+(2m) - h_\varepsilon^-(2m))_\varepsilon = \\ -\frac{\varepsilon}{2m} - \frac{\varepsilon}{2m} = -\varepsilon/m. \end{gathered} \qquad (2.5.23)$$

From Eq.(2.5.23) and Eq.(2.5.17) we obtain

$$(\bar{h}_\varepsilon'(r))_\varepsilon = \begin{cases} (h_\varepsilon^{+\prime}(r))_\varepsilon + ([h_\varepsilon]_{2m})_\varepsilon\delta(r-2m) & r \geq 2m \\ \left(h_\varepsilon^{-\prime}(r)\right)_\varepsilon + ([h_\varepsilon]_{2m})_\varepsilon\delta(r-2m) & r < 2m \end{cases} \qquad (2.5.24)$$

where $([h]_{2m})_\varepsilon = -(\varepsilon)_\varepsilon/m$ and therefore

$$\bar{h}_\varepsilon'(r) = \begin{cases} (h_\varepsilon^{+\prime}(r))_\varepsilon - \frac{(\varepsilon)_\varepsilon}{m}\delta(r-2m) & r \geq 2m \\ \left(h_\varepsilon^{-\prime}(r)\right)_\varepsilon - \frac{(\varepsilon)_\varepsilon}{m}\delta(r-2m) & r < 2m \end{cases} \qquad (2.5.25)$$

From Eq.(3.1.5) and Eq.(3.1.22) we obtain

$$\left[h_\varepsilon'\right]_{2m} = (h_\varepsilon^{+\prime}(2m+0))_\varepsilon - (h_\varepsilon^{-\prime}(2m-0))_\varepsilon = (h_\varepsilon^{+\prime}(2m))_\varepsilon - (h_\varepsilon^{-\prime}(2m))_\varepsilon =$$
$$\frac{(\varepsilon)_\varepsilon}{4m^2} - \left(-\frac{(\varepsilon)_\varepsilon}{4m^2}\right) = \frac{(\varepsilon)_\varepsilon}{2m^2}. \tag{2.5.26}$$

From Eq.(3.1.5),Eq.(3.1.22) and Eq.(2.5.7) we obtain

$$\left(\bar{h}_\varepsilon''(r)\right)_\varepsilon = \begin{cases} h_\varepsilon^{+\prime\prime}(r) + [h_\varepsilon]_{2m}\delta'(r-2m) + \left[h_\varepsilon'\right]_{2m}\delta(r-2m) & r \geq 2m \\ h_\varepsilon^{-\prime\prime}(r) + [h_\varepsilon]_{2m}\delta(r-2m) + \left[h_\varepsilon'\right]_{x_0}\delta(r-2m) & r < 2m \end{cases} \tag{2.5.27}$$

where $[h_\varepsilon']_{2m} = (\varepsilon)_\varepsilon/2m^2$ and therefore

$$\bar{h}_\varepsilon''(r) = \begin{cases} (h_\varepsilon^{+\prime\prime}(r))_\varepsilon - \frac{(\varepsilon)_\varepsilon}{m}\delta'(r-2m) + \frac{(\varepsilon)_\varepsilon}{2m^2}\delta(r-2m) & r \geq 2m \\ h_\varepsilon^{-\prime\prime}(r) - \frac{(\varepsilon)_\varepsilon}{m}\delta(r-2m) + \frac{(\varepsilon)_\varepsilon}{2m^2}\delta(r-2m) & r < 2m \end{cases} \tag{2.5.28}$$

Inserting Eqs.(2.5.26)-(2.5.28) into Eqs.(2.5.22) finally we get for $r \geq 2m$:

$$\begin{gathered} \left([\mathbf{R}_\varepsilon^+(r)]_0^0\right)_\varepsilon = \left([\mathbf{R}_\varepsilon^+(r)]_1^1\right)_\varepsilon = \frac{1}{2}\left((\bar{h}_\varepsilon''(r))_\varepsilon + \frac{2}{r}(\bar{h}_\varepsilon'(r))_\varepsilon\right) = \\ \frac{1}{2}\left((h_\varepsilon^{+\prime\prime}(r))_\varepsilon + \frac{2}{r}(h_\varepsilon^{+\prime}(x))_\varepsilon\right) - \\ -\frac{(\varepsilon)_\varepsilon}{2m}\delta'(r-2m) + \frac{(\varepsilon)_\varepsilon}{2m^2}\delta(r-2m) - \frac{2(\varepsilon)_\varepsilon}{rm}\delta(r-2m), \\ \left([\mathbf{R}_\varepsilon^+(r)]_2^2\right)_\varepsilon = \left([\mathbf{R}_\varepsilon^+(r)]_3^3\right)_\varepsilon = \frac{\left(\bar{h}_\epsilon'(r)\right)_\epsilon}{r} + \frac{1+(h_\epsilon(r))_\epsilon}{r^2} = \\ \frac{(h_\varepsilon^{+\prime}(r))_\varepsilon}{r} + \frac{1+(h_\varepsilon^+(r))_\varepsilon}{r^2} - \frac{(\varepsilon)_\varepsilon}{rm}\delta(r-2m). \end{gathered} \tag{2.5.29}$$

And finally we get for $r < 2m$:

$$\begin{gathered} \left([\mathbf{R}_\varepsilon^-(r)]_0^0\right)_\varepsilon = \left([\mathbf{R}_\varepsilon^-(r)]_1^1\right)_\varepsilon = \frac{1}{2}\left((\bar{h}_\varepsilon''(r))_\varepsilon + \frac{2}{r}(\bar{h}_\varepsilon'(r))_\varepsilon\right) = \\ \frac{1}{2}\left((h_\varepsilon^{-\prime\prime}(r))_\varepsilon + \frac{2}{r}(h_\varepsilon^{-\prime}(x))_\varepsilon\right) - \\ -\frac{(\varepsilon)_\varepsilon}{2m}\delta'(r-2m) + \frac{(\varepsilon)_\varepsilon}{2m^2}\delta(r-2m) - \frac{2(\varepsilon)_\varepsilon}{rm}\delta(r-2m), \\ \left([\mathbf{R}_\varepsilon^-(r)]_2^2\right)_\varepsilon = \left([\mathbf{R}_\varepsilon^-(r)]_3^3\right)_\varepsilon = \frac{\left(\bar{h}_\epsilon'(r)\right)_\epsilon}{r} + \frac{1+(h_\epsilon(r))_\epsilon}{r^2} = \\ \frac{(h_\varepsilon^{-\prime}(r))_\varepsilon}{r} + \frac{1+(h_\varepsilon^-(r))_\varepsilon}{r^2} - \frac{(\varepsilon)_\varepsilon}{rm}\delta(r-2m). \end{gathered} \tag{2.5.30}$$

Note that in $\mathcal{D}'(\mathbb{R})$:

$$\text{w-}\lim_{\varepsilon\to 0}\frac{\pi^{-1}\varepsilon}{(r-2m)^2+\varepsilon^2} = \delta(r-2m), \tag{2.5.31}$$

and therefore in $\mathcal{D}'(\mathbb{R})$:

$$\text{w-}\lim_{\varepsilon\to 0}\frac{d}{dr}\frac{\varepsilon}{(r-2m)^2+\varepsilon^2} = \text{w-}\lim_{\varepsilon\to 0}\frac{2\pi^{-1}\varepsilon(r-2m)}{\left[(r-2m)^2+\varepsilon^2\right]^2} = \delta'(r-2m)\,. \quad (2.5.32)$$

By using Eq.(2.5.31) and Eq.(2.5.32) one obtains the natural imbeddings

$$\delta(r-2m) \hookrightarrow \mathcal{G}(\mathbb{R}) : \delta(r-2m) \triangleq \pi^{-1}\left(\frac{\varepsilon}{(r-2m)^2+\varepsilon^2}\right)_{\varepsilon} \in \mathcal{G}(\mathbb{R}) \qquad (2.5.33)$$

and

$$\delta'(r-2m) \hookrightarrow \mathcal{G}(\mathbb{R}) : \delta'(r-2m) \triangleq 2\pi^{-1}\left(\frac{\varepsilon(r-2m)}{\left[(r-2m)^2+\varepsilon^2\right]^2}\right)_{\varepsilon} \in \mathcal{G}(\mathbb{R}) \qquad (2.5.34)$$

correspondingly.

Remark 2.5.3

Remark 2.5.4. Note that in $\mathcal{D}'([2m,\infty))$:

$$\text{w-}\lim_{\varepsilon\to 0}\frac{\varepsilon^2(r-2m)}{\left[(r-2m)^2+\varepsilon^2\right]^2} = c_1\delta(r-2m). \qquad (2.5.35)$$

2.6.Superdistributional general relativity

We briefly summarize the basics of superdistributional general relativity, as a preliminary to latter discussion.In the classical theory of gravitation one is led to consider the Einstein field equations which are,in general,quasilinear partial differential equations involving second order derivatives for the metric tensor. Hence, continuity of the first fundamental form is expected and at most, discontinuities in the second fundamental form, the coordinate independent statements appropriate to consider 3-surfaces of discontinuity in the spacetime manifolfd of General Relativity.

In standard general relativity, the space-time is assumed to be a four-dimensional differentiable manifold M endowed with the Lorentzian metric $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ $(\mu,\nu = 0,1,2,3)$. At each point p of space-time M, the metric can be diagonalized as

$ds_p^2 = \eta_{\mu\nu}(dX^{\mu})_p(dX^{\nu})_p$ with $\eta_{\mu\nu} \triangleq (-1,1,1,1)$, by choosing the coordinate system $\{X^{\mu};\mu = 0,1,2,3\}$ appropriately.

In superdistributional general relativity the space-time is assumed to be a four-dimensional differentiable manifold $M\backslash\Sigma$, where $\dim(M) = 4, \dim(\Sigma) \leqslant 3$ endowed with the Lorentzian supergeneralized metric

$$(ds_{\epsilon}^2)_{\epsilon} = (g_{\mu\nu}(\epsilon)dx^{\mu}dx^{\nu})_{\epsilon}\,;\mu,\nu = 0,1,2,3). \qquad (2.27)$$

At each point $p \in M\backslash\Sigma$, the metric can be diagonalized as

$$(ds_p^2(\epsilon))_{\epsilon} = (\eta_{\mu\nu}(dX_{\epsilon}^{\mu})_p(dX_{\epsilon}^{\nu})_p)_{\epsilon} \text{ with } \eta_{\mu\nu} \triangleq (-1,1,1,1), \qquad (2.28)$$

by choosing the generalized coordinate system $\{(X_{\epsilon}^{\mu})_{\epsilon};\mu = 0,1,2,3\}$ appropriately.

The classical smooth curvature tensor is given by

$$R^{\rho}{}_{\sigma\mu\nu} \triangleq \partial_{\mu}\left\{\frac{\rho}{\sigma\,\nu}\right\} - \partial_{\nu}\left\{\frac{\rho}{\sigma\,\mu}\right\} + \left\{\frac{\rho}{\lambda\,\mu}\right\}\left\{\frac{\lambda}{\sigma\,\nu}\right\} - \left\{\frac{\rho}{\lambda\,\nu}\right\}\left\{\frac{\lambda}{\sigma\,\mu}\right\} \qquad (2.29)$$

with $\left\{\frac{\rho}{\sigma\,\nu}\right\}$ being the smooth Christoffel symbol.The supergeneralized nonsmooth curvature tensor is given by

$$\begin{cases} (R^{\rho}{}_{\sigma\mu\nu}(\epsilon))_{\epsilon} \triangleq \partial_{\mu}\left(\left\{\frac{\rho}{\sigma\,\nu}\right\}_{\epsilon}\right)_{\epsilon} - \partial_{\nu}\left(\left\{\frac{\rho}{\sigma\,\mu}\right\}_{\epsilon}\right)_{\epsilon} + \left(\left\{\frac{\rho}{\lambda\,\mu}\right\}_{\epsilon}\right)_{\epsilon}\left(\left\{\frac{\lambda}{\sigma\,\nu}\right\}_{\epsilon}\right)_{\epsilon} - \\ -\left(\left\{\frac{\rho}{\lambda\,\nu}\right\}_{\epsilon}\right)_{\epsilon}\left(\left\{\frac{\lambda}{\sigma\,\mu}\right\}_{\epsilon}\right)_{\epsilon} \end{cases} \tag{2.30}$$

with $\left(\left\{\frac{\rho}{\sigma\,\nu}\right\}_{\epsilon}\right)_{\epsilon}$ being the supergeneralized Christoffel symbol.The fundamental classical action integral $\mathbb{I}$ is

$$\mathbb{I} = \frac{1}{c}\int(\bar{\mathbb{L}}_G + \mathbb{L}_M)d^4x, \tag{2.31}$$

where $\mathbb{L}_M$ is the Lagrangian density of a gravitational source and $\bar{\mathbb{L}}_G$ is the gravitational Lagrangian density given by

$$\bar{\mathbb{L}}_G = \frac{1}{2\kappa}\mathbb{G}\ . \tag{2.32}$$

Here κ is the Einstein gravitational constant $\kappa = 8\pi G/c^4$ and $\mathbb{G}$ is defined by

$$\mathbb{G} = \sqrt{-g}\,g^{\mu\nu}\left(\left\{\frac{\lambda}{\mu\,\rho}\right\}\left\{\frac{\rho}{\nu\,\lambda}\right\} - \left\{\frac{\lambda}{\mu\,\nu}\right\}\left\{\frac{\rho}{\lambda\,\rho}\right\}\right) \tag{2.33}$$

with $g = \det(g_{\mu\nu})$. There exists the relation

$$\sqrt{-g}\,R = \mathbb{G} + \partial_{\mu}\mathbb{D}^{\mu}\ , \tag{2.34}$$

with

$$\mathbb{D}^{\mu} = -\sqrt{-g}\left(g^{\mu\nu}\left\{\frac{\lambda}{\nu\,\lambda}\right\} - g^{\nu\lambda}\left\{\frac{\mu}{\nu\,\lambda}\right\}\right)\ . \tag{2.35}$$

Thus the supergeneralized fundamental action integral $(\mathbb{I}_{\epsilon})_{\epsilon}$ is

$$(\mathbb{I}_{\epsilon})_{\epsilon} = \frac{1}{c}\int((\bar{\mathbb{L}}_G(\epsilon))_{\epsilon} + (\mathbb{L}_M(\epsilon))_{\epsilon})d^4x\ , \tag{2.36}$$

where $(\mathbb{L}_M(\epsilon))_{\epsilon}$ is the supergeneralized Lagrangian density of a gravitational source and $(\bar{\mathbb{L}}_G(\epsilon))_{\epsilon}$ is the supergeneralized gravitational Lagrangian density given by

$$(\bar{\mathbb{L}}_G(\epsilon))_{\epsilon} = \frac{1}{2\kappa}(\mathbb{G}_{\epsilon})_{\epsilon}\ . \tag{2.37}$$

Here κ is the Einstein gravitational constant $\kappa = 8\pi G/c^4$ and $(\mathbb{G}_{\epsilon})_{\epsilon}$ is defined by

$$(\mathbb{G}_{\epsilon})_{\epsilon} = \sqrt{-(g_{\epsilon})_{\epsilon}}\,(g_{\epsilon}^{\mu\nu})_{\epsilon}\left(\left(\left\{\frac{\lambda}{\mu\,\rho}\right\}_{\epsilon}\right)\left(\left\{\frac{\rho}{\nu\,\lambda}\right\}_{\epsilon}\right) - \left(\left\{\frac{\lambda}{\mu\,\nu}\right\}_{\epsilon}\right)\left(\left\{\frac{\rho}{\lambda\,\rho}\right\}_{\epsilon}\right)\right) \tag{2.38}$$

with $g_{\epsilon} = \det[(g_{\mu\nu}(\epsilon))_{\epsilon}]$. There exists the relation

$$\sqrt{-(g_{\epsilon})_{\epsilon}}\,(\mathbf{R}_{\epsilon})_{\epsilon} = (\mathbb{G}_{\epsilon})_{\epsilon} + \partial_{\mu}(\mathbb{D}_{\epsilon}^{\mu})_{\epsilon}\ , \tag{2.39}$$

with

$$(\mathbb{D}_{\epsilon}^{\mu})_{\epsilon} = -\sqrt{-(g_{\epsilon})_{\epsilon}}\left((g_{\epsilon}^{\mu\nu})_{\epsilon}\left(\left\{\frac{\lambda}{\nu\,\lambda}\right\}_{\epsilon}\right)_{\epsilon} - (g_{\epsilon}^{\nu\lambda})_{\epsilon}\left(\left\{\frac{\mu}{\nu\,\lambda}\right\}_{\epsilon}\right)_{\epsilon}\right)\ . \tag{2.40}$$

Also, we have defined the classical scalar curvature by

$$\mathbf{R} = R^{\mu}{}_{\mu} \tag{2.41}$$

with the smooth Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}\ . \tag{2.42}$$

From the action $\mathbb{I}$, the classical Einstein equation

$$G_\mu{}^\nu = R_\mu{}^\nu - \frac{1}{2}\delta_\mu{}^\nu R = \kappa T_\mu{}^\nu \ , \tag{2.43}$$

follows, where $T_\mu{}^\nu$ is defined by

$$T_\mu{}^\nu = \frac{\widetilde{\mathbf{T}}_\mu{}^\nu}{\sqrt{-g}} \tag{2.44}$$

with

$$\widetilde{\mathbf{T}}_\mu^{\ \nu} \triangleq 2 g_{\mu\lambda} \frac{\delta \mathbb{L}_M}{\delta g_{\lambda\nu}} \tag{2.45}$$

being the energy-momentum density of the classical gravity source. Thus we have defined the supergeneralized scalar curvature by

$$(\mathbf{R}_\epsilon)_\epsilon = (R^\mu{}_\mu(\epsilon))_\epsilon \tag{2.46}$$

with the supergeneralized Ricci tensor

$$(\mathbf{R}_{\mu\nu}(\epsilon))_\epsilon = (R^\lambda{}_{\mu\lambda\nu}(\epsilon))_\epsilon \ . \tag{2.47}$$

From the action $(\mathbb{I}_\epsilon)_\epsilon$, the generalized Einstein equation

$$(G_\mu{}^\nu(\epsilon))_\epsilon = (R_\mu{}^\nu(\epsilon))_\epsilon - \frac{1}{2}\delta_\mu{}^\nu(\mathbf{R}_\epsilon)_\epsilon = \kappa (T_\mu{}^\nu(\epsilon))_\epsilon \ , \tag{2.48}$$

follows, where $(T_\mu{}^\nu(\epsilon))_\epsilon$ is defined by

$$(T_\mu{}^\nu(\epsilon))_\epsilon = \frac{\left(\widetilde{\mathbf{T}}_\mu{}^\nu(\epsilon)\right)_\epsilon}{\sqrt{-(g_\epsilon)_\epsilon}} \tag{2.49}$$

with

$$\left(\widetilde{\mathbf{T}}_\mu^{\ \nu}(\epsilon)\right)_\epsilon \triangleq 2(g_{\mu\lambda}(\epsilon))_\epsilon \frac{\delta(\mathbb{L}_M(\epsilon))_\epsilon}{\delta(g_{\lambda\nu}(\epsilon))_\epsilon} \tag{2.50}$$

being the supergeneralized energy-momentum density of the supergeneralized gravity source.The classical energy-momentum pseudo-tensor density $\widetilde{\mathbf{t}}_\mu^{\ \nu}$ of the gravitational field is defined by

$$\widetilde{\mathbf{t}}_\mu^{\ \nu} = \delta_\mu{}^\nu \bar{\mathbb{L}}_G - \frac{\partial \bar{\mathbb{L}}_G}{\partial g_{\sigma\tau,\nu}} g_{\sigma\tau,\mu} \tag{2.51}$$

with $g_{\sigma\tau,\nu} = \partial g_{\sigma\tau}/\partial x^\nu$.The supergeneralized energy-momentum pseudo-tensor density $\widetilde{\mathbf{t}}_\mu^{\ \nu}$ of the gravitational field is defined by

$$\left(\widetilde{\mathbf{t}}_\mu^{\ \nu}(\epsilon)\right)_\epsilon = \delta_\mu{}^\nu(\bar{\mathbb{L}}_G(\epsilon))_\epsilon - \left(\frac{\partial \bar{\mathbb{L}}_G(\epsilon)}{\partial g_{\sigma\tau,\nu}(\epsilon)}\right)_\epsilon (g_{\sigma\tau,\mu}(\epsilon))_\epsilon \tag{2.52}$$

with $(g_{\sigma\tau,\nu}(\epsilon))_\epsilon = (\partial g_{\sigma\tau}(\epsilon)/\partial x^\nu)_\epsilon$.

3.Distributional Schwarzschild Geometry from nonsmooth regularization via Horizon

3.1.Calculation of the stress-tensor by using nonsmooth regularization via Horizon

In this section we leave the neighborhood of the singularity at the origin and turn to the singularity at the horizon. The question we are aiming at is the following: using distributional geometry (thus without leaving Schwarzschild coordinates), is it possible to show that the horizon singularity of the Schwarzschild metric is not merely only a coordinate singularity. In order to investigate this issue we calculate the distributional curvature at horizon in Schwarzschild coordinates. In the usual Schwarzschild coordinates (t,r,θ,ϕ), $r \neq 2m$ the Schwarzschild metric (1.12) takes the form above horizon $r > 2m$ and below horizon $r < 2m$ correspondingly

$$\begin{cases} \text{above horizon } r > 2m : \\ ds^{+2} = h^{+}(r)dt^2 - [h^{+}(r)]^{-1}dr^2 + r^2 d\Omega^2, \\ h^{+}(r) = -1 + \frac{2m}{r} = -\frac{r-2m}{r} \\ \text{below horizon } r < 2m : \\ ds^{-2} = h^{-}(r)dt^2 - h^{-}(r)^{-1}dr^2 + r^2 d\Omega^2, \\ h^{-}(r) = -1 + \frac{2m}{r} = \frac{2m-r}{r} \end{cases} \tag{3.1.1}$$

Remark 3.1.1. Following the above discussion we consider the metric coefficients $h^{+}(r), [h^{+}(r)]^{-1}$ $h^{-}(r)$, and $[h^{-}(r)]^{-1}$ as an element of $\mathcal{D}'(\mathbb{R}^3)$ and embed it into $(\mathcal{G}(\mathbb{R}^3))$ by replacements above horizon $r \geqslant 2m$ and below horizon $r \leqslant 2m$ correspondingly

$$r \geqslant 2m : r - 2m \mapsto \sqrt{(r-2m)^2 + \epsilon^2},$$
$$r \leqslant 2m : 2m - r \mapsto \sqrt{(2m-r)^2 + \epsilon^2}.$$

Remark 3.1.2. Note that, accordingly, we have fixed the differentiable structure of the manifold: the usual Schwarzschild coordinates and the Cartesian coordinates associated with the spherical Schwarzschild coordinates in (3.1.1) are extended on $r = 2m$ through the horizon. Therefore we have above horizon $r \geqslant 2m$ and below horizon $r \leqslant 2m$ correspondingly

$$h(r) = \left\{ \begin{array}{c} -\frac{r-2m}{r} \text{ if } r \geqslant 2m \\ 0 \text{ if } r \leqslant 2m \end{array} \right\} \mapsto (h_\epsilon^+(r))_\epsilon = \left(-\frac{\sqrt{(r-2m)^2+\epsilon^2}}{r} \right)_\epsilon,$$

where $(h_\epsilon^+(r))_\epsilon \in \widetilde{\mathcal{G}}(\mathbb{R}^3, B^+(2m,R)), B^+(2m,R) = \{x \in \mathbb{R}^3 | 2m \leqslant \|x\| \leqslant R\}$.

$$h^{-1}(r) = \left\{ \begin{array}{c} -\frac{r}{r-2m}, r > 2m \\ \infty, r = 2m \end{array} \right\} \mapsto (h_\epsilon^+)^{-1}(r)$$

$$(3.1.2)$$

$$h^-(r) = \left\{ \begin{array}{c} -\frac{r-2m}{r} \text{ if } r \leqslant 2m \\ 0 \text{ if } r \geq 2m \end{array} \right\} \mapsto h_\epsilon^-(r) = \frac{\sqrt{(2m-r)^2+\epsilon^2}}{r}$$

where $h^-(r) \in \widetilde{\mathcal{G}}(\mathbb{R}^3, B^-(0,2m)),\ B^-(0,2m) = \{x \in \mathbb{R}^3 | 0 < \|x\| \leqslant 2m\}$

$$h^{-1}(r) = \left\{ \begin{array}{c} -\frac{r}{r-2m}, r < 2m \\ \infty, r = 2m \end{array} \right\} \mapsto (h_\epsilon^-)^{-1}(r) =$$

$$= \left(\frac{r}{\sqrt{(r-2m)^2+\epsilon^2}} \right)_\epsilon \in \widetilde{\mathcal{G}}(\mathbb{R}^3, B^-(0,2m))$$

Inserting (3.2) into (3.1) we obtain a generalized object modeling the singular Schwarzschild metric above (below) gorizon, i.e.,

$$\left\{ \begin{array}{l} (ds_\epsilon^{+2})_\epsilon = (h_\epsilon^+(r)dt^2)_\epsilon - \left([h_\epsilon^+(r)]^{-1}dr^2\right)_\epsilon + r^2 d\Omega^2, \\ (ds_\epsilon^{-2})_\epsilon = (h_\epsilon^-(r)dt^2)_\epsilon - \left([h_\epsilon^-(r)]^{-1}dr^2\right)_\epsilon + r^2 d\Omega^2 \end{array} \right. \quad (3.1.3)$$

The generalized Ricci tensor above horizon $[\mathbf{R}^+]_\alpha^\beta$ may now be calculated componentwise using the classical formulae

$$\left\{ \begin{array}{l} \left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon = \left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon = \frac{1}{2}\left((h_\epsilon^{+''})_\epsilon + \frac{2}{r}(h_\epsilon^{+'})_\epsilon\right) \\ \left([\mathbf{R}_\epsilon^+]_2^2\right)_\epsilon = \left([\mathbf{R}_\epsilon^+]_3^3\right)_\epsilon = \frac{(h_\epsilon^{+'})_\epsilon}{r} + \frac{1+(h_\epsilon^+)_\epsilon}{r^2}. \end{array} \right. \quad (3.1.4)$$

From (3.1.2) we obtain

$$h_\epsilon^{+\prime}(r) = -\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2},$$

$$r(h_\epsilon^{+\prime})_\epsilon + 1 + (h_\epsilon^{+})_\epsilon =$$

$$r\left\{-\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2}\right\} + 1 - \frac{\sqrt{(r-2m)^2+\epsilon^2}}{r} =$$

$$-\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r} + 1 - \frac{\sqrt{(r-2m)^2+\epsilon^2}}{r} =$$

$$-\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + 1.$$

$$h_\epsilon^{+\prime\prime}(r) = -\left(\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}}\right)' + \left(\frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2}\right)' =$$

$$= -\frac{1}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{(r-2m)^2}{r\left[(r-2m)^2+\epsilon^2\right]^{3/2}} + \frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} +$$

$$+\frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^3}.$$

$$r^2(h_\epsilon^{+\prime\prime})_\epsilon + 2r(h_\epsilon^{+\prime})_\epsilon = \tag{3.1.5}$$

$$r^2\left\{-\frac{1}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{(r-2m)^2}{r\left[(r-2m)^2+\epsilon^2\right]^{3/2}} + \frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} +\right.$$

$$\left.+\frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^3}\right\} +$$

$$+2r\left\{-\frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2}\right\} =$$

$$-\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}} + \frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} +$$

$$+\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r} +$$

$$-\frac{2(r-2m)}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r} =$$

$$-\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} + \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}.$$

Investigating the weak limit of the angular components of the Ricci tensor (using the abbreviation

$$\tilde{\Phi}(r) = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \Phi(x)$$

and let $\Phi(x)$ be the function $\Phi(x) \in \mathcal{S}^+_{2m}(\mathbb{R}^3)$, where by $\mathcal{S}^+_{2m}(\mathbb{R}^3)$ we denote the class of all functions $\Phi(x)$ with compact support such that:

(i) **supp**$(\Phi(x)) \subset \{x \| \|x\| \geq 2m\}$ (ii) $\tilde{\Phi}(r) \in C^{\infty}(\mathbb{R})$.

Then for any function $\Phi(x) \in \mathcal{S}_{2m}(\mathbb{R}^3)$ we get:

$$\int_K \left([\mathbf{R}^+_{\epsilon}]^2_2\right)_{\epsilon} \Phi(\vec{x}) d^3x = \int_K \left([\mathbf{R}^+_{\epsilon}]^3_3\right)_{\epsilon} \Phi(\vec{x}) d^3x =$$

$$\int_{2m}^{R} \left(r(h^{+\prime}_{\epsilon})_{\epsilon} + 1 + (h^+_{\epsilon})_{\epsilon}\right) \tilde{\Phi}(r) dr = \int_{2m}^{R} \left\{ -\frac{r - 2m}{\left[(r-2m)^2 + \epsilon^2\right]^{1/2}} \right\} \tilde{\Phi}(r) dr + \int_{2m}^{R} \tilde{\Phi}(r) dr. \tag{3.1.6}$$

By replacement $r - 2m = u$, from (3.1.6) we obtain

$$\int_K \left([\mathbf{R}^+_{\epsilon}]^2_2\right)_{\epsilon} \Phi(x) d^3x = \int_K \left([\mathbf{R}^+_{\epsilon}]^3_3\right)_{\epsilon} \Phi(x) d^3x =$$

$$- \int_0^{R-2m} \frac{u\tilde{\Phi}(u+2m) du}{(u^2 + \epsilon^2)^{1/2}} + \int_0^{R-2m} \tilde{\Phi}(u + 2m) du. \tag{3.1.7}$$

By replacement $u = \epsilon\eta$, from (3.1.7) we obtain the expression

$$\begin{cases} \mathbf{I}^+_3(\epsilon) = \int_K \left([\mathbf{R}^+_{\epsilon}]^3_3\right)_{\epsilon} \Phi(x) d^3x = \mathbf{I}^+_2(\epsilon) = \int_K \left([\mathbf{R}^+_{\epsilon}]^2_2\right)_{\epsilon} \Phi(\vec{x}) d^3x = \\ -\epsilon \times \left(\int_0^{\frac{R-2m}{\epsilon}} \frac{\eta\tilde{\Phi}(\epsilon\eta + 2m) d\eta}{(\eta^2 + 1)^{1/2}} - \int_0^{\frac{R-2m}{\epsilon}} \tilde{\Phi}(\epsilon\eta + 2m) d\eta \right). \end{cases} \tag{3.1.8}$$

From Eq.(3.1.8) we obtain

$$\begin{gathered} \mathbf{I}^+_3(\epsilon) = \mathbf{I}^+_2(\epsilon) = -\epsilon \frac{\tilde{\Phi}(2m)}{0!} \int_0^{\frac{R-2m}{\epsilon}} \left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1 \right] d\eta - \\ -\frac{\epsilon^2}{1!} \int_0^{\frac{R-2m}{\epsilon}} \left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta = \\ -\epsilon\tilde{\Phi}(2m) \left[\sqrt{\left(\frac{R-2m}{\epsilon}\right)^2 + 1} - \left(\frac{R-2m}{\epsilon} - 1\right) \right] - \\ -\frac{\epsilon^2}{1} \int_0^{\frac{R-2m}{\epsilon}} \left[\frac{\eta}{(\eta^2+1)^{1/2}} - 1 \right] \tilde{\Phi}^{(1)}(\xi) \eta d\eta, \end{gathered} \tag{3.1.9}$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta + 2m)$ as

$$\begin{cases} \widetilde{\Phi}(\epsilon\eta + 2m) = \sum_{l=0}^{n-1} \frac{\Phi^{(l)}(2m)}{l!} (\epsilon\eta)^l + \frac{1}{n!} (\epsilon\eta)^n \Phi^{(n)}(\xi) , \\ \xi \triangleq \theta\epsilon\eta + 2m , \quad 1 > \theta > 0 , \quad n = 1 \end{cases} \tag{3.1.10}$$

with $\tilde{\Phi}^{(l)}(\xi) \triangleq d^l\tilde{\Phi}/d\xi^l$. Equations (3.1.9)-(3.1.10) gives

$$\lim_{\epsilon\to 0}\mathbf{I}_3^+(\epsilon) = \lim_{\epsilon\to 0}\mathbf{I}_2^+(\epsilon) =$$

$$\lim_{\epsilon\to 0}\left\{-\epsilon\tilde{\Phi}(2m)\left[\sqrt{\left(\frac{R-2m}{\epsilon}\right)^2+1}+1-\frac{R-2m}{\epsilon}\right]\right\}+$$

$$+\lim_{\epsilon\to 0}\left\{-\frac{\epsilon^2}{1}\int_0^{\frac{R-2m}{\epsilon}}\left[\frac{\eta}{(\eta^2+1)^{1/2}}-1\right]\tilde{\Phi}^{(1)}(\xi)\eta d\eta\right\} = 0. \tag{3.1.11}$$

Thus in $\mathcal{S}'_{2m}(B_R^+(2m)) \subset \mathcal{S}'_{2m}(\mathbb{R}^3) \subset \mathcal{D}'(\mathbb{R}^3)$, where $B^+(2m,R) = \{x \in \mathbb{R}^3 | 2m \leqslant \|x\| \leqslant R\}$ from Eq.(3.1.11) we obtain

$$\begin{aligned} w-\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^+]_3^3 &= \lim_{\epsilon\to 0}\mathbf{I}_3^+(\epsilon) = 0,\\ w-\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^+]_2^2 &= \lim_{\epsilon\to 0}\mathbf{I}_2^+(\epsilon) = 0. \end{aligned} \tag{3.1.12}$$

For $\left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon, \left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon$ we get:

$$2\int_K \left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon \Phi(x)d^3x = 2\int_K \left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon \Phi(x)d^3x =$$

$$\int_{2m}^{R}\left(r^2(h_\epsilon^{+''})_\epsilon + 2r(h_\epsilon^{+'})_\epsilon\right)\tilde{\Phi}(r)dr =$$

$$= \int_{2m}^{R}\left\{-\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}+\frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}\right\}\tilde{\Phi}(r)dr. \tag{3.1.13}$$

By replacement $r-2m = u$, from (3.1.13) we obtain

$$\begin{cases} \mathbf{I}_1^+(\epsilon) = 2\int_K \left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon \Phi(x)d^3x = I_2^+(\epsilon) = 2\int_K \left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon \Phi(x)d^3x \\ \quad = \int_{2m}^{R}\left(r^2(h_\epsilon^{+''})_\epsilon + 2r(h_\epsilon^{+'})_\epsilon\right)\tilde{\Phi}(r)dr = \\ \quad = \int_0^{R-2m}\left\{-\frac{u+2m}{(u^2+\epsilon^2)^{1/2}}+\frac{u^2(u+2m)}{(u^2+\epsilon^2)^{3/2}}\right\}\tilde{\Phi}(u+2m)du. \end{cases} \tag{3.1.14}$$

By replacement $u = \epsilon\eta$, from (3.1.14) we obtain

$$2\int_K \left([\mathbf{R}_\epsilon^+]_1^1\right)_\epsilon \Phi(x)d^3x = 2\int_K \left([\mathbf{R}_\epsilon^+]_0^0\right)_\epsilon \Phi(x)d^3x =$$

$$= \int_{2m}^{R} \left(r^2(h_\epsilon^{+\prime\prime})_\epsilon + 2r(h_\epsilon^{+\prime})_\epsilon\right)\tilde{\Phi}(r)dr =$$

$$= \epsilon \int_0^{\frac{R-2m}{\epsilon}} \left\{-\frac{\epsilon\eta + 2m}{(\epsilon^2\eta^2 + \epsilon^2)^{1/2}} + \frac{\epsilon^2\eta^2(\epsilon\eta + 2m)}{(\epsilon^2\eta^2 + \epsilon^2)^{3/2}}\right\}\tilde{\Phi}(\epsilon\eta + 2m)d\eta =$$

$$- \int_0^{\frac{R-2m}{\epsilon}} \frac{\epsilon^2\eta\tilde{\Phi}(\epsilon\eta + 2m)d\eta}{(\epsilon^2\eta^2 + \epsilon^2)^{1/2}} - 2m \int_0^{\frac{R-2m}{\epsilon}} \frac{\epsilon\tilde{\Phi}(\epsilon\eta + 2m)d\eta}{(\epsilon^2\eta^2 + \epsilon^2)^{1/2}} + \tag{3.1.15}$$

$$\int_0^{\frac{R-2m}{\epsilon}} \frac{\epsilon^4\eta^3\tilde{\Phi}(\epsilon\eta + 2m)d\eta}{(\epsilon^2\eta^2 + \epsilon^2)^{3/2}} + 2m \int_0^{\frac{R-2m}{\epsilon}} \frac{\epsilon^3\eta^2\tilde{\Phi}(\epsilon\eta + 2m)d\eta}{(\epsilon^2\eta^2 + \epsilon^2)^{3/2}} =$$

$$\epsilon\left[-\int_0^{\frac{R-2m}{\epsilon}} \frac{\eta\tilde{\Phi}(\epsilon\eta + 2m)d\eta}{(\eta^2 + 1)^{1/2}} + \int_0^{\frac{R-2m}{\epsilon}} \frac{\eta^3\tilde{\Phi}(\epsilon\eta + 2m)d\eta}{(\eta^2 + 1)^{3/2}}\right] +$$

$$2m\left[-\int_0^{\frac{R-2m}{\epsilon}} \frac{\tilde{\Phi}(\epsilon\eta + 2m)d\eta}{(\eta^2 + 1)^{1/2}} + \int_0^{\frac{R-2m}{\epsilon}} \frac{\eta^2\tilde{\Phi}(\epsilon\eta + 2m)d\eta}{(\eta^2 + 1)^{3/2}}\right].$$

From Eq.(3.1.15) we obtain

$$\mathbf{I}_0^+(\epsilon) = \mathbf{I}_1^+(\epsilon) = 2m\frac{\tilde{\Phi}(2m)}{0!} \int_0^{\frac{R-2m}{\epsilon}} \left[-\frac{1}{(\eta^2 + 1)^{1/2}} + \frac{\eta^2}{(\eta^2 + 1)^{3/2}}\right]d\eta +$$

$$+\frac{\epsilon}{1!} \int_0^{\frac{R-2m}{\epsilon}} \tilde{\Phi}^{(1)}(\xi)\left[-\frac{1}{(\eta^2 + 1)^{1/2}} + \frac{\eta^2}{(\eta^2 + 1)^{3/2}}\right]\eta d\eta +$$

$$+\frac{\epsilon\tilde{\Phi}(2m)}{0!} \int_{\frac{-2m}{\epsilon}}^{\frac{R-2m}{\epsilon}} \left[-\frac{1}{(\eta^2 + 1)^{1/2}} + \frac{\eta^2}{(\eta^2 + 1)^{3/2}}\right]d\eta + \tag{3.1.16}$$

$$+\frac{\epsilon^2}{1!} \int_0^{\frac{R-2m}{\epsilon}} \tilde{\Phi}^{(1)}(\xi)\left[-\frac{1}{(\eta^2 + 1)^{1/2}} + \frac{\eta^2}{(\eta^2 + 1)^{3/2}}\right]\eta d\eta,$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta + 2m)$ as

$$\begin{cases} \widetilde{\Phi}(\epsilon\eta + 2m) = \sum_{l=0}^{n-1} \frac{\Phi^{\alpha\beta(l)}(2m)}{l!}(\epsilon\eta)^l + \frac{1}{n!}(\epsilon\eta)^n\Phi^{\alpha\beta(n)}(\xi)\ , \\ \xi \triangleq \theta\epsilon\eta + 2m\ ,\quad 1 > \theta > 0\ ,\quad n = 1 \end{cases} \tag{3.1.17}$$

with $\tilde{\Phi}^{(l)}(\xi) \triangleq d^l\tilde{\Phi}/d\xi^l$.Equation (3.1.17) gives

$$w\text{ - }\lim_{\epsilon\to 0}\mathbf{I}_0^{+}(\epsilon) = w\text{ - }\lim_{\epsilon\to 0}\mathbf{I}_1^{+}(\epsilon) =$$

$$2m\tilde{\Phi}(2m)\lim_{\epsilon\to 0}\left\{\int_0^{\frac{R-2m}{\epsilon}}\left[-\frac{1}{(\eta^2+1)^{1/2}}+\frac{\eta^2}{(\eta^2+1)^{3/2}}\right]d\eta\right\} = \tag{3.1.18}$$

$$2m\tilde{\Phi}(2m)\lim_{s\to\infty}\left[\int_0^s\frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}}-\int_0^s\frac{d\eta}{(\eta^2+1)^{1/2}}\right] =$$

$$= -2m\tilde{\Phi}(2m).$$

where use is made of the relation

$$\lim_{s\to\infty}\left[\int_0^s\frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}}-\int_0^s\frac{d\eta}{(u^2+1)^{1/2}}\right] = -1 \tag{3.1.19}$$

Thus in $\mathcal{S}'_{2m}(B^{+}(2m,R)) \subset \mathcal{S}'_{2m}(\mathbb{R}^3)$ we obtain

$$w\text{ - }\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^{+}]_1^1 = w\text{ - }\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^{+}]_0^0 = -m\tilde{\Phi}(2m). \tag{3.1.20}$$

The Colombeau generalized Ricci tensor below horizon $[\mathbf{R}_\epsilon^{-}]_\alpha^\beta = [\mathbf{R}_\epsilon^{-}]_\alpha^\beta$ may now be calculated componentwise using the classical formulae

$$\begin{cases}\left([\mathbf{R}_\epsilon^{-}]_0^0\right)_\epsilon = \left([\mathbf{R}_\epsilon^{-}]_1^1\right)_\epsilon = \frac{1}{2}\left((h_\epsilon^{-''})_\epsilon+\frac{2}{r}(h_\epsilon^{-'})_\epsilon\right),\\ \left([\mathbf{R}_\epsilon^{-}]_2^2\right)_\epsilon = \left([\mathbf{R}_\epsilon^{-}]_3^3\right)_\epsilon = \frac{(h_\epsilon^{-'})_\epsilon}{r}+\frac{1+(h_\epsilon^{-})_\epsilon}{r^2}.\end{cases} \tag{3.1.21}$$

From Eq.(3.1.21) we obtain

$$h_\epsilon^{-}(r) = -\frac{r-2m}{r} \mapsto h_\epsilon^{-}(r) = \left(\frac{\sqrt{(2m-r)^2+\epsilon^2}}{r}\right) = -h_\epsilon^{+}(r), r < 2m.$$

$$h_\epsilon^{-'}(r) = -h_\epsilon^{+'}(r) = \frac{r-2m}{r\left[(r-2m)^2+\epsilon^2\right]^{1/2}}-\frac{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^2},$$

$$r(h_\epsilon^{-'})_\epsilon+1+(h_\epsilon^{-})_\epsilon = -r(h_\epsilon^{+'})_\epsilon+1-(h_\epsilon^{+})_\epsilon =$$

$$\frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}+1. \tag{3.1.22}$$

$$h_\epsilon^{-''}(r) = -h_\epsilon^{+''}(r) =$$

$$-\frac{r-2m}{r^2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}+\frac{2\left[(r-2m)^2+\epsilon^2\right]^{1/2}}{r^3}.$$

$$r^2(h_\epsilon^{-''})_\epsilon+2r(h_\epsilon^{-'})_\epsilon = -r^2(h_\epsilon^{+''})_\epsilon-2r(h_\epsilon^{+'})_\epsilon =$$

$$\frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}}-\frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}}.$$

Investigating the weak limit of the angular components of the Ricci tensor (using the abbreviation $\tilde{\Phi}(r) = \int_0^\pi \sin\theta d\theta\int_0^{2\pi}d\phi\Phi(x)$ where $\Phi(\vec{x}) \in C^\infty(\mathbb{R}^3)$, $\Phi(x)$ is a function with compact support K such that $K \subseteq B^{-}(0,2m) = \{x \in \mathbb{R}^3 | 0 \leqslant \|x\| \leqslant 2m\}$ we get:

$$\int_K \left([\mathbf{R}_\epsilon^-]_2^2\right)_\epsilon \Phi(\vec{x}) d^3x = \int_K \left([\mathbf{R}_\epsilon^-]_3^3\right)_\epsilon \Phi(\vec{x}) d^3x =$$
$$\int_0^{2m} \left(r(h_\epsilon^{-\prime})_\epsilon + 1 + (h_\epsilon^-)_\epsilon\right)\tilde{\Phi}(r)dr = \int_0^{2m} \left\{ \frac{r-2m}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} \right\} \tilde{\Phi}(r)dr + \int_0^{2m} \tilde{\Phi}(r)dr. \tag{3.1.23}$$

By replacement $r - 2m = u$, from Eq.(3.1.23) we obtain

$$\int_K \left([\mathbf{R}_\epsilon^-]_2^2\right)_\epsilon \Phi(x) d^3x = \int_K \left([\mathbf{R}_\epsilon^-]_3^3\right)_\epsilon \Phi(x) d^3x =$$
$$\int_{-2m}^{0} \frac{u\tilde{\Phi}(u+2m)du}{(u^2+\epsilon^2)^{1/2}} + \int_{-2m}^{0} \tilde{\Phi}(u+2m)du. \tag{3.1.24}$$

By replacement $u = \epsilon\eta$, from (3.1.23) we obtain

$$\mathbf{I}_3^-(\epsilon) = \int_K \left([\mathbf{R}_\epsilon^-]_3^3\right)_\epsilon \Phi(x) d^3x = \mathbf{I}_2^-(\epsilon) = \int_K \left([\mathbf{R}_\epsilon^-]_2^2\right)_\epsilon \Phi(\vec{x}) d^3x =$$
$$\epsilon \times \left(\int_{-\frac{2m}{\epsilon}}^{0} \frac{\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} + \int_{-\frac{2m}{\epsilon}}^{0} \tilde{\Phi}(\epsilon\eta+2m)d\eta \right), \tag{3.1.25}$$

which is calculated to give

$$\mathbf{I}_3^-(\epsilon) = \mathbf{I}_2^-(\epsilon) = \epsilon \frac{\tilde{\Phi}(2m)}{0!} \int_{-\frac{2m}{\epsilon}}^{0} \left[\frac{\eta}{(\eta^2+1)^{1/2}} + 1 \right] d\eta +$$
$$+ \frac{\epsilon^2}{1!} \int_{-\frac{2m}{\epsilon}}^{0} \left[\frac{\eta}{(\eta^2+1)^{1/2}} + 1 \right] \tilde{\Phi}^{(1)}(\xi)\eta d\eta = \tag{3.1.26}$$
$$\epsilon\tilde{\Phi}(2m)\left[1 - \sqrt{\left(\frac{2m}{\epsilon}\right)^2 + 1} + \frac{2m}{\epsilon} \right] + + \frac{\epsilon^2}{1} \int_{-\frac{2m}{\epsilon}}^{0} \left[\frac{\eta}{(\eta^2+1)^{1/2}} + 1 \right] \tilde{\Phi}^{(1)}(\xi)\eta d\eta,$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta + 2m)$ as

$$\widetilde{\Phi}(\epsilon\eta+2m) = \sum_{l=0}^{n-1} \frac{\Phi^{(l)}(2m)}{l!}(\epsilon\eta)^l + \frac{1}{n!}(\epsilon\eta)^n \Phi^{(n)}(\xi) \ ,$$
$$\xi \triangleq \theta\epsilon\eta + 2m \ , \quad 1 > \theta > 0 \ , \quad n = 1 \tag{3.1.27}$$

with $\tilde{\Phi}^{(l)} \triangleq d^l\tilde{\Phi}/dr^l$. Equation (3.1.27) gives

$$\lim_{\epsilon\to 0} \mathbf{I}_3^-(\epsilon) = \lim_{\epsilon\to 0} \mathbf{I}_2^-(\epsilon) =$$
$$\lim_{\epsilon\to 0} \left\{ \epsilon\tilde{\Phi}(2m)\left[1 - \sqrt{\left(\frac{2m}{\epsilon}\right)^2 + 1} + \frac{2m}{\epsilon} \right] \right\} + \tag{3.1.28}$$
$$+ \lim_{\epsilon\to 0} \left\{ \frac{\epsilon^2}{2} \int_{-\frac{2m}{\epsilon}}^{0} \left[\frac{\eta}{(\eta^2+1)^{1/2}} + 1 \right] \tilde{\Phi}^{(1)}(\xi)\eta d\eta \right\} = 0.$$

Thus in $\mathcal{S}'_{2m}(B_R^-(2m)) \subset \mathcal{S}'_{2m}(\mathbb{R}^3)$, where $B^-(0,2m) = \{x \in \mathbb{R}^3 | 0 \leqslant \|x\| \leqslant 2m\}$ from Eq.(3.1.28) we obtain

$$\begin{aligned} w-\lim_{\epsilon\to 0} [\mathbf{R}_\epsilon^-]_3^3 &= \lim_{\epsilon\to 0} \mathbf{I}_3^-(\epsilon) = 0. \\ w-\lim_{\epsilon\to 0} [\mathbf{R}_\epsilon^-]_2^2 &= \lim_{\epsilon\to 0} \mathbf{I}_2^-(\epsilon) = 0. \end{aligned} \tag{3.1.29}$$

For $\left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon, \left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon$ we get:

$$\begin{aligned} &2\int_K \left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon \Phi(x) d^3x = 2\int_K \left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon \Phi(x) d^3x = \\ &\int_0^{2m} \left(r^2 (h_\epsilon^{-\prime\prime})_\epsilon + 2r(h_\epsilon^{-\prime})_\epsilon\right)\tilde{\Phi}(r) dr = \\ &= \int_0^{2m} \left\{ \frac{r}{\left[(r-2m)^2+\epsilon^2\right]^{1/2}} - \frac{r(r-2m)^2}{\left[(r-2m)^2+\epsilon^2\right]^{3/2}} \right\} \tilde{\Phi}(r) dr. \end{aligned} \tag{3.1.30}$$

By replacement $r - 2m = u$, from (3.1.30) we obtain

$$\begin{aligned} &I_1^+(\epsilon) = 2\int \left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon \Phi(x) d^3x = I_2^+(\epsilon) = 2\int \left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon \Phi(x) d^3x \\ &= \int_0^{2m} \left(r^2 (h_\epsilon^{-\prime\prime})_\epsilon + 2r(h_\epsilon^{-\prime})_\epsilon\right)\tilde{\Phi}(r) dr = \\ &= \int_{-2m}^{0} \left\{ \frac{u+2m}{(u^2+\epsilon^2)^{1/2}} - \frac{u^2(u+2m)}{(u^2+\epsilon^2)^{3/2}} \right\} \tilde{\Phi}(u+2m) du. \end{aligned} \tag{3.1.31}$$

By replacement $u = \epsilon\eta$, from (3.1.31) we obtain

$$2\int_K\left([\mathbf{R}_\epsilon^-]_1^1\right)_\epsilon \Phi(x)d^3x = 2\int_K\left([\mathbf{R}_\epsilon^-]_0^0\right)_\epsilon \Phi(x)d^3x =$$

$$\int\limits_{-\frac{2m}{\epsilon}}^{0}\left(r^2(h_\epsilon^{-\prime\prime})_\epsilon + 2r(h_\epsilon^{-\prime})_\epsilon\right)\tilde{\Phi}(r)dr =$$

$$= \epsilon\int\limits_{-\frac{2m}{\epsilon}}^{0}\left\{\frac{\epsilon\eta+2m}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} - \frac{\epsilon^2\eta^2(\epsilon\eta+2m)}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}}\right\}\tilde{\Phi}(\epsilon\eta+2m)d\eta =$$

$$\int\limits_{-\frac{2m}{\epsilon}}^{0}\frac{\epsilon^2\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} + 2m\int\limits_{-\frac{2m}{\epsilon}}^{0}\frac{\epsilon\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{1/2}} - \tag{3.32}$$

$$-\int\limits_{-\frac{2m}{\epsilon}}^{0}\frac{\epsilon^4\eta^3\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} - 2m\int\limits_{-\frac{2m}{\epsilon}}^{0}\frac{\epsilon^3\eta^2\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\epsilon^2\eta^2+\epsilon^2)^{3/2}} =$$

$$\epsilon\int\limits_{-\frac{2m}{\epsilon}}^{0}\frac{\eta\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} - \int\limits_{-\frac{2m}{\epsilon}}^{0}\frac{\eta^3\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{3/2}} +$$

$$+2m\left[\int\limits_{-\frac{2m}{\epsilon}}^{0}\frac{\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{1/2}} - \int\limits_{-\frac{2m}{\epsilon}}^{0}\frac{\eta^2\tilde{\Phi}(\epsilon\eta+2m)d\eta}{(\eta^2+1)^{3/2}}\right].$$

which is calculated to give

$$\mathbf{I}_0^-(\epsilon) = \mathbf{I}_1^-(\epsilon) = 2m\frac{\tilde{\Phi}(2m)}{0!}\epsilon^l\int\limits_{-\frac{2m}{\epsilon}}^{0}\left[\frac{1}{(\eta^2+1)^{1/2}} - \frac{\eta^2}{(\eta^2+1)^{3/2}}\right]d\eta +$$
$$+\frac{\epsilon}{1!}\int\limits_{0}^{\frac{2m}{\epsilon}}\tilde{\Phi}^{(1)}(\xi)\left[\frac{1}{(\eta^2+1)^{1/2}} - \frac{\eta^2}{(\eta^2+1)^{3/2}}\right]\eta d\eta + O(\epsilon^2). \tag{3.33}$$

where we have expressed the function $\widetilde{\Phi}(\epsilon\eta+2m)$ as

$$\widetilde{\Phi}(\epsilon\eta+2m) = \sum_{l=0}^{n-1}\frac{\Phi^{\alpha\beta(l)}(2m)}{l!}(\epsilon\eta)^l + \frac{1}{n!}(\epsilon\eta)^n\Phi^{\alpha\beta(n)}(\xi)\ ,$$
$$\xi \triangleq \theta\epsilon\eta+2m\ ,\quad 1>\theta>0\ ,\quad n=1 \tag{3.34}$$

with $\tilde{\Phi}^{(l)}(\xi) \triangleq d^l\tilde{\Phi}/d\xi^l$.Equation (3.34) gives

$$\lim_{\epsilon\to0}\mathbf{I}_0^-(\epsilon) = \lim_{\epsilon\to0}\mathbf{I}_1^-(\epsilon) =$$

$$2m\lim_{\epsilon\to0}\left\{\frac{\tilde{\Phi}(2m)}{0!}\int\limits_{-\frac{2m}{\epsilon}}^{0}\left[\frac{1}{(\eta^2+1)^{1/2}} - \frac{\eta^2}{(\eta^2+1)^{3/2}}\right]d\eta\right\} =$$
$$2m\tilde{\Phi}(2m)\lim_{s\to0}\left[\int_{-s}^{0}\frac{d\eta}{(\eta^2+1)^{1/2}} - \int_{-s}^{0}\frac{\eta^2d\eta}{(\eta^2+1)^{3/2}}\right] =$$
$$= 2m\tilde{\Phi}(2m). \tag{3.35}$$

where use is made of the relation

$$\lim_{s\to\infty}\left[\int_{-s}^{0}\frac{d\eta}{(u^2+1)^{1/2}}-\int_{-s}^{0}\frac{\eta^2 d\eta}{(\eta^2+1)^{3/2}}\right]=1. \tag{3.36}$$

Thus in $\mathcal{S}'(B^-(0,2m))\subset\mathcal{S}'(\mathbb{R}^3)$ we obtain

$$w\text{ - }\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^-]_1^1 = w\text{ - }\lim_{\epsilon\to 0}[\mathbf{R}_\epsilon^-]_0^0 = m\tilde{\Phi}(2m). \tag{3.37}$$

Using Egs. (3.12),(3.20),(3.29),(3.37) we obtain

$$\iint\left[\left(\mathbf{T}_r^{+r}+\mathbf{T}_\theta^{+\theta}+\mathbf{T}_\phi^{+\phi}+\mathbf{T}_t^{+t}\right)+\left(\mathbf{T}_r^{-r}+\mathbf{T}_\theta^{-\theta}+\mathbf{T}_\phi^{-\phi}+\mathbf{T}_t^{-t}\right)\right]\sqrt{-g}\,d^3x=0 \tag{3.38}$$

Thus the Tolman formula [3],[4] for the total energy of a static and asymptotically flat spacetime with g the determinant of the four dimensional metric and d^3x the coordinate volume element, gives

$$E_T=\int\left(\mathbf{T}_r^r+\mathbf{T}_\theta^\theta+\mathbf{T}_\phi^\phi+\mathbf{T}_t^t\right)\sqrt{-g}\,d^3x=m, \tag{3.39}$$

We revrite now the Schwarzschild metric (3.3) in the form

$$\begin{cases}(ds_\epsilon^{\pm 2})_\epsilon=(h_\epsilon^\pm(r)dt^2)_\epsilon-((1+C_\epsilon^\pm(r))dr^2)_\epsilon+r^2d\Omega^2\\ C_\epsilon^\pm(r)=-1+[h_\epsilon^\pm(r)]^{-1}.\end{cases} \tag{3.40}$$

Using Eq.(A.5) from Eq.(3.40) one obtains for $r\asymp 2m$

$$\begin{gathered}(\mathbf{R}^{\pm\mu\nu}(\epsilon)\mathbf{R}^\pm_{\mu\nu}(\epsilon))_\epsilon=\\ \left(\left(\frac{1}{2}(h_\epsilon^\pm)''+\frac{1}{r}(h_\epsilon^\pm)'\right)^2\right)_\epsilon+2\left(\left[-\frac{(h_\epsilon^\pm)'}{r}+\frac{1}{r^2}\right]^2\right)_\epsilon\asymp\\ \asymp\left(\frac{\varepsilon^4}{4m^4\left[\varepsilon^2+(r_\varepsilon-2m)^2\right]^3}\right)_\varepsilon,\end{gathered} \tag{3.41}$$

and

$$[(\mathbf{R}^{\rho\sigma\mu\nu}(r,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r,\varepsilon))_\varepsilon]\asymp K(r_s)+\left(\frac{\varepsilon^4}{4m^4\left[\varepsilon^2+(r_\varepsilon-2m)^2\right]^3}\right)_\varepsilon. \tag{3.42}$$

3.2.Examples of distributional geometries. Calculation of the distributional quadratic scalars by using nonsmooth regularization via Horizon

Let us consider again the Schwarzschild metric (3.1)

$$\begin{cases}ds^2=h(r)dt^2-h(r)^{-1}dr^2+r^2d\Omega^2,\\ h(r)=-1+\dfrac{2m}{r}=-\dfrac{r-2m}{r},\\ h^{-1}(r)=-\dfrac{r}{r-2m}.\end{cases} \tag{3.43}$$

We revrite now the Schwarzschild metric (3.43) above Horizon ($r \geq 2m$) in the form

$$\begin{cases} ds^{+2} = -A^{+}(r)dt^2 + (A^{+}(r))^{-1}(r)dr^2 + r^2 d\Omega^2, \\ A^{+}(r) = \dfrac{r-2m}{r}, \\ (A^{+}(r))^{-1} = \dfrac{r}{r-2m}. \end{cases} \quad (3.44)$$

Following the above discussion we consider the singular metric coefficient $A^{-1}(r)$ as an element of $\mathcal{D}'(\mathbb{R}^3)$ and embed it into $(\mathcal{G}(\mathbb{R}^3))$ by replacement

$$r - 2m \mapsto \sqrt{r^2+\epsilon^2} - 2m. \quad (3.45)$$

Thus above Horizon ($r \geq 2m$) the corresponding distributional metric $\left(\widetilde{d}s_\epsilon^{+2}\right)_\epsilon$ takes the form

$$\begin{cases} \left(\widetilde{d}s_\epsilon^{+2}\right)_\epsilon = \left(-A_\epsilon^{+}(r)dt^2 + (A_\epsilon^{+}(r))^{-1}dr^2\right)_\epsilon + r^2 d\Omega^2, \\ (A_\epsilon^{+}(r))_\epsilon = \left(\dfrac{\sqrt{r^2+\epsilon^2} - 2m}{r}\right)_\epsilon, \\ \left((A_\epsilon^{+}(r))^{-1}\right)_\epsilon = \left(\dfrac{r}{\sqrt{r^2+\epsilon^2} - 2m}\right)_\epsilon. \end{cases} \quad (3.46)$$

We revrite now the Schwarzschild metric (3.43) below Horizon ($r < 2m$) in the form

$$\begin{cases} ds^{-2} = A^{-}(r)dt^2 - (A^{-}(r))^{-1}dr^2 + r^2 d\Omega^2, \\ A^{-}(r) = \dfrac{2m-r}{r}, (A^{-}(r))^{-1} = \dfrac{r}{2m-r}. \end{cases} \quad (3.47)$$

Following the above discussion we consider the singular metric coefficient $A^{-1}(r)$ as an element of $\mathcal{D}'(\mathbb{R}^3)$ and embed it into $(\mathcal{G}(\mathbb{R}^3))$ by replacement

$$2m - r \mapsto 2m - \sqrt{r^2+\epsilon^2}. \quad (3.48)$$

Thus belov Horizon ($r < 2m$) the corresponding distributional metric $\left(\widetilde{d}s_\epsilon^{-2}\right)_\epsilon$ takes the form

$$\begin{cases} (d\widetilde{s}_\epsilon^{-2})_\epsilon = \left(A_\epsilon^{-}(r)dt^2 - (A_\epsilon^{-}(r))^{-1}dr^2\right)_\epsilon + r^2 d\Omega^2, \\ (A_\epsilon^{-}(r))_\epsilon = \left(\dfrac{2m - \sqrt{r^2+\epsilon^2}}{r}\right)_\epsilon, \left((A_\epsilon^{-}(r))^{-1}\right)_\epsilon = \left(\dfrac{r}{2m - \sqrt{r^2+\epsilon^2}}\right)_\epsilon. \end{cases} \quad (3.49)$$

From Eq.(3.46) one obtains

$$\begin{cases} (A_\epsilon^{+})' = \left(-\dfrac{\sqrt{r^2+\epsilon^2} - 2m}{r}\right)' = -\dfrac{1}{\sqrt{r^2+\epsilon^2}} + \dfrac{\sqrt{r^2+\epsilon^2} - 2m}{r^2} \\ (A_\epsilon^{+})'' = \dfrac{r}{(r^2+\epsilon^2)^{3/2}} - 2\dfrac{\sqrt{r^2+\epsilon^2} - 2m}{r^3} + \dfrac{1}{r\sqrt{r^2+\epsilon^2}} \end{cases} \quad (3.50)$$

From Eq.(3.46) using Eq.(A.5) one obtains

$$(\mathbf{R}(\epsilon))_\epsilon = \left(-\frac{4A'_\epsilon}{r} + \frac{2A_\epsilon C_\epsilon}{r^2} - A''_\epsilon\right)_\epsilon =$$
$$-\frac{4}{r}\left(-\frac{1}{\sqrt{r^2+\epsilon^2}} + \frac{\sqrt{r^2+\epsilon^2} - 2m}{r^2}\right)_\epsilon + \frac{2}{r^2} -$$
$$-\left(\frac{r}{(r^2+\epsilon^2)^{3/2}} - 2\frac{\sqrt{r^2+\epsilon^2} - 2m}{r^3} + \frac{1}{r\sqrt{r^2+\epsilon^2}}\right)_\epsilon. \tag{3.51}$$

From Eq.(3.51) for $r = 2m$ one obtains

$$(\mathbf{R}(\epsilon))_\epsilon = c_1\delta(r-2m), \tag{3.52}$$

see subsect. 2.5,Remark 2.5.4.

Remark 3.3. Note that curvature scalar $(\mathbf{R}(\epsilon))_\epsilon$ again nonzero but nonsingular.

Let us introduce now the general metric which has the form [11]:

$$\begin{cases} ds^2 = -A(r)(dx^0)^2 - 2D(r)dx^0dr + (B(r) + C(r)(dr)^2 \\ \quad +B(r)r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]\ , \end{cases} \tag{3.53}$$

where

$$\begin{cases} A(r) = \Omega^2\left(1 - \frac{a}{K(r)}\right)\ ,\quad B(r) = \frac{K^2(r)}{\rho^2(r)}\ , \\ C(r) = \left(1 - \frac{a}{K(r)}\right)^{-1}(K'(r))^2 - \frac{K^2(r)}{\rho^2(r)} - \left(1 - \frac{a}{K(r)}\right)(f')^2\ , \\ D(r) = \Omega\left(1 - \frac{a}{K(r)}\right)f', K'(r) \triangleq dK(r)/dr, f'(r) \triangleq df(r)/dr, \\ K(r) = \rho(r) - |a|, \\ a < 0. \end{cases} \tag{3.54}$$

Note that the coordinates $t = x^0/c$ and r are time and space coordinates, respectively, only if

$$1 - \frac{a}{K} > 0\ ,\quad \left(1 - \frac{a}{K}\right)^{-1}(K')^2 - \left(1 - \frac{a}{K}\right)(f')^2 > 0\ . \tag{3.55}$$

In the Cartesian coordinate system $\{x^\mu; \mu = 0,1,2,3\}$ with

$$x^1 = r\cos\phi\sin\theta, x^2 = r\sin\phi\sin\theta, x^3 = r\cos\theta, \tag{3.56}$$

the metric (3.53)-(3.55) takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{3.57}$$

with $g_{\mu\nu}$ given by

$$g_{00} = -A\ ,\quad g_{0\alpha} = -D\frac{x^\alpha}{r}\ ,\quad g_{\alpha\beta} = B\delta^{\alpha\beta} + C\frac{x^\alpha x^\beta}{r^2}\ . \tag{3.58}$$

From Eq.(3.54) one obtain

$$A(r) = \Omega^2\left(\frac{\rho(r)}{\rho(r)-|a|}\right), \quad B = \frac{(\rho(r)-|a|)^2}{\rho^2(r)},$$
$$C(r) = \left(\frac{\rho(r)-|a|}{\rho(r)}\right) - \frac{(\rho(r)-|a|)^2}{\rho^2(r)} - \left(\frac{\rho(r)}{\rho(r)-|a|}\right)(f'(r))^2, \tag{3.59}$$
$$D(r) = \Omega\left(\frac{\rho(r)}{\rho(r)-|a|}\right)f'(r), f'(r) \triangleq df(r)/dr.$$

Regularizing the function $(\rho(r)-|a|)^{-1}$ above gorizon (under condition $\rho(r)-|a| \geq 0$) such as

$$\rho(r)-|a| \geq 0:$$
$$\left(\rho(r)-|a|\right)^{-1} \mapsto (\rho_\epsilon(r)-|a|)^{-1} = \left(\sqrt{\rho^2(r)+\epsilon^2}-|a|\right)^{-1} \tag{3.60}$$

with $\epsilon \in (0,1]$ from Eq.(3.59)-Eq.(3.60) one obtains

$$A_\epsilon^+(r) = \Omega^2\left(\frac{\rho_\epsilon(r)}{\rho_\epsilon(r)-|a|}\right), \quad B_\epsilon^+(r) = \frac{(\rho_\epsilon(r)-|a|)^2}{\rho_\epsilon^2(r)},$$
$$C_\epsilon^+(r) = \left(\frac{\rho_\epsilon(r)-|a|}{\rho_\epsilon(r)}\right) - \frac{(\rho_\epsilon(r)-|a|)^2}{\rho_\epsilon^2(r)} - \left(\frac{\rho_\epsilon(r)}{\rho_\epsilon(r)-|a|}\right)(f'(r))^2, \tag{3.61}$$
$$D_\epsilon^+(r) = \Omega\left(\frac{\rho_\epsilon(r)}{\rho_\epsilon(r)-|a|}\right)f'(r), f'(r) \triangleq df(r)/dr.$$

Regularizing the function $(|a|-\rho(r))^{-1}$ below gorizon (under condition $|a|-\rho(r) \geq 0$) such as

$$|a|-\rho(r) \geq 0:$$
$$(|a|-\rho(r))^{-1} \mapsto (|a|-\rho_\epsilon(r)) = \left(|a|-\sqrt{r^2+\epsilon^2}\right)^{-1} \tag{3.62}$$

with $\epsilon \in (0,1]$ from Eq.(3.59),Eq.(3.62) one obtains

$$A_\epsilon^-(r) = -\Omega^2\left(\frac{\rho_\epsilon(r)}{|a|-\rho_\epsilon(r)}\right), \quad B_\epsilon^-(r) = \frac{(|a|-\rho_\epsilon(r))^2}{\rho_\epsilon^2(r)},$$
$$C_\epsilon^-(r) = -\left(\frac{|a|-\rho_\epsilon(r)}{\rho_\epsilon(r)}\right) - \frac{(|a|-\rho_\epsilon(r))^2}{\rho_\epsilon^2(r)} + \left(\frac{\rho_\epsilon(r)}{|a|-\rho_\epsilon(r)}\right)(f'(r))^2, \tag{3.63}$$
$$D_\epsilon^-(r) = -\Omega\left(\frac{\rho_\epsilon(r)}{|a|-\rho_\epsilon(r)}\right)f'(r), f'(r) \triangleq df(r)/dr.$$

Remark 3.4. Finally the metric (3.57) becomes the Colombeau object of the form

$$(ds_\epsilon^2)_\epsilon = (g_{\mu\nu}^\pm(\epsilon)dx^\mu dx^\nu)_\epsilon \tag{3.64}$$

with $g_{\mu\nu}(\epsilon)$ given by

$$g_{00}^\pm(\epsilon) = -A_\epsilon^\pm(r), \quad g_{0\alpha}^\pm(\epsilon) = -D_\epsilon^\pm(r)\frac{x^\alpha}{r},$$
$$g_{\alpha\beta}^\pm(\epsilon) = B_\epsilon^\pm(r)\delta^{\alpha\beta} + C_\epsilon^\pm(r)\frac{x^\alpha x^\beta}{r^2}. \tag{3.65}$$

Using now Eq. A2 one obtains that the Colombeau curvature scalars $(\mathbf{R}^\pm(\epsilon))_\epsilon$ in terms of

Colombeau generalized functions $(A_\epsilon^\pm(r))_\epsilon, (B_\epsilon^\pm(r))_\epsilon, (C_\epsilon^\pm(r))_\epsilon, (D_\epsilon^\pm(r))_\epsilon$ is expressed as

$$
\begin{aligned}
(\mathbf{R}^+(\epsilon))_\epsilon &= \left(\frac{r^2+\epsilon^2}{\left(\sqrt{r^2+\epsilon^2}-|a|\right)^2} \left[\frac{9a\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} - \frac{2a\epsilon^2}{r^2(r^2+\epsilon^2)^{\frac{3}{2}}} \right] \right)_\epsilon \\
(\mathbf{R}^-(\epsilon))_\epsilon &= -\left(\frac{r^2+\epsilon^2}{\left(\sqrt{r^2+\epsilon^2}-|a|\right)^2} \left[\frac{9a\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} - \frac{2a\epsilon^2}{r^2(r^2+\epsilon^2)^{\frac{3}{2}}} \right] \right)_\epsilon
\end{aligned} \tag{3.66}
$$

Remark 3.6. Note that (i) on horizon $r = a$ Colombeau scalars $(\mathbf{R}^\pm(\epsilon))_\epsilon$ well defined and becomes to infinite large Colombeau generalized numbers

$$
\begin{cases}
(\mathbf{R}^+(\epsilon))_\epsilon = \\
\left(\frac{a^2+\epsilon^2}{\left(\sqrt{a^2+\epsilon^2}-|a|\right)^2} \left[\frac{9a\epsilon^2}{(a^2+\epsilon^2)^{\frac{5}{2}}} - \frac{2a\epsilon^2}{a^2(a^2+\epsilon^2)^{\frac{3}{2}}} \right] \right)_\epsilon = 7a^{-2}(\epsilon^{-2})_\epsilon \in \widetilde{\mathbb{R}}, \\
(\mathbf{R}^-(\epsilon))_\epsilon = \\
-\left(\frac{a^2+\epsilon^2}{\left(\sqrt{a^2+\epsilon^2}-|a|\right)^2} \left[\frac{9a\epsilon^2}{(a^2+\epsilon^2)^{\frac{5}{2}}} - \frac{2a\epsilon^2}{r^2(r^2+\epsilon^2)^{\frac{3}{2}}} \right] \right)_\epsilon = -7a^{-2}(\epsilon^{-2})_\epsilon \in \widetilde{\mathbb{R}}
\end{cases} \tag{3.67}
$$

(ii) for $r \neq a$ Colombeau scalars $(\mathbf{R}^\pm(\epsilon))_\epsilon$ well defined and becomes to infinite small Colombeau generalized numbers $(\mathbf{R}^\pm(\epsilon))_\epsilon \approx \pm(\epsilon^2)_\epsilon$.

Using now Eq. A2 one obtains that the Colombeau scalars $(\mathbf{R}^{\pm\mu\nu}(\epsilon)\mathbf{R}_{\mu\nu}^\pm(\epsilon))_\epsilon$ in terms of Colombeau generalized functions $(A_\epsilon^\pm(r))_\epsilon, (B_\epsilon^\pm(r))_\epsilon, (C_\epsilon^\pm(r))_\epsilon$, $(D_\epsilon^\pm(r))_\epsilon$ is expressed as

$$
\begin{cases}
(\mathbf{R}^{\pm\mu\nu}(\epsilon)\mathbf{R}_{\mu\nu}^\pm(\epsilon))_\epsilon = \\
\pm\left(\frac{(r^2+\epsilon^2)^2}{\left(\sqrt{r^2+\epsilon^2}-|a|\right)^4} \left\{ \left[\frac{5}{2}\left[\frac{3a\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} \right]^2 - \frac{2a\epsilon^2}{r^2(r^2+\epsilon^2)^{\frac{3}{2}}} \right] + \right.\right. \\
\left.\left. 2\left[\frac{3a\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}} + \frac{a\epsilon^2}{r^2(r^2+\epsilon^2)^{\frac{3}{2}}} \right]^2 \right\} \right)_\epsilon
\end{cases} \tag{3.68}
$$

Remark 3.7. Note that (i) on horizon $r = a$ Colombeau scalars $(\mathbf{R}^{\pm\mu\nu}(\epsilon)\mathbf{R}_{\mu\nu}^\pm(\epsilon))_\epsilon$ well defined and becomes to infinite large Colombeau generalized numbers, (ii) for $r \neq a$ Colombeau scalars $(\mathbf{R}^\pm(\epsilon))_\epsilon$ well defined and becomes to infinite small Colombeau generalized numbers.

Using now Eq. A2 one obtains that the Colombeau scalars $(\mathbf{R}^{\pm\rho\sigma\mu\nu}(\epsilon)\mathbf{R}_{\rho\sigma\mu\nu}^\pm(\epsilon))_\epsilon$ in terms of Colombeau generalized functions $(A_\epsilon^\pm(r))_\epsilon, (B_\epsilon^\pm(r))_\epsilon, (C_\epsilon^\pm(r))_\epsilon$, $(D_\epsilon^\pm(r))_\epsilon$ is expressed as

$$(\mathbf{R}^{\pm\rho\sigma\mu\nu}(\epsilon)\mathbf{R}^{\pm}_{\rho\sigma\mu\nu}(\epsilon))_\epsilon = \left(\frac{12a^2}{\left(\sqrt{r^2+\epsilon^2}-|a|\right)^6}\left[1+\frac{a\epsilon^2}{(r^2+\epsilon^2)^{\frac{3}{2}}}\right]^2 \mp\right.$$

$$\mp\frac{4a^2}{\left(\sqrt{r^2+\epsilon^2}-|a|\right)^5}\left[1+\frac{a\epsilon^2}{(r^2+\epsilon^2)^{\frac{3}{2}}}\right]^2\left[\frac{2\epsilon^2}{r^2(r^2+\epsilon^2)}+\frac{9\epsilon^2}{(r^2+\epsilon^2)^{\frac{5}{2}}}\right]+ \quad (3.69)$$

$$\frac{a^2}{\left(\sqrt{r^2+\epsilon^2}-|a|\right)^4}\left[\frac{4\epsilon^4}{r^4(r^2+\epsilon^2)}+\frac{81\epsilon^4}{(r^2+\epsilon^2)^3}\right].$$

Remark 3.8. Note that (i) on horizon $r = a$ Colombeau scalars $(\mathbf{R}^{\pm\rho\sigma\mu\nu}(\epsilon)\mathbf{R}^{\pm}_{\rho\sigma\mu\nu}(\epsilon))_\epsilon$ well defined and becomes to infinite large Colombeau generalized numbers, (ii) for $r \neq a$ Colombeau scalars finite

$$(\mathbf{R}^{\pm\rho\sigma\mu\nu}(\epsilon)\mathbf{R}^{\pm}_{\rho\sigma\mu\nu}(\epsilon))_\epsilon = \frac{12a^2}{(r^2-|a|)^6} \quad (3.70)$$

and tends to zero in the limit $r \to \infty$.

Remark 3.9. Note that under generalized transformatios such as

$$dt = \left(d\left[\frac{\sqrt{r^2+\epsilon^2}-2m}{r}\right]v_\epsilon^+\right)_\epsilon + \left(\frac{\sqrt{r^2+\epsilon^2}-2m}{r}dv_\epsilon^+\right)_\epsilon, \quad (3.71)$$

and

$$dt = \left(d\left[\frac{2m-\sqrt{r^2+\epsilon^2}}{r}\right]v_\epsilon^-\right)_\epsilon + \left(\frac{2m-\sqrt{r^2+\epsilon^2}}{r}dv_\epsilon^-\right)_\epsilon, \quad (3.72)$$

the metric given by Eq.(3.61)-Eq.(3.64) becomes to Colombeau metric of the form

$$\begin{cases} ds_\varepsilon^{\pm 2} = \mp A^\pm(r,\varepsilon)(dv_\epsilon^\pm)^2 - 2v_\epsilon^\pm D_2^\pm(r,\varepsilon)dv_\epsilon^\pm dr + [B^\pm(r,\varepsilon)+C_1^\pm(v_\epsilon^\pm,r,\varepsilon)](dr)^2 + \\ +B^\pm(r,\varepsilon)r^2[(d\theta)^2+\sin^2\theta(d\phi)^2]. \end{cases} \quad (3.73)$$

4. Quantum scalar field in curved distributional space-time

4.1 Canonical quantization in curved distributional space-time

Much of formalism can be explained with Colombeau generalized scalar field.The basic concepts and methods extend straightforwardly to distributional tensor and distributional spinor fields. To being with let's take a spacetime of arbitrary dimension D, with a metric $g_{\mu\nu}$ of signature$(+-\ldots-)$. The action for the Colombeau generalized scalar field $(\varphi_\varepsilon)_\varepsilon \in \mathcal{G}(M)$ is

$$(S_\varepsilon)_\varepsilon = \left(\int_M d^Dx\frac{1}{2}\sqrt{|g_\varepsilon|}\,(g_\varepsilon^{\mu\nu}\partial_\mu\varphi_\varepsilon\partial_\nu\varphi_\varepsilon) - (m^2+R_\varepsilon)\varphi_\varepsilon^2\right)_\varepsilon. \quad (4.1)$$

The corresponding equation of motion is

$$([\Box_\varepsilon + m^2 + \xi R_\varepsilon]\varphi_\varepsilon)_\varepsilon, \varepsilon \in (0,1]. \quad (4.2)$$

Here

$$(\Box_\varepsilon\varphi_\varepsilon)_\varepsilon = \left(|g_\varepsilon|^{-1/2}\partial_\mu|g_\varepsilon|^{1/2}g_\varepsilon^{\mu\nu}\partial_\mu\varphi_\varepsilon\right)_\varepsilon. \quad (4.3)$$

With $\hbar$ explicit, the mass m should be replaced by $m/\hbar$. Separating out a time coordinate

$x^0,\ x^\mu = (x^0, x^i), i = 1,2,3$ we can write the action as

$$(S_\varepsilon)_\varepsilon = \left(\int dx^0 L_\varepsilon\right)_\varepsilon, (L_\varepsilon)_\varepsilon = \left(\int d^{D-1}x\mathcal{L}_\varepsilon\right)_\varepsilon. \quad (4.4)$$

The canonical momentum at a time x^0 is given by

$$(\pi_\varepsilon(\underline{x}))_\varepsilon = (\delta L_\varepsilon/\delta(\partial_0\varphi_\varepsilon(\underline{x})))_\varepsilon = \left(|h_\varepsilon|^{1/2} n^\mu \partial_\mu \varphi_\varepsilon(\underline{x})\right)_\varepsilon, \quad (4.5)$$

where $\underline{x}$ labels a point on a surface of constant x^0, the x^0 argument of $(\varphi_\varepsilon)_\varepsilon$ is suppressed, n^μ is the unit normal to the surface, and $(|h_\varepsilon|)_\varepsilon$ is the determinant of the induced spatial metric $(h_{ij}(\varepsilon))_\varepsilon$. To quantize, the Colombeau generalized field $(\varphi_\varepsilon)_\varepsilon$ and its conjugate momentum $(\pi_\varepsilon(\underline{x}))_\varepsilon$ are now promoted to hermitian operators and required to satisfy the canonical commutation relation,

$$\left([\varphi_\varepsilon(\underline{x}), \pi_\varepsilon(\underline{y})]\right)_\varepsilon = i\hbar\delta^{D-1}(\underline{x},\underline{y}), \varepsilon \in (0,1]. \quad (4.6)$$

Here $\int d^{D-1}y\delta^{D-1}(\underline{x},\underline{y})f(\underline{y}) = f(\underline{x})$ for any scalar function $f \in D(\mathbb{R}^3)$, without the use of a metric volume element. We form now a conserved bracket from two complex Colombeau solutions to the scalar wave equation (4.2) by

$$(\langle\varphi_\varepsilon, \phi_\varepsilon\rangle)_\varepsilon = \left(\int_\Sigma d\Sigma_\mu j_\varepsilon^\mu\right)_\varepsilon, \varepsilon \in (0,1]. \quad (4.7)$$

where

$$(j_\varepsilon^\mu(\varphi_\varepsilon, \phi_\varepsilon))_\varepsilon = (i/\hbar)\left(|g_\varepsilon|^{1/2} g_\varepsilon^{\mu\nu}(\overline{\varphi}_\varepsilon \partial_\nu \phi_\varepsilon - \varphi_\varepsilon \partial_\nu \overline{\phi}_\varepsilon)\right)_\varepsilon. \quad (4.8)$$

This bracket is called the generalized Klein-Gordon inner product, and $(\langle\phi_\varepsilon, \phi_\varepsilon\rangle)_\varepsilon$ the generalized Klein Gordon norm of $(\varphi_\varepsilon)_\varepsilon$. The generalized current density $(j_\varepsilon^\mu(\varphi_\varepsilon, \phi_\varepsilon))_\varepsilon$ is divergenceless,i.e. $(\partial_\mu j_\varepsilon^\mu(\varphi_\varepsilon, \phi_\varepsilon))_\varepsilon = 0$ when the Colombeau generalized functions $(\varphi_\varepsilon)_\varepsilon$ and $(\phi_\varepsilon)_\varepsilon$ satisfy the KG equation (4.2), hence the value of the integral in (4.7) is independent of the spacelike surface Σ over which it is evaluated, provided the functions vanish at spatial infinity. The generalized KG inner product satisfies the relations

$$\left(\overline{\langle\varphi_\varepsilon, \phi_\varepsilon\rangle}\right)_\varepsilon = -(\langle\overline{\varphi}_\varepsilon, \overline{\phi}_\varepsilon\rangle)_\varepsilon = (\langle\phi_\varepsilon, \varphi_\varepsilon\rangle)_\varepsilon, \varepsilon \in (0,1]. \quad (4.9)$$

We define now the annihilation operator associated with a complex Colombeau solution $(\phi_\varepsilon)_\varepsilon$ by the bracket of $(\phi_\varepsilon)_\varepsilon$ with the generalized field operator $(\varphi_\varepsilon)_\varepsilon$:

$$(a(\phi_\varepsilon))_\varepsilon = (\langle\phi_\varepsilon, \varphi_\varepsilon\rangle)_\varepsilon. \quad (4.10)$$

It follows from the hermiticity of $(\varphi_\varepsilon)_\varepsilon$ that the hermitian conjugate of $(a(\phi_\varepsilon))_\varepsilon$ is given by

$$(a^\dagger(\phi_\varepsilon))_\varepsilon = -(a(\overline{\phi}_\varepsilon))_\varepsilon. \quad (4.11)$$

From Eq.(4.5) and CCR (4.6) one obtain

$$([a(\varphi_\varepsilon), a^\dagger(\phi_\varepsilon)])_\varepsilon = (\langle\varphi_\varepsilon, \phi_\varepsilon\rangle)_\varepsilon. \quad (4.12)$$

Note that from Eq.(4.11) follows

$$([a(\varphi_\varepsilon), a(\phi_\varepsilon)])_\varepsilon = -(\langle\varphi_\varepsilon, \overline{\phi}_\varepsilon\rangle)_\varepsilon, ([a^\dagger(\varphi_\varepsilon), a^\dagger(\phi_\varepsilon)])_\varepsilon = -(\langle\overline{\varphi}_\varepsilon, \phi_\varepsilon\rangle) \quad (4.13)$$

Note that if $(\phi_\varepsilon)_\varepsilon$ is a positive norm solution with unit norm and with, then $(a(\varphi_\varepsilon))_\varepsilon$ and $a^\dagger(\varphi_\varepsilon)$ satisfy the commutation relation $([a^\dagger(\varphi_\varepsilon), a(\phi_\varepsilon)])_\varepsilon = 1.$ Suppose now that $|\Psi\rangle$ is a normalized quantum state satisfying $(a(\phi_\varepsilon)|\Psi\rangle)_\varepsilon = 1,$ then for each $n,$ the state

$|n,\Psi\rangle = \left((1/\sqrt{n!})(a(\phi_\varepsilon))^n|\Psi\rangle\right)_\varepsilon$ is a normalized eigenstate of the number operator $(N[(\phi_\varepsilon)])_\varepsilon = (a^\dagger(\varphi_\varepsilon)a(\varphi_\varepsilon))_\varepsilon$ with eigenvalue $n.$ The span of all these states defines a Fock space of the distributional $(\varphi_\varepsilon)_\varepsilon$- wavepacket "$n$-particle excitations" above the state $|\Psi\rangle$. If we want to construct the full Hilbert space of the field theory in curved distributional spacetime,how can we proceed? We should find a decomposition of the space of complex Colombeau solutions to the wave equation (4.2) $\mathbf{S}$ into a direct sum of a positive norm subspace $\mathbf{S_p}$ and its complex conjugate $\mathbf{S_p}$, such that all brackets between solutions from the two subspaces vanish. That is, we must find a direct sum decomposition:

$$\mathbf{S} = \mathbf{S_p} \oplus \overline{\mathbf{S_p}} \tag{4.14}$$

such that

$$(\langle\varphi_\varepsilon,\varphi_\varepsilon\rangle)_\varepsilon > 0, \forall(\varphi_\varepsilon)_\varepsilon \in \mathbf{S_p} \tag{4.15}$$

and

$$(\langle\varphi_\varepsilon,\phi_\varepsilon\rangle)_\varepsilon > 0, \forall(\varphi_\varepsilon)_\varepsilon,(\phi_\varepsilon)_\varepsilon \in \mathbf{S_p}. \tag{4.16}$$

The condition (4.15) implies that each $(\varphi_\varepsilon)_\varepsilon$ in $\mathbf{S_p}$ can be scaled to define its own harmonic oscillator sub-albegra. The second condition implies, according to (4.13), that the annihilators and creators for $(\varphi_\varepsilon)_\varepsilon$ and $(\phi_\varepsilon)_\varepsilon$ in the subspace $\mathbf{S_p}$ commute amongst themselves:

$$([a(\varphi_\varepsilon),a(\phi_\varepsilon)])_\varepsilon = ([a^\dagger(\varphi_\varepsilon),a^\dagger(\phi_\varepsilon)])_\varepsilon = 0. \tag{4.17}$$

Given such a decompostion a total Hilbert space $\mathbf{H}$ for the field theory can be defined as the space of finite norm sums of possibly infinitely many states of the form

$$(a^\dagger(\phi_{1,\varepsilon})\ldots a^\dagger(\phi_{n,\varepsilon})|0\rangle)_\varepsilon, \tag{4.18}$$

where $|0\rangle$ is a state such that $(a(\phi_{n,\varepsilon})|0\rangle)_\varepsilon = 0$ for all $(\phi_\varepsilon)_\varepsilon$ in $\mathbf{S_p}$. The state $|0\rangle$, as in classical case, is called a Fock vacuum and Hilbert space $\mathbf{H}$ is called a Fock space. The representation of the field operator on this Fock space is hermitian and satisfies the canonical commutation relations in sense of Colombeau generalized function.

4.2 Defining distributional outgoing modes

For illustration we consider the non-rotating,uncharged d-dimensional SAdS BH with a distributional line element

$$(ds_\varepsilon^2)_\varepsilon = \left(-f_\varepsilon dt^2 + f_\varepsilon^{-1}dr^2\right)_\varepsilon + r^2 d\Omega_{d-2}^2, \varepsilon \in (0,1], \tag{4.19}$$

where

$$\begin{cases} f_\varepsilon \neq 0, \varepsilon \in (0,1], \\ f_{\varepsilon=0} = 1 + \dfrac{r^2}{L^2} - \dfrac{r_0^{d-3}}{r^{d-3}}, \end{cases} \tag{4.20}$$

where $d\Omega_{d-2}^2$ is the metric of the $(d-2)$-sphere, and the AdS curvature radius squared L^2 is related to the cosmological constant by $L^2 = -(d-2)(d-1)/2\Lambda$. The parameter r_0 is proportional to the mass M of the spacetime: $M = (d-2)A_{d-2}r_0^{d-3}/16\pi$, where $A_{d-2} = 2\pi^{(d-1)/2}/\Gamma[(d-1)/2]$. The distributional Schwarzschild geometry corresponds to $L \to \infty$. The corresponding equation of motion (4.2) for massless case are

$$(\nabla_\mu \nabla^\mu \varphi_\varepsilon)_\varepsilon = \frac{(d-2)\gamma}{4(d-1)}(R_\varepsilon)_\varepsilon,$$
$$(G_{\varepsilon,\mu\nu})_\varepsilon + \Lambda(g_{\varepsilon,\mu\nu})_\varepsilon = 8\pi G(T_{\varepsilon,\mu\nu})_\varepsilon, \tag{4.21}$$
$$(T_{\varepsilon,\mu\nu})_\varepsilon \sim \delta(x).$$

The time-independence and the spherical symmetry of the metric imply the canonical decomposition

$$(\varphi_\varepsilon(t,r,\theta))_\varepsilon = e^{-i\omega t}\left(\sum \frac{\Psi_{lm,\varepsilon}(r)Y_{lm}(\theta)}{r^{(d-2)/2}+\varepsilon}\right)_\varepsilon, \tag{4.22}$$

where $Y_{lm}(\theta)$ denotes the d-dimensional scalar spherical harmonics, satisfying

$$\Delta\Omega_{d-2}Y_{lm}(\theta) = -l(l+d-3)Y_{lm}(\theta), \tag{4.23}$$

the Laplace-Beltrami operator.Substituting the decomposition (4.22) into Eq. (6) one get a radial wave equation

$$\left(f_\varepsilon^2 \frac{d^2\Psi_{lm,\varepsilon}(r)}{dr^2} + f_\varepsilon f_\varepsilon' \frac{d\Psi_{lm,\varepsilon}(r)}{dr} + (\omega^2 - V_{lm,\varepsilon}(r))\Psi_{lm,\varepsilon}(r)\right)_\varepsilon = 0. \tag{4.24}$$

We define now a “tortoise” distributional coordinate $(r_\varepsilon^*)_\varepsilon = (r_\varepsilon^*(r))_\varepsilon$ by the relation

$$\left(\frac{dr_\varepsilon^*}{dr}\right)_\varepsilon = \left(f_\varepsilon^{-1}(r)\right)_\varepsilon. \tag{4.25}$$

By using a “tortoise” distributional coordinate the Eq.(4.24) can be written in the form of a Schrödinger equation with the potential $V_{lm,\varepsilon}(r)$

$$\left(\frac{d\Psi_\varepsilon(r_\varepsilon^*)}{dr_\varepsilon^*}\right)_\varepsilon + ((\omega^2 - V_\varepsilon(r_\varepsilon^*))\Psi_\varepsilon(r_\varepsilon^*))_\varepsilon = 0. \tag{4.26}$$

Note that the tortoise distributional coordinate $(r_\varepsilon^*(r))_\varepsilon$ becomes to infinite Colombeau constant $[(r_\varepsilon^*(r_+))_\varepsilon] = [(\ln\varepsilon)_\varepsilon]$ at the horizon, i.e. as $r \to r_+$, but its behavior at infinity is strongly dependent on the cosmological constant: $[(r_\varepsilon^*(r_+))_\varepsilon] = +\infty$ for asymptotically-flat spacetimes, and $[(r_\varepsilon^*(r_+))_\varepsilon]$ = finite Colombeau constant for the SAdS$_d$ geometry.

4.2.1. Boundary conditions at the horizon of the distributional SAdS BH geometry.

For most spacetimes of interest the potential $(V_\varepsilon(r_\varepsilon^*(r)))_\varepsilon = 0$ as $r = r_+$, i.e. $(|r_\varepsilon^*(r)|)_\varepsilon = +\infty$, and in this limit solutions to the wave equation (4.26) behave as

$$(\Psi_\varepsilon(t,r_\varepsilon^*))_\varepsilon \sim (\exp[-i\omega(t \pm r_\varepsilon^*(r))])_\varepsilon, \text{ as } r\sim r_+. \tag{4.27}$$

Note that classically nothing should leave the horizon and thus classically only ingoing modes (corresponding to a plus sign) should be present,i.e.

$$(\Psi_\varepsilon(t,r_\varepsilon^*))_\varepsilon \sim (\exp[-i\omega(t + r_\varepsilon^*(r))])_\varepsilon, \text{ as } r\sim r_+. \tag{4.28}$$

Note that for non-extremal spacetimes, the tortoise coordinate tends to

$$(r_\varepsilon^*(r))_\varepsilon = \left(\int f_\varepsilon^{-1}(r)dr\right)_\varepsilon \sim \left[\left(f_\varepsilon'(r_+)\right)_\varepsilon\right]^{-1} (\ln(|r-r_+|+\varepsilon))_\varepsilon \text{ as } r\sim r_+, \tag{4.29}$$

where $\left(f_\varepsilon'(r_+)\right)_\varepsilon > 0$. Therefore near the horizon, outgoing modes behave as

$$\begin{cases} (\exp[-i\omega(t-r_\varepsilon^*(r))])_\varepsilon = \{(\exp[-i\omega v_\varepsilon^*(t,r)])_\varepsilon\}(\exp[-2i\omega r_\varepsilon^*(r)])_\varepsilon = \\ = \{(\exp[-i\omega v_\varepsilon^*(t,r)])_\varepsilon\}\left\{\left([|r-r_+|+\varepsilon]^{2i\omega/f_\varepsilon'(r_+)}\right)_\varepsilon\right\}, \end{cases} \tag{4.30}$$

where $(v_\varepsilon^*(t,r))_\varepsilon = t + (r_\varepsilon^*(r))_\varepsilon$. Now Eq. (4.30) shows that outgoing modes is Colombeau generalized function of class $\mathcal{G}(\mathbb{R})$.

5. Energy-momentum tensor calculation by using Colombeau distributional modes

We shall assume now any distributional spacetime which is conformally static in both the asymptotic past and future. We will be considered distributional spacetime which is conformally flat in the asymptotic past,i.e.

$$\begin{cases} ds_\varepsilon^2 \sim (f_{\varepsilon,in}^2(-dt^2+d\vec{x}^2))_\varepsilon & \text{asymp. past} \\ ds_\varepsilon^2 \sim (f_{\varepsilon,out}^2(-dt^2+h_{\varepsilon,ij}dx^idx^j))_\varepsilon, & \text{asymp. future} \end{cases} \tag{5.1}$$

where $\varepsilon \in (0,1]$ $(f_{\varepsilon,\mathbf{J}})_\varepsilon = (f_{\varepsilon,\mathbf{J}}(t,\vec{x}))_\varepsilon > 0, \mathbf{J} \in \{in, out\}$, are smooth functions and $h_{\varepsilon,ij} = h_{\varepsilon,ij}(\vec{x}), i,j = 1,2,3$, are the components of an arbitrary distributional spatial metric. Note that we use the same labels t and $\vec{x} = (x^1,x^2,x^3)$ for coordinates in the asymptotic past and future only for simplicity; they are obviously defined on non-intersecting regions of the spacetime.) In each of these asymptotic regions the distributional field $(\Phi_\varepsilon)_\varepsilon$ can be written as $(\Phi_\varepsilon)_\varepsilon = (\tilde{\Phi}_\varepsilon/f_{\varepsilon,\mathbf{J}})_\varepsilon$, where $(\tilde{\Phi}_\varepsilon)_\varepsilon$ satisfies

$$-\left(\frac{\partial^2}{\partial t^2}\tilde{\Phi}\right)_\varepsilon = -(\Delta_{\varepsilon,\mathbf{J}}\tilde{\Phi}_\varepsilon)_\varepsilon + (V_{\varepsilon,\mathbf{J}}\tilde{\Phi}_\varepsilon)_\varepsilon, \tag{5.2}$$

where $(\Delta_{\varepsilon,in})_\varepsilon$ is the flat Laplace operator, $(\Delta_{\varepsilon,out})_\varepsilon$ is the Laplace operator associated with the spatial metric $(h_{\varepsilon ij})_\varepsilon$, and the effective potential V_J is given by

$$\begin{cases} (V_{\varepsilon,\mathbf{J}})_\varepsilon = \left(\dfrac{\Delta_{\varepsilon\mathbf{J}}f_{\varepsilon,\mathbf{J}}}{f_{\varepsilon,\mathbf{J}}}\right)_\varepsilon + \left(f_{\varepsilon,\mathbf{J}}^2(m^2+\xi R_\varepsilon)\right)_\varepsilon = \\ (1-6\xi)\left(\dfrac{\Delta_{\varepsilon\mathbf{J}}f_{\varepsilon,\mathbf{J}}}{f_{\varepsilon,\mathbf{J}}}\right)_\varepsilon + m^2\left(f_{\varepsilon,\mathbf{J}}^2\right)_\varepsilon + \xi(K_{\varepsilon,\mathbf{J}})_\varepsilon, \end{cases} \tag{5.3}$$

with $(K_{\varepsilon,in})_\varepsilon = 0$, $K_{\varepsilon,out} = K_{\varepsilon,out}(\vec{x})$ the scalar curvature associated with the spatial distributional metric $(h_{\varepsilon,ij})_\varepsilon$.

We assume now this condition: (i) the massless ($m = 0$) field with arbitrary coupling ξ in spacetimes which are asymptotically flat in the past and asymptotically static in the future,i.e. $f_{in} = 1$ and $f_{\varepsilon,out} = f_{\varepsilon,out}(\vec{x})$, as those describing the formation of a static BH from matter initially scattered throughout space, and (ii) the massless, conformally coupled field ($m = 0$ and $\xi = 1/6$). With this assumptions for the potential, two different sets of positive-norm distributional modes, $\left(u_{\varepsilon,\vec{k}}^{(+)}\right)_\varepsilon$ and $\left(v_{\varepsilon,\alpha}^{(+)}\right)_\varepsilon$, can be naturally defined by the requirement that they are the solutions of Eq.(5.2) which satisfy the asymptotic conditions:

$$\left(u_{\varepsilon,\vec{k}}^{(+)}\right)_\varepsilon \underset{\text{past}}{\asymp} (16\pi^3\omega_{\vec{k}})^{-1/2}\ e^{-i(\omega_{\vec{k}}t-\vec{k}\cdot\vec{x})}\left(f_{\varepsilon,in}^{-1}\right)_\varepsilon \tag{5.4}$$

and

$$\left(v^{(+)}_{\varepsilon,\alpha}\right)_\varepsilon \underset{\text{future}}{\asymp} (2\varpi_\alpha)^{-1/2}e^{-i\varpi_\alpha t}\left(f^{-1}_{\varepsilon,out}\ F_{\varepsilon,\alpha}(\vec{x})\right)_\varepsilon, \tag{5.5}$$

where $\vec{k} \in \mathbb{R}^3$, $\omega_{\vec{k}} := \|\vec{k}\|$, $\varpi_\alpha > 0$, and $(F_{\varepsilon,\alpha}(\vec{x}))_\varepsilon$ are Colombeau solutions of

$$\begin{gathered}([-\Delta_{\varepsilon,out} + V_{\varepsilon,out}(\vec{x})]F_{\varepsilon,\alpha}(\vec{x}))_\varepsilon = \varpi^2_\alpha(F_{\varepsilon,\alpha}(\vec{x}))_\varepsilon,\\ (F_{\varepsilon,\alpha}(\vec{x}))_\varepsilon|_{\varepsilon=0} \in C^\infty(\mathbb{R}^3)\end{gathered} \tag{5.6}$$

satisfying the normalization

$$\left(\int_{\Sigma_{out}} d^3x\sqrt{h}\ \ F_{\varepsilon,\alpha}(\vec{x})^*F_{\varepsilon,\beta}(\vec{x})\right)_\varepsilon = \delta_{\alpha,\beta} \tag{5.7}$$

on a Cauchy surface Σ_{out} in the asymptotic future. Note that each $F_{\varepsilon,\alpha}, \varepsilon \in [0,1]$ can be chosen to be real without loss of generality. There are reasonable situations where the distributional modes $\left(v^{(+)}_{\varepsilon,\alpha}\right)_\varepsilon$, given in Eq. (5.5), together with distributional modes $\left(v^{(-)}_{\varepsilon,\alpha}\right)_\varepsilon$ fail to form a complete set of distributional normal modes. This happens whenever the operator $([-\Delta_{\varepsilon,out} + V_{\varepsilon,out}(\vec{x})])_\varepsilon$ in Eq. (5.2) happens to possess normalizable i.e., satisfying Eq. (5.7) eigenfunctions with negative eigenvalues, $\varpi^2_\alpha = -\Omega^2_\alpha < 0$. In this case, additional positive-norm modes $\left(w^{(+)}_{\varepsilon,\alpha}\right)_\varepsilon$ with the asymptotic behavior

$$\left(w^{(+)}_{\varepsilon,\alpha}\ \right)_\varepsilon \underset{\text{future}}{\asymp} (e^{\Omega_\alpha t - i\pi/12} + e^{-\Omega_\alpha t + i\pi/12})\left(\frac{F_{\varepsilon,\alpha}(\vec{x})}{\sqrt{2\Omega_\alpha}\ \ f_{\varepsilon,out}(\vec{x})}\right)_\varepsilon \tag{5.8}$$

and their complex conjugates $\left(w^{(-)}_{\varepsilon,\alpha}\right)_\varepsilon$ are necessary in order to expand an arbitrary Colombeau solution of Eq.(5.1) As a direct consequence, at least some of the in-modes $\left(u^{(\pm)}_{\varepsilon,\vec{k}}\right)_\varepsilon$ (typically those with low $\omega_{\vec{k}}$) eventually undergo an exponential growth.This asymptotic divergence is reflected on the unbounded increase of the vacuum fluctuations,

$$(\langle\Phi^2_\varepsilon(\vec{x})\rangle)_\varepsilon \underset{\text{future}}{\asymp} \frac{\kappa\ e^{2\bar{\Omega}t}}{2\bar{\Omega}}\left[\left(\frac{\bar{F}_\varepsilon(\vec{x})}{f_{\varepsilon,out}(\vec{x})}\right)_\varepsilon\right]^2[1 + \mathcal{O}(e^{-\epsilon t})], \tag{5.9}$$

where $\bar{F}(\vec{x})$ is the eigenfunction of Eq. (5.6) associated with the lowest negative eigenvalue allowed, $\varpi^2_\alpha = -\bar{\Omega}^2$, ϵ is some positive constant, and κ is a dimensionless constant (typically of order unity) whose exact value depends globally on the spacetime structure (since it crucially depends on the projection of each $\left(u^{(\pm)}_{\varepsilon,\vec{k}}\right)_\varepsilon$ on the mode $\left(w^{(\pm)}_{\varepsilon,\alpha}\right)_\varepsilon$ whose $\varpi^2_\alpha = -\bar{\Omega}^2$; κ also depends on the initial state, here assumed to be the vacuum $|0\rangle_{in}$). As one would expect, these wild quantum fluctuations give an important contribution to the vacuum energy stored in the field. In fact, the expectation value of its distributional energy-momentum tensor, $(\langle T_{\varepsilon,\mu\nu}(\vec{x})\rangle)_\varepsilon, \varepsilon \in (0,1]$, in the asymptotic future is found to be dominated by this exponential growth:

$$\begin{cases}(\langle T_{\varepsilon,00}(\vec{x})\rangle)_\varepsilon \underset{\text{future}}{\asymp} \{(\langle\Phi^2_\varepsilon(\vec{x})\rangle)_\varepsilon\}\left\{\frac{(1-4\xi)}{2}\left(\bar{\Omega}^2 + \frac{(D\bar{F}_\varepsilon)^2}{\bar{F}^2} + m^2f^2_\varepsilon + \xi K_\varepsilon\right)_\varepsilon\right.\\ \qquad\left. +(1-6\xi)\left(\frac{2\xi D^2 f_\varepsilon}{f_\varepsilon} + \frac{(Df_\varepsilon)^2}{2f^2_\varepsilon} - \frac{D_i f_\varepsilon D^i\bar{F}_\varepsilon}{f_\varepsilon\bar{F}_\varepsilon}\right)_\varepsilon + \mathcal{O}(e^{-\epsilon t})\right\},\end{cases} \tag{5.10}$$

$$\begin{cases} ((\langle T_{\varepsilon,0i}(\vec{x})\rangle)_\varepsilon \underset{\text{future}}{\asymp} \\ \{((\langle\Phi_\varepsilon^2(\vec{x})\rangle)_\varepsilon\}\left\{(1-4\xi)\left(\frac{\bar{\Omega}D_i\bar{F}_\varepsilon}{\bar{F}_\varepsilon}\right)_\varepsilon-(1-6\xi)\left(\frac{\bar{\Omega}D_if_\varepsilon}{f_\varepsilon}\right)_\varepsilon+\mathcal{O}(e^{-\epsilon t})\right\}, \end{cases} \tag{5.11}$$

$$\begin{gathered} ((\langle T_{\varepsilon,ij}(\vec{x})\rangle)_\varepsilon \underset{\text{future}}{\asymp} \\ \{((\langle\Phi_\varepsilon^2(\vec{x})\rangle)_\varepsilon\}\left\{(1-2\xi)\left(\frac{D_i\bar{F}_\varepsilon D_j\bar{F}_\varepsilon}{\bar{F}_\varepsilon^2}\right)_\varepsilon-2\xi\left(\frac{D_iD_j\bar{F}_\varepsilon}{F_\varepsilon}\right)_\varepsilon+\xi\left(\widetilde{R}_{\varepsilon,ij}\right)_\varepsilon\right. \\ +\frac{(1-4\xi)h_{ij}}{2}\left(\bar{\Omega}^2-\left(\frac{(D\bar{F}_\varepsilon)^2}{\bar{F}_\varepsilon^2}-m^2f_\varepsilon^2-\xi K_\varepsilon\right)_\varepsilon\right) \\ +(1-6\xi)\left[\left(\frac{D_if_\varepsilon D_jf_\varepsilon}{f_\varepsilon^2}-\frac{D_ifD_j\bar{F}}{f_\varepsilon\bar{F}_\varepsilon}-\frac{D_jf_\varepsilon D_i\bar{F}_\varepsilon}{f_\varepsilon\bar{F}_\varepsilon}\right)_\varepsilon+\right. \\ \left.+(h_{\varepsilon,ij})_\varepsilon\left(\frac{2\xi D^2f_\varepsilon}{f_\varepsilon}-\frac{(Df_\varepsilon)^2}{2f_\varepsilon^2}+\frac{D_kf_\varepsilon D^k\bar{F}_\varepsilon}{f_\varepsilon\bar{F}_\varepsilon}\right)_\varepsilon\right]+\mathcal{O}(e^{-\epsilon t}), \end{gathered} \tag{5.12}$$

where D_i is the derivative operator compatible with the distributional metric $(h_{\varepsilon,ij})_\varepsilon$ (so that $\Delta_{out}=D^2$), $\left(\widetilde{R}_{\varepsilon,ij}\right)_\varepsilon$ is the associated distributional Ricci tensor so that $(K_{\varepsilon,out})_\varepsilon=\left(h_\varepsilon^{ij}\widetilde{R}_{\varepsilon,ij}\right)_\varepsilon$, and we have omitted the subscript *out* in $(f_{\varepsilon,out})_\varepsilon$ and $(K_{\varepsilon,out})_\varepsilon$ for simplicity. The Eqs. (5.10-5.12), together with Eq.(5.9), imply that on time scales determined by $\bar{\Omega}^{-1}$, the vacuum fluctuations of the field should overcome any other classical source of energy, therefore taking control over the evolution of the background geometry through the semiclassical Einstein equations (in which $(\langle T_{\varepsilon,\mu\nu}\rangle)_\varepsilon$ is included as a source term for the distributional Einstein tensor). We are then confronted with a startling situation where the quantum fluctuations of a field, whose energy is usually negligible in comparison with classical energy components, are forced by the distributional background spacetime to play a dominant role. We are still left with the task of showing that there exist indeed well-behaved distributional background spacetimes in which the operator $[(-\Delta_{\varepsilon,out}+V_{\varepsilon,out}(\vec{x}))_\varepsilon]$ possesses negative eigenvalues $\varpi_\alpha^2<0$, condition on which depends Eq(5.9). Experience from quantum mechanics tells us that this typically occurs when $(V_{\varepsilon,out})_\varepsilon$ gets sufficiently negative over a sufficiently large region. It is easy to see from Eq. (5.3) that, except for very special geometries (as the flat one), one can generally find appropriate values of $\xi\in\mathbb{R}$ which make $(V_{\varepsilon,out})_\varepsilon$ as negative as would be necessary in order to guarantee the existence of negative eigenvalues. For distributional BH spacetime using Eq.(5.9)-Eq.(5.12) one obtains

$$((\langle\Phi_\varepsilon^2(r)\rangle)_\varepsilon \underset{\text{future}}{\asymp} \frac{\kappa\ e^{2\bar{\Omega}t}}{2\bar{\Omega}}\left[\left(\frac{r^{1/2}\bar{F}_\varepsilon(r)}{\left((r-r_+)^2+\varepsilon^2\right)^{1/4}}\right)_\varepsilon\right]^2, r\to r_+ \tag{5.13}$$

$$\begin{gathered}\left(\langle T_{\varepsilon,00}(r)\rangle\right)_{\varepsilon} \underset{\text{future}}{\asymp} \\ \{(\langle\Phi_{\varepsilon}^{2}(r)\rangle)_{\varepsilon}\}\left\{\frac{(1-4\xi)}{2}\left(\bar{\Omega}^{2}+\frac{(D\bar{F}_{\varepsilon}(r_{+}))^{2}}{\bar{F}^{2}(r_{+})}+m^{2}\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/2}+\xi K_{\varepsilon}\right)_{\varepsilon}\right. \\ \left.+(1-6\xi)\left(-\frac{2\xi D^{2}f_{\varepsilon}(r)}{\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/4}}+\frac{(Df_{\varepsilon}(r))^{2}}{\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/4}}-\frac{D_{i}f_{\varepsilon}(r)D^{i}\bar{F}_{\varepsilon}}{\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/4}\bar{F}_{\varepsilon}}\right)_{\varepsilon}\right\}, \\ r\to r_{+}, f_{\varepsilon}(r)=\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/4}\end{gathered} \tag{5.14}$$

$$\begin{cases}\left(\langle T_{\varepsilon,0i}(r)\rangle\right)_{\varepsilon} \underset{\text{future}}{\asymp} \\ \{(\langle\Phi_{\varepsilon}^{2}(r)\rangle)_{\varepsilon}\}\left\{(1-4\xi)\left(\frac{\bar{\Omega}D_{i}\bar{F}_{\varepsilon}(r_{+})}{\bar{F}_{\varepsilon}(r_{+})}\right)_{\varepsilon}-(1-6\xi)\left(\frac{\bar{\Omega}D_{i}f_{\varepsilon}(r)}{\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/4}}\right)_{\varepsilon}\right\}, \\ r\to r_{+}, f_{\varepsilon}(r)=\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/4}\end{cases} \tag{5.15}$$

$$\begin{gathered}\left(\langle T_{\varepsilon,ij}(r)\rangle\right)_{\varepsilon} \underset{\text{future}}{\asymp} \\ \{(\langle\Phi_{\varepsilon}^{2}(r)\rangle)_{\varepsilon}\}\left\{(1-2\xi)\left(\frac{D_{i}\bar{F}_{\varepsilon}D_{j}\bar{F}_{\varepsilon}}{\bar{F}_{\varepsilon}^{2}}\right)_{\varepsilon}-2\xi\left(\frac{D_{i}D_{j}\bar{F}_{\varepsilon}}{F_{\varepsilon}}\right)_{\varepsilon}+\xi\left(\widetilde{R}_{\varepsilon,ij}\right)_{\varepsilon}\right. \\ +\frac{(1-4\xi)h_{ij}}{2}\left(\bar{\Omega}^{2}-\left(\frac{(D\bar{F}_{\varepsilon}(r_{+}))^{2}}{\bar{F}_{\varepsilon}^{2}(r_{+})}-m^{2}\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/2}-\xi K_{\varepsilon}\right)_{\varepsilon}\right) \\ +(1-6\xi)\left[\left(\frac{D_{i}f_{\varepsilon}D_{j}f_{\varepsilon}}{\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/2}}-\frac{D_{i}fD_{j}\bar{F}}{\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/4}\bar{F}_{\varepsilon}}-\frac{D_{j}f_{\varepsilon}D_{i}\bar{F}_{\varepsilon}}{\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/2}\bar{F}_{\varepsilon}}\right)_{\varepsilon}+\right. \\ \left.+(h_{\varepsilon,ij})_{\varepsilon}\left(\frac{2\xi D^{2}f_{\varepsilon}}{\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/4}}-\frac{(Df_{\varepsilon})^{2}}{2\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/2}}+\frac{D_{k}f_{\varepsilon}D^{k}\bar{F}_{\varepsilon}}{\left((r-r_{+})^{2}+\varepsilon^{2}\right)^{1/4}\bar{F}_{\varepsilon}}\right)_{\varepsilon}\right], \\ r\to r_{+}.\end{gathered} \tag{5.16}$$

Remark 5.1.Note that in spite of the unbounded growth at $r\to r_{+}$ in Eq.(5.13)-Eq.(5.16), $(\langle T_{\varepsilon,\mu\nu}\rangle)_{\varepsilon}$ is covariantly conserved: $(\nabla_{\mu}\langle T_{\varepsilon,\nu}^{\mu}\rangle)_{\varepsilon}=0$. In the static case $(f_{\varepsilon,out})_{\varepsilon}=(f_{\varepsilon,out}(\vec{x}))_{\varepsilon}$, for instance for distributional BH geometry, this implies that the total vacuum energy is kept constant, although it continuously flows from spatial regions where its density is negative to spatial regions where it is positive.

Remark 5.2. Note that the singular behavior at $r\to r_{+}$ appearing in Eq.(5.13)-Eq.(5.16) leads only to asymptotic divergences, i.e. all the quantities remain finite everywhere except horizon.

6. Distributional SAdS BH spacetime-induced vacuum dominance

6.1. Adiabatic expansion of Green functions

Using equation of motion Eq.(5.2) one can obtain corresponding distributional generalization of the canonical Green functions equations. In particular for the distributional propagator

$$iG_\varepsilon^\pm(x,x') = \langle 0|T(\varphi_\varepsilon^\pm(x)\varphi_\varepsilon^\pm(x'))|0\rangle, \varepsilon \in (0,1] \tag{6.1}$$

one obtains directly

$$([\Box_{\varepsilon,x} + m^2 + \xi\mathbf{R}^\pm(x,\varepsilon)]G_\varepsilon^\pm(x,x'))_\varepsilon = -[-g^\pm(x,\varepsilon)]^{-1/2}\delta^n(x-x'). \tag{6.2}$$

Special interest attaches to the short distance behaviour of the Green functions, such as $(G_\varepsilon^\pm(x,x'))_\varepsilon$ in the limit $\|x-x'\| \to 0$ with a fixed $\varepsilon \in (0,1]$. We obtan now an adiabatic expansion of $(G_\varepsilon^\pm(x,x'))_\varepsilon$. Introducing Riemann normal coordinates y^μ for the point x, with origin at the point x' we have expanding

$$\begin{aligned}(g_{\mu\upsilon}^\pm(x,\varepsilon))_\varepsilon &= \eta_{\mu\upsilon} + \frac{1}{3}\left[(\mathbf{R}_{\mu\alpha\upsilon\beta}^\pm(\varepsilon))_\varepsilon\right]y^\alpha y^\beta - \frac{1}{6}\left[(\mathbf{R}_{\mu\alpha\upsilon\beta;\gamma}^\pm(\varepsilon))_\varepsilon\right]y^\alpha y^\beta y^\gamma + \\ &+\left[\frac{1}{20}(\mathbf{R}_{\mu\alpha\upsilon\beta;\gamma\delta}^\pm(\varepsilon))_\varepsilon + \frac{2}{45}\left[(\mathbf{R}_{\alpha\mu\beta\lambda}^\pm(\varepsilon))_\varepsilon\right](\mathbf{R}_{\gamma\upsilon\delta}^{\pm\lambda}(\varepsilon))_\varepsilon\right]y^\alpha y^\beta y^\gamma y^\delta + \ldots\end{aligned} \tag{6.3}$$

where $\eta_{\mu\upsilon}$ is the Minkowski metric tensor, and the coefficients are all evaluated at $y = 0$. Defining now

$$(\mathcal{L}_\varepsilon^\pm(x,x'))_\varepsilon = \left[\left((-g_{\mu\upsilon}^\pm(x,\varepsilon))^{1/4}\right)_\varepsilon\right](G_\varepsilon^\pm(x,x'))_\varepsilon \tag{6.4}$$

and its Colombeau-Fourier transform by

$$(\mathcal{L}_\varepsilon^\pm(x,x'))_\varepsilon = (2\pi)^{-n}\left(\int d^n k e^{-iky}\mathcal{L}_\varepsilon^\pm(k)\right)_\varepsilon \tag{6.5}$$

where $ky = \eta^{\alpha\beta}k_\alpha y_\beta$, one can work in a sort of localized momentum space. Expanding (6.2) in normal coordinates and converting to k-space, $(\mathcal{L}_\varepsilon^\pm(k))_\varepsilon$ can readily be solved by iteration to any adiabatic order. The result to adiabatic order four (i.e., four derivatives of the metric) is

$$\begin{aligned}(\mathcal{L}_\varepsilon^\pm(k))_\varepsilon &= (k^2-m^2)^{-1} - \left(\frac{1}{6}-\xi\right)(k^2-m^2)^{-2}(\mathbf{R}^\pm(\varepsilon))_\varepsilon + \\ &+\frac{i}{2}\left(\frac{1}{6}-\xi\right)\partial^\alpha(k^2-m^2)^{-2}(\mathbf{R}_{;\alpha}^\pm(\varepsilon))_\varepsilon - \\ &-\frac{1}{3}\left[(a_{\alpha\beta}^\pm(\varepsilon))_\varepsilon\right]\partial^\alpha\partial^\beta(k^2-m^2)^{-2} + \left[\left(\frac{1}{6}-\xi\right)^2(\mathbf{R}^{\pm2}(\varepsilon))_\varepsilon + \frac{2}{3}\left(a_\lambda^{\pm\lambda}(\varepsilon)\right)_\varepsilon\right](k^2-m^2)^{-3}\end{aligned} \tag{6.6}$$

where $\partial_\alpha = \partial/\partial k^\alpha$,

$$\begin{aligned}(a_{\alpha\beta}^\pm(\varepsilon))_\varepsilon &\asymp \left(\frac{1}{2}-\xi\right)(\mathbf{R}_{;\alpha\beta}^\pm(\varepsilon))_\varepsilon + \frac{1}{120}(\mathbf{R}_{;\alpha\beta}^\pm(\varepsilon))_\varepsilon - \frac{1}{140}\left(\mathbf{R}_{\alpha\beta;\lambda}^{\pm\ \ \ \lambda}(\varepsilon)\right)_\varepsilon - \\ &-\frac{1}{30}\left[\left(\mathbf{R}_\alpha^{\pm\ \lambda}(\varepsilon)\right)_\varepsilon\right](\mathbf{R}_{\lambda\beta}^\pm(\varepsilon))_\varepsilon + \frac{1}{60}\left[\left(\mathbf{R}^{\pm\kappa\ \lambda}{}_{\alpha\ \beta}(\varepsilon)\right)_\varepsilon\right](\mathbf{R}_{\kappa\lambda}^\pm(\varepsilon))_\varepsilon + \\ &+\frac{1}{60}\left[\left(\mathbf{R}^{\pm\ \lambda\mu\kappa}{}_\alpha(\varepsilon)\right)_\varepsilon\right](\mathbf{R}_{\lambda\mu\kappa\beta}^\pm(\varepsilon))_\varepsilon,\end{aligned} \tag{6.7}$$

and we are using the symbol $\asymp$ to indicate that this is an asymptotic expansion. One ensures that Eq.(6.5) represents a time-ordered product by performing the k^0 integral along the appropriate contour in Pic.3. This is equivalent to replacing m^2 by $m^2 - i\epsilon$. Similarly, the adiabatic expansions of other Green functions can be obtained by using the other contours in Pic.3.Substituting Eq.(6.6) into Eq.(6.5) gives

$$\begin{cases} (\mathcal{L}_{\varepsilon}^{\pm}(x,x'))_{\varepsilon} = (2\pi)^{-n} \times \\ \left(\int d^n k e^{-iky}\left[a_0^{\pm}(x,x';\varepsilon) + a_1^{\pm}(x,x';\varepsilon)\left(-\frac{\partial}{\partial m^2}\right) + a_2^{\pm}(x,x';\varepsilon)\left(\frac{\partial}{\partial m^2}\right)^2\right](k^2 - m^2)^{-1}\right)_{\varepsilon} \end{cases} \quad (6.8)$$

where

$$(a_0^{\pm}(x,x';\varepsilon))_{\varepsilon} = 1 \quad (6.9)$$

and, to adiabatic order 4,

$$\begin{cases} (a_1^{\pm}(x,x';\varepsilon))_{\varepsilon} = \\ \left(\frac{1}{6} - \xi\right)(\mathbf{R}^{\pm}(\varepsilon))_{\varepsilon} - \frac{i}{2}\left(\frac{1}{6} - \xi\right)[(\mathbf{R}^{\pm}_{;\alpha}(\varepsilon))_{\varepsilon}]y^{\alpha} - \frac{1}{3}\left[(a^{\pm}_{\alpha\beta}(\varepsilon))_{\varepsilon}\right]y^{\alpha}y^{\beta} \\ (a_2^{\pm}(x,x';\varepsilon))_{\varepsilon} = \frac{1}{2}\left(\frac{1}{6} - \xi\right)(\mathbf{R}^{\pm 2}(\varepsilon))_{\varepsilon} + \frac{1}{3}\left(a^{\pm\lambda}_{\lambda}(\varepsilon)\right)_{\varepsilon} \end{cases} \quad (6.10)$$

with all geometric quantities on the right-hand side of Eq.(6.10) evaluated at x'.

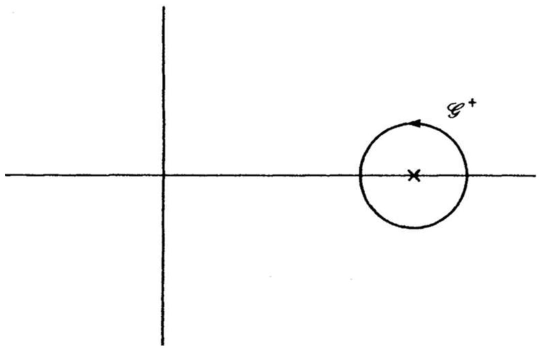

Fig.4.The contour in the complex k^0plane $\mathbb{C}$ to be used in the evaluation of the integral giving $\mathcal{L}^+$.The cross indicates the pole at $k^0 = (|\mathbf{k}|^2 + m^2)^{1/2}$.

If one uses the canonical integral representation

$$(k^2 - m^2 + i\epsilon)^{-1} = -i\int_0^{\infty} ds e^{is(k^2 - m^2 + i\epsilon)} \quad (6.11)$$

in Eq.(6.8), then the $d^n k$ integration may be interchanged with the ds integration, and performed explicitly to yield (dropping the $i\epsilon$)

$$(\mathcal{L}_{\varepsilon}^{\pm}(x,x'))_{\varepsilon} = -i(4\pi)^{-n/2}\left(\int_0^{\infty} ids(is)^{-n/2}\exp\left[-im^2 s + \frac{\sigma(x,x')}{2is}\right]\mathcal{F}_{\varepsilon}^{\pm}(x,x';is)\right)_{\varepsilon} \quad (6.12)$$

$$\sigma(x,x') = \frac{1}{2}y_{\alpha}y^{\alpha}.$$

The function $\sigma(x,x')$ which is one-half of the square of the proper distance between x and x', while the function $(\mathcal{F}_{\varepsilon}(x,x';is))_{\varepsilon}$ has the following asymptotic adiabatic expansion

$$(\mathcal{F}^{\pm}_{\varepsilon}(x,x';is))_{\varepsilon} \asymp (a^{\pm}_{0}(x,x';\varepsilon))_{\varepsilon} + is(a^{\pm}_{1}(x,x';\varepsilon))_{\varepsilon} + (is)^{2}(a^{\pm}_{2}(x,x';\varepsilon))_{\varepsilon} + \ldots \tag{6.13}$$

Using Eq.(6.4), equation (6.12) gives a representation of $(G^{\pm}_{\varepsilon}(x,x'))_{\varepsilon}$:

$$\begin{cases} (G^{\pm}_{\varepsilon}(x,x'))_{\varepsilon} = \\ -i(4\pi)^{-n/2}\left(\left[\left(\Delta^{1/2}_{\pm}(x,x';\varepsilon)\right)_{\varepsilon}\right]\int\limits_{0}^{\infty} ids(is)^{-n/2}\exp\left[-im^{2}s + \frac{\sigma(x,x')}{2is}\right]\mathcal{F}_{\varepsilon}(x,x';is)\right)_{\varepsilon} \end{cases} \tag{6.14}$$

where $(\Delta_{\pm}(x,x';\varepsilon))_{\varepsilon}$ is the distributional Van Vleck determinant

$$(\Delta_{\pm}(x,x';\varepsilon))_{\varepsilon} = -\det[\partial_{\mu}\partial_{\nu}\sigma(x,x')]\left([g^{\pm}(x,\varepsilon)g^{\pm}(x',\varepsilon)]^{-1/2}\right)_{\varepsilon} \tag{6.15}$$

In the normal coordinates about x' that we are currently using, $(\Delta_{\pm}(x,x';\varepsilon))_{\varepsilon}$ reduces to $\left([-g^{\pm}(x,\varepsilon)]^{-1/2}\right)_{\varepsilon}$. The full asymptotic expansion of $(\mathcal{F}^{\pm}_{\varepsilon}(x,x';is))_{\varepsilon}$ to all adiabatic orders are

$$(\mathcal{F}^{\pm}_{\varepsilon}(x,x';is))_{\varepsilon} \asymp \sum_{j=0}^{\infty}(is)^{j}(a^{\pm}_{2}(x,x';\varepsilon))_{\varepsilon} \tag{6.16}$$

with $(a^{\pm}_{0}(x,x';\varepsilon))_{\varepsilon} = 1$, the other $(a^{\pm}_{j}(x,x';\varepsilon))_{\varepsilon}$ being given by canonical recursion relations which enable their adiabatic expansions to be obtained. The expansions (6.13) and (6.16) are, however, only asymptotic approximations in the limit of large adiabatic parameter T.

If (6.16) is substituted into (6.14) the integral can be performed to give the adiabatic expansion of the Feynman propagator in coordinate space:

$$\begin{aligned}(G^{\pm}_{\varepsilon}(x,x'))_{\varepsilon} \asymp -(4\pi i)^{-n/2}\Bigg(&\Delta^{1/2}_{\pm}(x,x';\varepsilon)\sum_{j=0}^{\infty}a^{\pm}_{j}(x,x';\varepsilon)\left(-\frac{\partial}{\partial m^{2}}\right)^{j}\times \\ &\times\left[\left(-\frac{2m^{2}}{\sigma}\right)^{\frac{n-2}{4}}H^{(2)}_{(n-2)/2}\left((2m^{2}\sigma)^{\frac{1}{2}}\right)\right]\Bigg)_{\varepsilon}\end{aligned} \tag{6.17}$$

which, strictly, a small imaginary part $i\epsilon$ should be subtracted from σ.Since we have not imposed global boundary conditions on the distributional Green function Colombeau solution of (6.2), the expansion (6.17) does not determine the particular vacuum state in (6.1). In particular, the "$i\epsilon$" in the expansion of $(G^{\pm}_{\varepsilon}(x,x'))_{\varepsilon}$ only ensures that (6.17) represents the expectation value, in some set of states, of a time-ordered product of fields. Under some circumstances the use of "$i\epsilon$" in the exact representation (6.14) may give additional information concerning the global nature of the states

6.2. Effective action for the quantum matter fields in curved distributional spcetime

As in classical case one can obtain Colombeau generalized quantity $(W_{\varepsilon})_{\varepsilon}$, called the effective action for the quantum matter fields in curved distributional spcetime, which, when functionally differentiated, yields

$$\left(\frac{2}{(-g(\varepsilon))^{\frac{1}{2}}}\frac{\delta W_\varepsilon}{\delta g^{\mu\nu}(\varepsilon)}\right)_\varepsilon = (\langle \mathbf{T}_{\mu\nu}(\varepsilon)\rangle)_\varepsilon \tag{6.18}$$

To discover the structure of $(W_\varepsilon)_\varepsilon$, let us return to first principles, recalling the Colombeau path-integral quantization procedure such as developed in [34],[35],[36]. Our notation will imply a treatment for the scalar field, but the formal manipulations are identical for fields of higher spins. Note that the generating functional

$$(Z_\varepsilon[\mathbf{J}_\varepsilon])_\varepsilon = \left(\int D[\varphi_\varepsilon]\exp\left\{iS_{\mathbf{m}}(\varepsilon) + i\int \mathbf{J}_\varepsilon(x)\varphi_\varepsilon(x)d^n x\right\}\right)_\varepsilon \tag{6.19}$$

was interpreted physically as the vacuum persistence amplitude $(\langle \mathbf{out}_\varepsilon, 0|0, \mathbf{in}_\varepsilon\rangle)_\varepsilon$. The presence of the external distributional current density $(\mathbf{J}_\varepsilon)_\varepsilon$ can cause the initial vacuum state $(|0,\mathbf{in}_\varepsilon\rangle)_\varepsilon$ to be unstable, i.e., it can bring about the production of particles. In flat space, in the limit $(\mathbf{J}_\varepsilon)_\varepsilon = 0,$ no particles are produced, and one have the normalization condition

$$(Z_\varepsilon[0])_\varepsilon = \left(\int D[\varphi_\varepsilon]\exp\left\{iS_{\mathbf{m}}(\varepsilon) + i\int \mathbf{J}_\varepsilon(x)\varphi_\varepsilon(x)d^n x\right\}\right)_\varepsilon\Big|_{\mathbf{J}=0} = (\langle 0_\varepsilon|0_\varepsilon\rangle)_\varepsilon = 1. \tag{6.20}$$

However, when distributional spacetime is curved, we have seen that, in general,

$$(|0,\mathbf{out}_\varepsilon\rangle)_\varepsilon \neq (|0,\mathbf{in}_\varepsilon\rangle)_\varepsilon, \tag{6.21}$$

even in the absence of source currents J. Hence (6.19) will no longer apply.
Path-integral quantization still works in curved distributional spacetime; one simply treats $(S_{\mathbf{m}}(\varepsilon))_\varepsilon$ in (6.19) as the curved distributional spacetime matter action, and $(\mathbf{J}_\varepsilon(x))_\varepsilon$ as a current density (a scalar density in the case of scalar fields). One can thus set $\mathbf{J}_\varepsilon = 0$ in (6.19) and examine the variation of $(Z_\varepsilon[0])_\varepsilon$:

$$(\delta Z_\varepsilon[0])_\varepsilon = i\int D[\varphi_\varepsilon]\delta S_{\mathbf{m}}(\varepsilon)\exp[iS_{\mathbf{m}}(\varphi_\varepsilon;\varepsilon)] = i(\langle \mathbf{out}_\varepsilon, 0|\delta S_{\mathbf{m}}(\varepsilon)|0,\mathbf{in}_\varepsilon\rangle)_\varepsilon. \tag{6.22}$$

Note that

$$\left(\frac{2}{(-g(\varepsilon))^{\frac{1}{2}}}\frac{\delta S_{\mathbf{m}}(\varepsilon)}{\delta g^{\mu\nu}(\varepsilon)}\right)_\varepsilon = (\mathbf{T}_{\mu\nu}(\varepsilon))_\varepsilon. \tag{6.23}$$

From (6.22) and (6.23) one obtains directly

$$\left(\frac{2}{(-g(\varepsilon))^{\frac{1}{2}}}\frac{\delta Z_\varepsilon[0]}{\delta g^{\mu\nu}(\varepsilon)}\right)_\varepsilon = i(\langle \mathbf{out}_\varepsilon, 0|\mathbf{T}_{\mu\nu}(\varepsilon)|0,\mathbf{in}_\varepsilon\rangle)_\varepsilon \tag{6.24}$$

Noting that the matter action $S_{\mathbf{m}}(\varepsilon)$ appears exponentiated in (6.19), one obtains directly

$$Z_\varepsilon[0] = (\exp(iW_\varepsilon))_\varepsilon \tag{6.25}$$

and

$$(\exp(W_\varepsilon))_\varepsilon = -i(\ln\langle \mathbf{out}_\varepsilon, 0|0,\mathbf{in}_\varepsilon\rangle)_\varepsilon. \tag{6.26}$$

Following canonical calculation one obtains

$$(Z_\varepsilon^{\pm}[0])_\varepsilon \propto \left([\det(-G_\varepsilon^{\pm}(x,x'))]^{\frac{1}{2}}\right)_\varepsilon \tag{6.27}$$

where the proportionality constant is metric-independent and can be ignored. Thus we obtain

$$(W_\varepsilon^{\pm})_\varepsilon = -i(\ln Z_\varepsilon^{\pm}[0])_\varepsilon = -\frac{i}{2}\left(\mathbf{tr}\left[\ln\left(-\hat{G}_\varepsilon^{\pm}\right)\right]\right)_\varepsilon. \tag{6.28}$$

In (6.28) $\left(\hat{G}_{\varepsilon}^{\pm}\right)_{\varepsilon}$ is to be interpreted as an Colombeau generalized operator which acts on an linear space $\Im$ of generalized vectors $|x\rangle$, normalized by

$$(\langle x|x'\rangle)_{\varepsilon} = \delta(x-x')\left([-g^{\pm}(x,\varepsilon)]^{-\frac{1}{2}}\right)_{\varepsilon} \tag{6.29}$$

in such a way that

$$(G_{\varepsilon}^{\pm}(x,x'))_{\varepsilon} = \left(\langle x|\hat{G}_{\varepsilon}^{\pm}|x'\rangle\right)_{\varepsilon}. \tag{6.30}$$

Remark 6.1.Note that the trace $(\mathbf{tr}[\Re_{\varepsilon}])_{\varepsilon}$ of an Colombeau generalized operator $(\Re_{\varepsilon})_{\varepsilon}$ which acts on a linear space $\Im$, is defined by

$$(\mathbf{tr}[\Re_{\varepsilon}])_{\varepsilon} = \left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}}\Re_{xx;\varepsilon}\right)_{\varepsilon} = \left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}}\langle x|\Re_{xx;\varepsilon}|x'\rangle\right)_{\varepsilon}. \tag{6.31}$$

Writing now the Colombeau generalized operator $\left(\hat{G}_{\varepsilon}^{\pm}\right)_{\varepsilon}$ as

$$\left(\hat{G}_{\varepsilon}^{\pm}\right)_{\varepsilon} = -(\mathcal{F}_{\varepsilon}^{\pm-1})_{\varepsilon} = -i\left(\int_0^{\infty} ds\exp[-s\mathcal{F}_{\varepsilon}^{\pm}]\right)_{\varepsilon}, \tag{6.32}$$

by Eq.(6.14) one obtains

$$\begin{cases} (\langle x|\exp[-s\mathcal{F}_{\varepsilon}^{\pm}]|x'\rangle) = \\ i(4\pi)^{-n/2}\left[\left(\Delta_{\pm}^{1/2}(x,x';\varepsilon)\right)_{\varepsilon}\right]\exp\left[-im^2 s+\dfrac{\sigma(x,x')}{2is}\right]\mathcal{F}_{\varepsilon}^{\pm}(x,x';is)(is)^{-n/2} \end{cases} \tag{6.33}$$

Now, assuming $(\mathcal{F}_{\varepsilon})_{\varepsilon}$ to have a small negative imaginary part, we obtains

$$\left(\int_{\Lambda}^{\infty} ds(is)^{-1} i\exp[-s\mathcal{F}_{\varepsilon}^{\pm}]\right)_{\varepsilon} = (\mathrm{Ei}(-i\Lambda\mathcal{F}_{\varepsilon}^{\pm}))_{\varepsilon} \tag{6.34}$$

where $\mathrm{Ei}(x)$ is the exponential integral function.

Remark 6.2.Note that for $x \to 0$

$$\mathrm{Ei}(x) = \gamma + \ln(-x) + O(x) \tag{6.35}$$

γ is the Euler's constant. Substituting now (6.35) into (6.34) and letting $\Lambda \to 0$ we obtain

$$\left(\ln\left(-\hat{G}_{\varepsilon}^{\pm}\right)\right)_{\varepsilon} = -(\ln(\mathcal{F}_{\varepsilon}))_{\varepsilon} = \left(\int_0^{\infty} ds\exp[-s\mathcal{F}_{\varepsilon}^{\pm}](is)^{-1}\right)_{\varepsilon}, \tag{6.36}$$

which is correct up to the addition of a metric-independent infinite large Colombeau constant $\Omega \in \widetilde{\mathbb{R}}$ that can be ignored in what follows. Thus, in the generalized De Witt-Schwinger representation (6.33) or (6.14) we have

$$\left(\langle x|\ln\left(-\hat{G}_{\varepsilon}^{\pm}\right)|x'\rangle\right)_{\varepsilon} = \left(\int_{m^2}^{\infty} G_{\varepsilon}^{\pm}(x,x';m^2)dm^2\right)_{\varepsilon}, \tag{6.37}$$

where the integral with respect to m^2 brings down the extra power of $(is)^{-1}$ that appears in Eq.(6.36). Returning now to the expression (6.28) for $(W_{\varepsilon}^{\pm})_{\varepsilon}$ using Eq.(6.37) and Eq.(6.31) we get

$$(W_{\varepsilon}^{\pm})_{\varepsilon} = \frac{i}{2}\left[\left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}}\right)_{\varepsilon}\right]\left(\lim_{x\to x'}\int_{m^2}^{\infty} G_{\varepsilon}^{\pm}(x,x';m^2)dm^2\right)_{\varepsilon} \tag{6.38}$$

Interchanging the order of integration and taking the limit $x \to x'$ one obtains

$$(W_{\varepsilon}^{\pm})_{\varepsilon} = \frac{i}{2}\left(\int_{m^2}^{\infty} dm^2 \int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}} G_{\varepsilon}^{\pm}(x,x;m^2)\right)_{\varepsilon}. \quad (6.39)$$

Colombeau quantity $(W_{\varepsilon}^{\pm})_{\varepsilon}$ is colled as the one-loop effective action. In the case of fermion effective actions, there would be a remaining trace over spinorial indices. From Eq.(6.39) we may define an effective Lagrangian density $\left(L_{\varepsilon;\mathbf{eff}}^{\pm}(x)\right)_{\varepsilon}$ by

$$(W_{\varepsilon}^{\pm})_{\varepsilon} = \left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}} L_{\varepsilon;\mathbf{eff}}^{\pm}(x)\right)_{\varepsilon} \quad (6.40)$$

whence one get

$$(L_{\varepsilon}^{\pm}(x))_{\varepsilon} = [-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}} \mathcal{L}_{\varepsilon;\mathbf{eff}}^{\pm}(x) = \frac{i}{2}\left(\lim_{x\to x'} \int_{m^2}^{\infty} dm^2 G_{\varepsilon}^{\pm}(x,x';m^2)\right)_{\varepsilon}. \quad (6.41)$$

6.3. Stress-tensor renormalization

Note that $(L_{\varepsilon}^{\pm}(x))_{\varepsilon}$ diverges at the lower end of the s integral because the $\sigma/2s$ damping factor in the exponent vanishes in the limit $x \to x'$. (Convergence at the upper end is guaranteed by the $-i\epsilon$ that is implicitly added to m^2 in the De Witt-Schwinger representation of $(L_{\varepsilon}^{\pm}(x))_{\varepsilon}$. In four dimensions, the potentially divergent terms in the DeWitt- Schwinger expansion of $(L_{\varepsilon}^{\pm}(x))_{\varepsilon}$ are

$$\begin{cases} (L_{\varepsilon;\mathrm{div}}^{\pm}(x))_{\varepsilon} = \\ -(32\pi^2)^{-1}\left(\lim_{x\to x'}\left[\left(\Delta_{\pm}^{1/2}(x,x';\varepsilon)\right)_{\varepsilon}\right]\int_0^{\infty}\frac{ds}{s^3}\exp\left[-im^2 s + \frac{\sigma(x,x')}{2is}\right]\times \right. \\ \left.\times\left[a_0^{\pm}(x,x';\varepsilon) + isa_1^{\pm}(x,x';\varepsilon) + (is)^2 a_2^{\pm}(x,x';\varepsilon)\right]\right)_{\varepsilon} \end{cases} \quad (6.42)$$

where the coefficients $a_0^{\pm}$, $a_1^{\pm}$ and $a_2^{\pm}$ are given by Eq.(6.9)-Eq.(6.10). The remaining terms in this asymptotic expansion, involving $a_3^{\pm}$ and higher, are finite in the limit $x \to x'$.

Let us determine now the precise form of the geometrical $(L_{\varepsilon;\mathrm{div}}^{\pm}(x))_{\varepsilon}$ terms, to compare them with the conventional gravitational Lagrangian that appears in (2.38). This is a delicate matter because (6.48) is, of course, infinite. What we require is to display the divergent terms in the form $\infty \times$ [geometrical object]. This can be done in a variety of ways. For example, in n dimensions, the asymptotic (adiabatic) expansion of $\left(L_{\varepsilon;\mathbf{eff}}^{\pm}(x)\right)_{\varepsilon}$ is

$$\begin{cases} \left(L_{\varepsilon;\mathbf{eff}}^{\pm}(x)\right)_{\varepsilon} \asymp \\ 2^{-1}(4\pi)^{-n/2}\left(\lim_{x\to x'}\left[\left(\Delta_{\pm}^{1/2}(x,x';\varepsilon)\right)_{\varepsilon}\right]\sum_{j=0}^{\infty} a_j(x,x';\varepsilon)\times\right. \\ \left.\times\int_0^{\infty} ids(is)^{j-1-n/2}\exp\left[-im^2 s + \frac{\sigma(x,x')}{2is}\right]\right)_{\varepsilon} \end{cases} \quad (6.43)$$

of which the first $n/2 + 1$ terms are divergent as $\sigma \to 0$. If n is treated as a variable which can be analytically continued throughout the complex plane, then we may take the $x \to x'$ limit

$$\begin{cases} \left(L^{\pm}_{\varepsilon;\mathbf{eff}}(x)\right)_{\varepsilon} \asymp 2^{-1}(4\pi)^{-n/2}\left(\sum_{j=0}^{\infty} a_j(x;\varepsilon)\int_0^{\infty} ids(is)^{j-1-n/2}\exp[-im^2 s]\right)_{\varepsilon} = \\ 2^{-1}(4\pi)^{-n/2}\sum_{j=0}^{\infty} a_j(x;\varepsilon)(m^2)^{n/2-j}\Gamma\left(j-\frac{n}{2}\right), \\ a_j(x;\varepsilon) = a_j(x,x;\varepsilon). \end{cases} \tag{6.44}$$

From Eq.(6.44) follows we shall wish to retain the units of $L^{\pm}_{\varepsilon;\mathbf{eff}}(x)$ as (length)$^{-4}$, even when $n \neq 4$. It is therefore necessary to introduce an arbitrary mass scale μ and to rewrite Eq.(6.44) as

$$\left(L^{\pm}_{\varepsilon;\mathbf{eff}}(x)\right)_{\varepsilon} \asymp 2^{-1}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4}\left(\sum_{j=0}^{\infty} a_j(x;\varepsilon)(m^2)^{4-2j}\Gamma\left(j-\frac{n}{2}\right)\right)_{\varepsilon}. \tag{6.45}$$

If $n \to 4$, the first three terms of Eq.(6.45) diverge because of poles in the Γ- functions:

$$\begin{cases} \Gamma\left(-\frac{n}{4}\right) = \frac{4}{n(n-2)}\left(\frac{2}{4-n}-\gamma\right)+O(n-4), \\ \Gamma\left(1-\frac{n}{2}\right) = \frac{4}{(2-n)}\left(\frac{2}{4-n}-\gamma\right)+O(n-4), \\ \Gamma\left(2-\frac{n}{2}\right) = \frac{2}{4-n}-\gamma+O(n-4). \end{cases} \tag{6.46}$$

Denoting these first three terms by $(L^{\pm}_{\varepsilon;\mathbf{div}}(x))_{\varepsilon}$, we have

$$(L^{\pm}_{\varepsilon;\mathbf{div}}(x))_{\varepsilon} = \\ (4\pi)^{-n/2}\left\{\frac{1}{n-4}+\frac{1}{2}\left[\gamma+\ln\left(\frac{m^2}{\mu^2}\right)\right]\right\}\left(\left[\frac{4m^4 a_0(x;\varepsilon)}{n(n-2)}-\frac{2m^2 a_1(x;\varepsilon)}{n-2}+a_2(x;\varepsilon)\right]\right)_{\varepsilon}. \tag{6.47}$$

The functions $a_0(x;\varepsilon), a_1(x;\varepsilon)$ and $a_2(x;\varepsilon)$ are given by taking the coincidence limits of (6.9)-(6.10)

$$\begin{cases} (a^{\pm}_0(x;\varepsilon))_{\varepsilon} = 1, \\ (a^{\pm}_1(x;\varepsilon))_{\varepsilon} = \left(\frac{1}{6}-\xi\right)(\mathbf{R}^{\pm}(\varepsilon))_{\varepsilon}, \\ (a^{\pm}_2(x;\varepsilon))_{\varepsilon} = \frac{1}{180}(\mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x,\varepsilon)\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x,\varepsilon))_{\varepsilon} - \frac{1}{180}(\mathbf{R}^{\pm\alpha\beta}(x,\varepsilon)\mathbf{R}^{\pm}_{\alpha\beta}(x,\varepsilon))_{\varepsilon} - \\ -\frac{1}{6}\left(\frac{1}{5}-\xi\right)(\Box\mathbf{R}^{\pm}(x,\varepsilon))_{\varepsilon} + \frac{1}{2}\left(\frac{1}{6}-\xi\right)(\mathbf{R}^{\pm 2}(x,\varepsilon))_{\varepsilon}. \end{cases} \tag{6.48}$$

Finally one obtains

$$(L^{\pm}_{\varepsilon;\mathbf{ren}}(x))_{\varepsilon} \asymp -\frac{1}{64\pi^2}\left(\int_0^{\infty} ids\ln(is)\frac{\partial^3}{\partial(is)^3}\left[\mathcal{F}^{\pm}_{\varepsilon}(x,x;is)e^{-ism^2}\right]\right)_{\varepsilon}. \tag{6.49}$$

Special interest attaches to field theories in distributional spasetime in which the classical action $(\mathbf{S}_{\varepsilon})_{\varepsilon}$ is invariant under distributional conformal transformations,i.e.,

$$(g_{\mu\nu}(x,\varepsilon))_{\varepsilon} \to (\Omega^2_{\varepsilon}(x)g_{\mu\nu}(x,\varepsilon))_{\varepsilon} \triangleq (\bar{g}^{\pm}_{\mu\nu}(x,\varepsilon))_{\varepsilon}. \tag{6.50}$$

From the definitions one has

$$(\mathbf{S}_{\varepsilon}[\bar{g}^{\pm}_{\mu\nu}(x,\varepsilon)])_{\varepsilon} = (\mathbf{S}_{\varepsilon}[g^{\pm}_{\mu\nu}(x,\varepsilon)])_{\varepsilon} + \left(\int d^n x \left(\frac{\delta \mathbf{S}_{\varepsilon}[\bar{g}^{\pm}_{\mu\nu}(x,\varepsilon)]}{\delta \bar{g}^{\pm\rho\sigma}(x,\varepsilon)}\right)\delta \bar{g}^{\pm\rho\sigma}(x,\varepsilon)\right)_{\varepsilon}. \quad (6.51)$$

From Eq.(6.51) one obtains

$$T^{\pm\ \rho}_{\rho}[g^{\pm}_{\mu\nu}(x,\varepsilon),\varepsilon] = -\left(\frac{\Omega^2_{\varepsilon}(x)}{[-g(x,\varepsilon)]^{\frac{1}{2}}}\frac{\delta \mathbf{S}_{\varepsilon}[\bar{g}^{\pm}_{\mu\nu}(x,\varepsilon)]}{\delta \Omega_{\varepsilon}(x)}\right)_{\varepsilon}\Bigg|_{\Omega_{\varepsilon}=1}, \quad (6.52)$$

and it is clear that if the classical action is invariant under the conformal transformations (6.50), then the classical stress-tensor is traceless.Because conformal transformations are essentially a rescaling of lengths at each spacetime point x, the presence of a mass and hence a fixed length scale in the theory will always break the conformal invariance. Therefore we are led to the massless limit of the regularization and renormalization procedures used in the previous section. Although all the higher order $(j > 2)$ terms in the DeWitt-Schwinger expansion of the effective Lagrangian (6.45) are infrared divergent at $n = 4$ as $m \to 0$, we can still use this expansion to yield the ultraviolet divergent terms arising from $j == 0,1,$ and 2 in the four-dimensional case. We may put $m = 0$ immediately in the $j = 0$ and 1 terms in the expansion, because they are of positive power for $n \sim 4$. These terms therefore vanish. The only nonvanishing potentially ultraviolet divergent term is therefore $j = 2$:

$$2^{-1}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4} a_2(x,\varepsilon)\Gamma\left(2-\frac{n}{2}\right), \quad (6.53)$$

which must be handled carefully. Substituting for $a_2(x)$ with $\xi = \xi(n)$ from (6.48), and rearranging terms, we may write the divergent term in the effective action arising from (6.53) as follows

$$\begin{gathered}(W^{\pm}_{\varepsilon,\mathbf{div}})_{\varepsilon} = 2^{-1}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4}\Gamma\left(2-\frac{n}{2}\right)\left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}} a_2(x,\varepsilon)\right)_{\varepsilon} = \\ 2^{-1}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4}\Gamma\left(2-\frac{n}{2}\right)\left(\int d^n x[-g^{\pm}(x,\varepsilon)]^{\frac{1}{2}}[\alpha \mathsf{F}^{\pm}_{\varepsilon}(x)+\beta G^{\pm}_{\varepsilon}(x)]\right)_{\varepsilon} + O(n-4)\end{gathered} \quad (6.54)$$

where

$$\begin{gathered}(\mathsf{F}_{\varepsilon}(x))_{\varepsilon} = (\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x,\varepsilon)\mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x,\varepsilon))_{\varepsilon} - 2(\mathbf{R}^{\pm\alpha\beta}(x,\varepsilon)\mathbf{R}^{\pm}_{\alpha\beta}(x,\varepsilon))_{\varepsilon} + \frac{1}{3}(\mathbf{R}^{\pm 2}(x,\varepsilon))_{\varepsilon}, \\ (G^{\pm}_{\varepsilon}(x))_{\varepsilon} = (\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x,\varepsilon)\mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x,\varepsilon))_{\varepsilon}\end{gathered} \quad (6.55)$$

and

$$\alpha = \frac{1}{120}, \beta = -\frac{1}{360}. \quad (6.56)$$

Finally one obtains

$$\begin{gathered}\langle T^{\mu}_{\mu}(x,\varepsilon)\rangle_{\mathbf{ren}} = -(1/2880\pi^2)\left[\alpha\left(\mathsf{F}_{\varepsilon}(x) - \tfrac{2}{3}\Box\mathbf{R}^{\pm}(x,\varepsilon)\right)_{\varepsilon} + \beta(G^{\pm}_{\varepsilon}(x))_{\varepsilon}\right] = \\ -(1/2880\pi^2)\left[(\mathbf{R}^{\pm}_{\alpha\beta\gamma\delta}(x,\varepsilon)\mathbf{R}^{\pm\alpha\beta\gamma\delta}(x,\varepsilon))_{\varepsilon} - (\mathbf{R}^{\pm}_{\alpha\beta}(x,\varepsilon)\mathbf{R}^{\pm\alpha\beta}(x,\varepsilon))_{\varepsilon} - \Box\mathbf{R}^{\pm}(x,\varepsilon)\right].\end{gathered} \quad (6.57)$$

Note that from Eq.(3.42) for $r \to 2m$ follows that

$$(\mathbf{R}^{\rho\sigma\mu\nu}(\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(\varepsilon))_{\epsilon} \asymp \left(\left[(r-2m)^2+\varepsilon^2\right]^{-1}\right)_{\varepsilon} + 4(2m)^4. \quad (6.58)$$

Thus for the case of the distributional Schwarzchild spesetime given by the distributional metric (3.40) using Eq.(6.57) and Eq.(6.58) for $r \to 2m$ one obtains

$$\langle T_{\mu}^{\mu}(x,\varepsilon)\rangle_{\mathbf{ren}} \asymp -(2880\pi^2)^{-1}\Big[\Big(\big[(r-2m)^2+\varepsilon^2\big]^{-1}\Big)_{\varepsilon}+4(2m)^4\Big]. \tag{6.59}$$

This result in a good agreement with Eq.(5.14)-Eq.(5.16).

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7.Novel explanation of the Active Galactic Nuclei.The Power Source of Quasars as a result of vacuum polarization by the gravitational singularities on BHs horizon.

7.1. The current paradigm for Active Galactic Nuclei.High energy emission from galactic jets.

Accretion of gas onto the supermassive Kerr black holes lurking at the center of active galactic nuclei (AGN) gives rise to powerful relativistic jets.

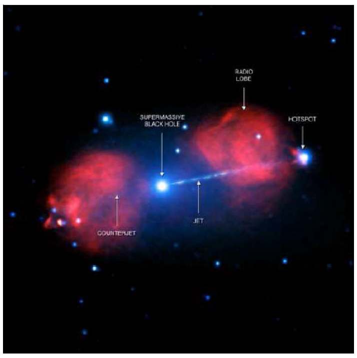


Fig.7.1.Jet from Black Hole in a Galaxy Pictor A
The active galaxy Pictor A lies nearly 500 million
light-years from Earth and contains a
supermassive black hole at its centre.
This is a composite radio and X-ray image.

We remind that in the standard theory of an accretion disk around a black hole it is assumed that there is no coupling between the disk and the central black hole [51]. However, in the presence of a magnetic field, a magnetic coupling between the disk and the black hole could exist and play an important role in the balance and transportation of energy and angular momentum.In the absence of the magnetic coupling, the energy source for the radiation of the disk is the gravitational energy of the disk (i.e., the gravitational binding energy between the disk and the black hole). But, if the magnetic coupling exists and the black hole is rotating, the rotational energy of the black hole provides an additional energy source for the radiation of the disk.With the magnetic coupling, the black hole exerts a torque on the disk, which transfers energy and angular momentum between the black hole and the disk. If the black hole rotates faster than the disk, energy and angular momentum are extracted from the black hole and transfered to the disk. The energy deposited into the disk is eventually radiated away by the disk,which will increase the efficiency of the disk and make the disk brighter than usual. If the black hole rotates slower than the disk, energy and angular momentum are transfered from the disk to the black hole, which will lower the efficiency of the disk and make the disk dimmer than usual. Therefore, the magnetic coupling between the black hole and the disk has important effects on the radiation properties of the disk [52]-[53].

The current paradigm for AGN is that their radio emission is explained by synchrotron radiation from relativistic electrons that are Doppler boosted through bulk motion. In this model, the intrinsic brightness temperatures cannot exceed 10^{11} to 10^{12} K [55]. Typical Doppler boosting is expected to be able to raise this temperature by a factor of 10.

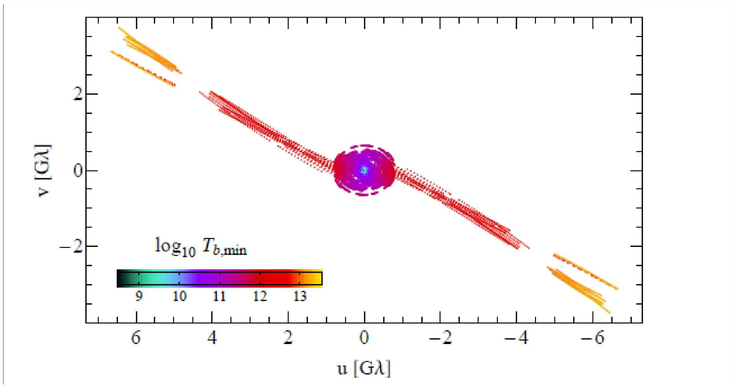


Fig.7.2.Fourier coverage (uv-coverage) of the fringe fitted data (i.e.,reliable fringe detections) of the Radio Astron observations of BL Lac on 2013 November 10-11at 22 GHz.
Color marks the lower limit of observed brightness temperature obtained from visibility amplitudes. Adapted from [54].

The observed brightness temperature of the most compact structures in BL Lac, constrained by baselines longer than $5.3G\lambda$, must indeed exceed 2×10^{13}K and can reach as high as $\sim 3 \times 10^{14}$K [55]. As follows from Fig. 7.2, these visibilities correspond to the structural scales of $30 - 40\ \mu as$ oriented along position angles of $25^\circ - 30^\circ$. These values are indeed close to the width of the inner jet and the normal to its direction.The observed, $T_{b,obs}$, and intrinsic, $T_{b,int}$, brightness temperatures are related by

$$T_{b,obs} = \delta(1+z)^{-1} T_{b,int} \qquad (7.1.1)$$

with $\delta = 7.2$. The estimeted by (7.1.1) a lower limit of the intrinsic brightness temperature in the core component of our Radio Astron observations of $T_{b,int} > 2.910^{12}$ K. It is commonly considered that inverse Compton losses limit the intrinsic brightness temperature for incoherent synchrotron sources, such as AGN, to about 10^{12}K [1].In case of a strong flare, the "Compton catastrophe" is calculated to take about one day to drive the brightness temperature below 10^{12}K [1]. The estimated lower limit for the intrinsic brightness temperature of the core in the Radio Astron image of $T_{b,int} > 2.910^{12}$K is therefore more than an order of magnitude larger than the equipartition brightness temperature limit established in [55] and at least several times larger than the limit established by inverse Compton cooling.

Remark 4.1.1. Note that if the estimate of the maximum brightness temperature given in

[53], is closer to actual values, it would imply $T_{b;int} 5 \times 10^{13}$K. This is difficult to reconcile

with current incoherent synchrotron emission models from relativistic electrons, requiring alternative models such as emission from relativistic protons.

Remark 4.1.2. However the proton, as we know, is 1836 times heavier than an electron and absolutely huge energy is required to accelerated it to sublight speed.

Remark 4.1.3.These alternative models such as emission from relativistic protons can be suported by semiclassical gravity effect finds its roots in the singular behavior of quantum fields on curved distributional spacetimes presented by rotating gravitational singularities.

7.2.The Colombeau distributional Kerr spacetime in Boyer-Lindquist form.

The classical Kerr metric in Boyer-Lindquist form reads

$$\begin{gathered} ds^2 = -\Xi(r,\theta)dt^2 - \frac{4mra\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta_a}dr^2 + \rho^2 d\theta^2 + \\ \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2, \end{gathered} \tag{7.2.1}$$

where

$$\begin{gathered} \rho^2 = r^2 + a^2\cos^2\theta, \Delta_a = r^2 - 2mr + a^2, \\ \Xi(r,\theta) = 1 - \frac{2mr}{\rho^2} = \frac{r^2 - 2mr + a^2\cos^2\theta}{\rho^2}. \end{gathered} \tag{7.2.2}$$

Remark 7.2.1.Note that For small values of the parameter a, where we may neglect terms of the order of a^2, we get from (7.2.1) the Lense-Thirring metric with $J_z = ma$

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{4ma\sin^2\theta}{r}dtd\phi. \tag{7.2.3}$$

I.Slow Kerr gravitational singularity: $a < m$.

Note that

$$\Xi(r,\theta) = \frac{r^2 - 2mr + a^2\cos^2\theta}{\rho^2} = \frac{(r - r_{E_+}(\theta))(r - r_{E_-}(\theta))}{\rho^2}, \tag{7.2.4}$$

where $r_{E_\pm}(\theta) = m \pm \sqrt{m^2 - a^2\cos^2\theta}$ and $\Delta_a = r^2 - 2mr + a^2 = (r - r_+)(r - r_-)$, where $r_\pm = m \pm \sqrt{m^2 - a^2}$.

Remark 7.2.2.Let $\Xi(\theta)$ be a submanifold given by equation ϕ = const, then metric (7.2.1) restricted on submanifold $\cup_\theta \Xi(\theta)$ reads

$$ds^2 = -\Xi(r,\theta)dt^2 + \frac{\rho^2}{\Delta_a}dr^2 + \rho^2 d\theta^2. \tag{7.2.5}$$

Note that: (i) the metric (7.2.5) is degenerate on outer ergosurfaces $r = r_{E_+}(\theta)$ and inner ergosurfaces $r - r_{E_-}(\theta)$, (ii) the metric (7.2.5) is singular on horizon $r = r_+$, (iii) the metric

(7.2.5) is singular on submanifold given by equation $r = r_-$.

Remark 7.2.3. Note that we will be consider the distributional Kerr spacetime not as full distributionel BH with Colombeau generalized metric (7.2.7), but only as gravitational singularity located on submanifold $\cup_\theta \Xi(\theta)$ which coincide with outer ergosurface of classical Kerr spacetime. In accordance with Eq.(7.2.11),Eq.(7.2.19) and Eq.(7.2.20) submanifold $\cup_\theta \Xi(\theta)$ presented the singular boundary of distributional spacetime presented by Colombeau generalized metric (7.2.7).

We introduce now the following regularized (above ergosurface $r = r_{E_+}(\theta)$) quantity

$$\Xi_\varepsilon^+(r_\varepsilon,\theta) = \frac{(r_\varepsilon - r_{E_-}(\theta))\sqrt{(r_\varepsilon - r_{E_+}(\theta))^2 + \varepsilon^2}}{\rho_\varepsilon^2(r_\varepsilon)},$$
$$\Delta_{a,\varepsilon} = r_\varepsilon^2 - 2mr_\varepsilon + a^2, \tag{7.2.6}$$

where $\rho_\varepsilon^2 = \rho_\varepsilon^2(r_\varepsilon) = r_\varepsilon^2 + a^2\cos^2\theta, \varepsilon \in (0,\delta],\ r_\varepsilon \geq r_{E_+}(\theta) \geq r_+$. Thus Colombeau generalized metric (above ergosurface $r = r_{E_+}(\theta)$) corresponding to classical Kerr metric

(7.2.1) reads

$$(ds_\varepsilon^{+2})_\varepsilon = -[(\Xi_\varepsilon(r_\varepsilon,\theta))_\varepsilon]dt^2 - \left[\left(\frac{4mr_\varepsilon a\sin^2\theta}{\rho_\varepsilon^2}\right)_\varepsilon\right]dtd\phi +$$
$$\left[\left(\frac{\rho_\varepsilon^2}{\Delta_{a,\varepsilon}}\right)_\varepsilon\right][(dr_\varepsilon^2)_\varepsilon] + (\rho_\varepsilon^2)_\varepsilon d\theta^2 + \tag{7.2.7}$$
$$\left(r_\varepsilon^2 + a^2 + \frac{2mra^2\sin^2\theta}{\rho_\varepsilon^2}\right)_\varepsilon \sin^2\theta d\phi^2.$$

Remark 7.2.4. Let $\Xi(\theta)$ be a submanifold given by equation $\phi = \text{const}$, then Colombeau

generalized metric (7.2.7) restricted on $\Xi(\theta)$ reads

$$(ds_\varepsilon^{+2})_\varepsilon = -[(\Xi_\varepsilon(r_\varepsilon,\theta))_\varepsilon]dt^2 + \left[\left(\frac{\rho_\varepsilon^2}{\Delta_{a,\varepsilon}}\right)_\varepsilon\right][(dr_\varepsilon^2)_\varepsilon] + [(\rho_\varepsilon^2)_\varepsilon]d\theta^2. \tag{7.2.8}$$

Note that Colombeau generalized metric (7.2.7) nondegenerate on outer ergosurfaces $(r_\varepsilon)_\varepsilon = r_{E_+}(\theta)$, see Pic.7.1.

Remark 7.2.4. Note that for small values of the parameter a, where we may neglect terms

of the order of a^2, we get from (7.2.7) effectively the following Colombeau generalized metric

$$(ds_\varepsilon^{+2})_\varepsilon = -\left(1 - \frac{r_{E_+}(\theta)}{(r_\varepsilon)_\varepsilon}\right)dt^2 + \left(1 - \frac{r_{E_+}(\theta)}{(r_\varepsilon)_\varepsilon}\right)^{-1}[(dr_\varepsilon^2)_\varepsilon] +$$
$$[(r_\varepsilon^2)_\varepsilon](d\theta^2 + \sin^2\theta d\phi^2) - \frac{4ma\sin^2\theta}{(r_\varepsilon)_\varepsilon}dtd\phi, \tag{7.2.9}$$

where $r_{E_+}(\theta) = m + \sqrt{m^2 - a^2\cos^2\theta}$.

Remark 7.2.5. Note that Colombeau generalized metric (7.2.9) restricted on $\Xi(\theta)$ reads

$$(ds_\varepsilon^{+2})_\varepsilon = -\left(1 - \frac{r_{E_+}(\theta)}{(r_\varepsilon)_\varepsilon}\right)dt^2 + \left(1 - \frac{r_{E_+}(\theta)}{(r_\varepsilon)_\varepsilon}\right)^{-1}[(dr_\varepsilon^2)_\varepsilon] + [(r_\varepsilon^2)_\varepsilon]d\theta^2 \quad (7.2.10)$$

(**I**)Let $(\mathbf{R}^{a\ll 1}(r_\varepsilon,\varepsilon))_\varepsilon$ be Colombeau generalized curvature scalar $(\mathbf{R}(r_\varepsilon,\varepsilon))_\varepsilon$ corresponding to the metric (7.2.10) with $a \ll 1$. Main singular part $\mathbf{sing}\left[(\mathbf{R}^{a\ll 1}(r_\varepsilon,\varepsilon))_\varepsilon\right]$ of the Colombeau generalized curvature scalar $(\mathbf{R}^{a\ll 1}(r_\varepsilon,\varepsilon))_\varepsilon$ corresponding to the metric (7.2.10) reads

$$\mathbf{sing}\left[(\mathbf{R}^{a\ll 1}(r_\varepsilon,\varepsilon))_\varepsilon\right] =_{\widetilde{\mathbb{R}}} \left(\frac{\varepsilon^2}{r_{E_+}(\theta)\left[(r_\varepsilon - r_{E_+}(\theta))^2 + \varepsilon^2\right]^{3/2}}\right)_\varepsilon, \quad (7.2.11)$$

where $\mathbf{cl}[(r_\varepsilon)_\varepsilon] \approx_{\widetilde{\mathbb{R}}} r_{E_+}(\theta)$, see Appendix Eq.(A.12).

(**II**) Let $\left(\mathbf{R}^{\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon$ be Colombeau generalized quadratic scalar $(\mathbf{R}^{\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon$ corresponding to the metric (7.2.10) with $a \ll 1$. From Eq.(7.2.10) and Eq.(A.1)-Eq.(A.2) one obtains that main singular part

$$\mathbf{sing}\left[\left(\mathbf{R}^{\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right]$$

of the quadratic scalar $\left(\mathbf{R}^{\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon$ reads:

$$\mathbf{sing}\left[\left(\mathbf{R}^{\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right] =_{\widetilde{\mathbb{R}}} \left(\frac{\varepsilon^4}{4(r_{E_+}(\theta))^4\left[\varepsilon^2 + (r_\varepsilon - 2m)^2\right]^3}\right)_\varepsilon. \quad (7.2.12)$$

(**III**) Let $\left(\mathbf{R}^{\rho\sigma\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon$ be Colombeau generalized quadratic scalar $(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon$ corresponding to the metric (7.2.10) with $a \ll 1$. From Eq.(7.2.10) and Eq.(A.1)-Eq.(A.2) one obtains that main singular part

$$\mathbf{sing}\left[\left(\mathbf{R}^{\rho\sigma\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right]$$

of the Colombeau generalized quadratic scalar $\left(\mathbf{R}^{\rho\sigma\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon$ reads

$$\mathbf{sing}\left[\left(\mathbf{R}^{\rho\sigma\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right] =_{\widetilde{\mathbb{R}}} \left(\frac{\varepsilon^4}{4(r_{E_+}(\theta))^4\left[\varepsilon^2 + (r_\varepsilon - 2m)^2\right]^3}\right)_\varepsilon. \quad (7.2.13)$$

Remark 7.2.6. Note that from Eq.(7.2.11)-Eq.(7.2.13) at outer ergosurfaces $(r_\varepsilon)_\varepsilon = r_{E_+}(\theta)$, (see Pic.7.1) follows that

$$\mathbf{sing}\left[(\mathbf{R}^{a\ll 1}(r_\varepsilon,\varepsilon))_\varepsilon\right]\Big|_{(r_\varepsilon)_\varepsilon = r_{E_+}(\theta)} =_{\widetilde{\mathbb{R}}} r_{E_+}^{-1}(\theta)(\varepsilon^{-1})_\varepsilon \approx_{\widetilde{\mathbb{R}}} \infty. \quad (7.2.14)$$

and

$$\begin{gathered}\mathbf{sing}\left[\left(\mathbf{R}^{\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right]\Big|_{(r_\varepsilon)_\varepsilon = r_{E_+}(\theta)} =_{\widetilde{\mathbb{R}}} r_{E_+}^{-4}(\theta)(\varepsilon^{-2})_\varepsilon \approx_{\widetilde{\mathbb{R}}} \infty, \\ \mathbf{sing}\left[\left(\mathbf{R}^{\rho\sigma\mu\nu(a\ll 1)}(r_\varepsilon,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}^{(a\ll 1)}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right]\Big|_{(r_\varepsilon)_\varepsilon = r_{E_+}(\theta)} =_{\widetilde{\mathbb{R}}} r_{E_+}^{-4}(\theta)(\varepsilon^{-2})_\varepsilon \approx_{\widetilde{\mathbb{R}}} \infty.\end{gathered} \quad (7.2.15)$$

Let $(\mathbf{R}^{a<m}(r_\varepsilon,\varepsilon))_\varepsilon$ be Colombeau generalized curvature scalar $(\mathbf{R}^{a<m}(r_\varepsilon,\varepsilon))_\varepsilon$ corresponding to the metric (7.2.8) with $a < m$. We let now that

$$\Delta_\varepsilon = A_\varepsilon(B_\varepsilon + C_\varepsilon) = A_\varepsilon\rho_\varepsilon^2\Delta_{a,\varepsilon}^{-1}, B_\varepsilon r_\varepsilon^2 = \rho_\varepsilon^2, B_\varepsilon + C_\varepsilon = \rho_\varepsilon^2\Delta_{a,\varepsilon}^{-1}, A_\varepsilon = \Xi_\varepsilon^{+}(r,\theta), D_\varepsilon = 0. \quad (7.2.16)$$

From Eq.(7.2.8),Eq.(7.2.16) and Eq.(A.1)-Eq.(A.2) we obtain

$$
\begin{aligned}
(\mathbf{R}^{a<m}(r_\varepsilon,\varepsilon))_\varepsilon = &\left(\frac{A_\varepsilon}{\Delta_\varepsilon}\left[\frac{2}{r_\varepsilon}\left(-2\frac{A'_\varepsilon}{A_\varepsilon}-3\frac{B'_\varepsilon}{B_\varepsilon}+\frac{\Delta'_\varepsilon}{\Delta_\varepsilon}\right)+\frac{2}{r^2}\frac{C_\varepsilon}{B_\varepsilon}-\frac{A''_\varepsilon}{A_\varepsilon}-2\frac{B''_\varepsilon}{B_\varepsilon}\right.\right.\\
&\left.\left.+\frac{1}{2}\left(\frac{B'_\varepsilon}{B_\varepsilon}\right)^2-2\frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon}+\left(\frac{1}{2}\frac{A'_\varepsilon}{A_\varepsilon}+\frac{B'_\varepsilon}{B_\varepsilon}\right)\frac{\Delta'_\varepsilon}{\Delta_\varepsilon}\right]\right)_\varepsilon =\\
&\left(\frac{\Delta_{a,\varepsilon}}{\rho^2_\varepsilon}\left[\frac{2}{r_\varepsilon}\left(-2\frac{A'_\varepsilon}{A_\varepsilon}-3\frac{B'_\varepsilon}{B_\varepsilon}+\frac{\Delta'_\varepsilon}{\Delta_\varepsilon}\right)+\frac{2}{r^2}\frac{C_\varepsilon}{B_\varepsilon}-\frac{A''_\varepsilon}{A_\varepsilon}-2\frac{B''_\varepsilon}{B_\varepsilon}\right.\right.\\
&\left.\left.+\frac{1}{2}\left(\frac{B'_\varepsilon}{B_\varepsilon}\right)^2-2\frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon}+\left(\frac{1}{2}\frac{A'_\varepsilon}{A_\varepsilon}+\frac{B'_\varepsilon}{B_\varepsilon}\right)\frac{\Delta'_\varepsilon}{\Delta_\varepsilon}\right]\right)_\varepsilon .
\end{aligned}
\tag{7.2.17}
$$

Note that

$$
\begin{aligned}
\frac{\partial}{\partial r_\varepsilon}\sqrt{(r_\varepsilon-r_{E_+}(\theta))^2+\varepsilon^2} &= \frac{r_\varepsilon-r_{E_+}(\theta)}{\sqrt{(r_\varepsilon-r_{E_+}(\theta))^2+\varepsilon^2}}\\
\frac{\partial^2}{\partial r^2_\varepsilon}\sqrt{(r_\varepsilon-r_{E_+})^2+\varepsilon^2} &= \frac{\varepsilon^2}{\left[(r_\varepsilon-r_{E_+})^2+\varepsilon^2\right]^{\frac{3}{2}}}.
\end{aligned}
\tag{7.2.18}
$$

From Eq.(7.2.17) and Eq.(7.2.18) one obtains that main singular part $\mathbf{sing}[(\mathbf{R}^{a<m}(r_\varepsilon,\varepsilon))_\varepsilon]$ of the Colombeau generalized curvature scalar $(\mathbf{R}^{a<m}(r_\varepsilon,\varepsilon))_\varepsilon$ corresponding to the metric (7.2.8) (mod nonsingular multiplier) reads

$$
\mathbf{sing}[(\mathbf{R}^{a<m}(r_\varepsilon,\varepsilon))_\varepsilon] =_{\widetilde{\mathbb{R}}} \left(\frac{\varepsilon^2}{\left[(r_\varepsilon-r_{E_+}(\theta))^2+\varepsilon^2\right]^2}\right)_\varepsilon . \tag{7.2.19}
$$

Remark 7.2.7.(I) Let $\left(\mathbf{R}^{\mu\nu(a<m)}(r_\varepsilon,\varepsilon)\mathbf{R}^{(a<m)}_{\mu\nu}(r_\varepsilon,\varepsilon)\right)_\varepsilon$ be Colombeau generalized quadratic

scalar $(\mathbf{R}^{\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon$ corresponding to the metric (7.2.8) with $a<m$. From Eq.(7.2.8)-Eq.(7.2.16) and Eq.(A.1)-Eq.(A.2) one obtains that main singular part

$$
\mathbf{sing}\left[\left(\mathbf{R}^{\mu\nu(a<m)}(r_\varepsilon,\varepsilon)\mathbf{R}^{(a<m)}_{\mu\nu}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right]
$$

of the Colombeau generalized quadratic scalar $\left(\mathbf{R}^{\mu\nu(a<m)}(r_\varepsilon,\varepsilon)\mathbf{R}^{(a<m)}_{\mu\nu}(r_\varepsilon,\varepsilon)\right)_\varepsilon$ reads

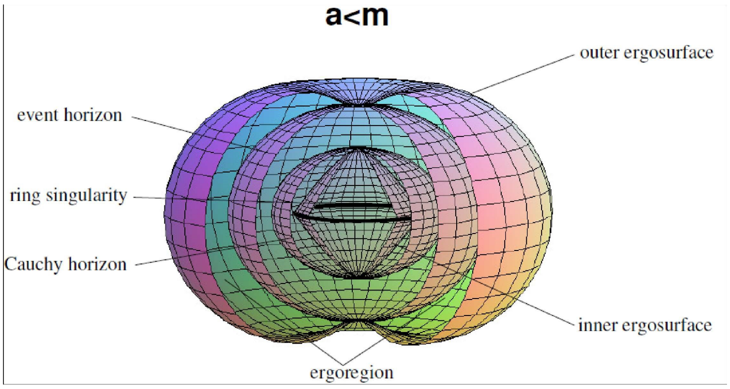
$$
\mathbf{sing}\left[\left(\mathbf{R}^{\mu\nu(a<m)}(r_\varepsilon,\varepsilon)\mathbf{R}^{(a<m)}_{\mu\nu}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right] =_{\widetilde{\mathbb{R}}} \left(\frac{\varepsilon^4}{4(r_{E_+}(\theta))^4\left[\varepsilon^2+(r_\varepsilon-2m)^2\right]^3}\right)_\varepsilon . \tag{7.2.20}
$$

(II) Let $\left(\mathbf{R}^{\rho\sigma\mu\nu(a<m)}(r_\varepsilon,\varepsilon)\mathbf{R}^{(a<m)}_{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon)\right)_\varepsilon$ be Colombeau generalized quadratic scalar $(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon$ corresponding to the metric (7.2.8) with $a<m$. From Eq.(7.2.8), Eq.(7.2.16) and Eq.(A.1)-Eq.(A.2) one obtains that main singular part

$$
\mathbf{sing}\left[\left(\mathbf{R}^{\rho\sigma\mu\nu(a<m)}(r_\varepsilon,\varepsilon)\mathbf{R}^{(a<m)}_{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right]
$$

of the Colombeau generalized quadratic scalar $\left(\mathbf{R}^{\rho\sigma\mu\nu(a<m)}(r_\varepsilon,\varepsilon)\mathbf{R}^{(a<m)}_{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon)\right)_\varepsilon$ reads

$$
\mathbf{sing}\left[\left(\mathbf{R}^{\rho\sigma\mu\nu(a<m)}(r_\varepsilon,\varepsilon)\mathbf{R}^{(a<m)}_{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon)\right)_\varepsilon\right] =_{\widetilde{\mathbb{R}}} \left(\frac{\varepsilon^4}{4(r_{E_+}(\theta))^4\left[\varepsilon^2+(r_\varepsilon-2m)^2\right]^3}\right)_\varepsilon . \tag{7.2.21}
$$

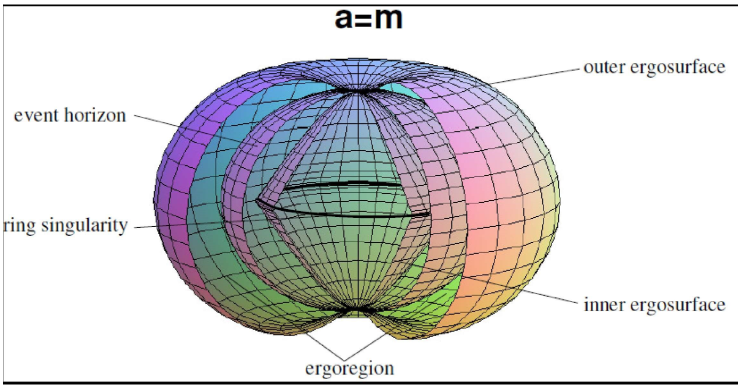


Pic.7.Ergosurface,horizon,and singularity for slow Kerr black hole.

II.Critical Kerr gravitational singularity: $a = m$.

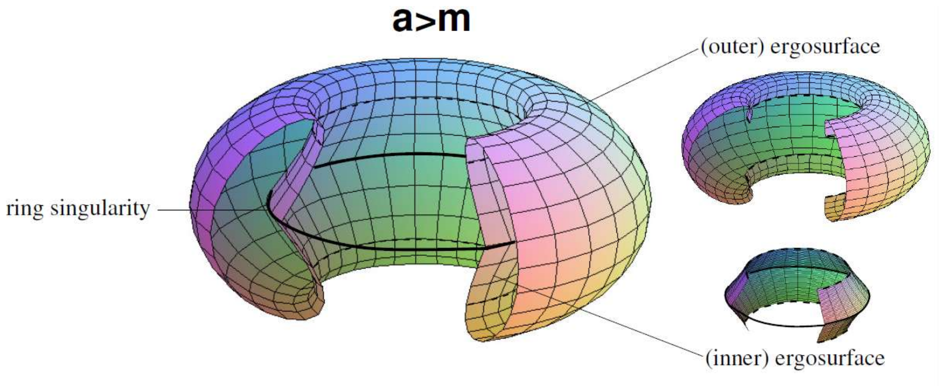
Note that in contrast with full distributional Kerr spacetime the case of the critical Kerr gravitational singularity considered in this subsection (see Remark 7.2.3) not principal different in comparison with a case of the slow Kerr gravitational singularity considered above. In particular the Eqs.(7.2.19)-(7.2.21) holds with $r_{E_+}(\theta)$ given by Eq.(7.2.22)

$$r_{E_+}(\theta) = m + \sqrt{m^2 - a^2\cos^2\theta} = m(1 + \sin^2\theta). \quad (7.2.22)$$



Pic.8.Ergosurface,horizon,and singularity for critical Kerr black hole.

III.Fast Kerr gravitational singularity: $a > m$.



Pic.9.Ergosurface,horizon,and singularity for fast
Kerr black hole.

Let $\Xi_\eta(\theta)$ be a submanifold given by equations (i) $\phi = \text{const}$ and (ii) $m^2 - a^2\cos^2\theta \geq 0$, i.e.

$$\cos^2\theta \leq \frac{m^2}{a^2} = \eta. \tag{7.2.23}$$

Let Θ_η be a set $\Theta_\eta = \{\theta | \cos^2\theta \leq \eta\}$ and let $\chi(\theta,\eta)$ be the indicator function of the set Θ_η,i.e. $\chi(\theta,\eta)$ is the function defined to be identically 1 on Θ_η, and is 0 elsewhere. We introduce now the following regularized (above ergosurface $r = r_{E_+}(\theta), \theta \in \Theta_\eta$) quantity

$$\Xi_\varepsilon^+(r_\varepsilon,\theta,\eta) = \frac{\chi(\theta,\eta)(r_\varepsilon - r_{E_-}(\theta))\sqrt{(r_\varepsilon - r_{E_+}(\theta))^2 + \varepsilon^2}}{\rho_\varepsilon^2(r_\varepsilon)}, \tag{7.2.25}$$

where $\rho_\varepsilon^2 = \rho_\varepsilon^2(r_\varepsilon) = r_\varepsilon^2 + a^2\cos^2\theta, \varepsilon \in (0,\delta]$, $r_\varepsilon \geq r_{E_+}(\theta) > 0$. Thus Colombeau generalized metric (above ergosurface $r = r_{E_+}(\theta)$) corresponding to classical Kerr metric (7.2.1) reads

$$(ds_\varepsilon^{+2})_\varepsilon = -\chi(\theta,\eta)[(\Xi_\varepsilon(r_\varepsilon,\theta,\eta))_\varepsilon]dt^2 - \left[\left(\frac{4mr_\varepsilon a\sin^2\theta}{\rho_\varepsilon^2}\right)_\varepsilon\right]dtd\phi + \left[\left(\frac{\rho_\varepsilon^2}{\Delta_{a,\varepsilon}}\right)_\varepsilon\right][(dr_\varepsilon^2)_\varepsilon] + (\rho_\varepsilon^2)_\varepsilon d\theta^2 + \left(r_\varepsilon^2 + a^2 + \frac{2mra^2\sin^2\theta}{\rho_\varepsilon^2}\right)_\varepsilon \sin^2\theta d\phi^2. \tag{7.2.26}$$

Remark 7.2.8. Note that we will be consider the distributional Kerr spacetime not as full distributionel BH with Colombeau generalized metric (7.2.7), but only as gravitational singularity located on submanifold $\cup_{\theta\in\Theta_\eta}\Xi_\eta(\theta)$ which coincide with an part of the outer ergosurface of classical Kerr spacetime. In accordance with Eq.(7.2.11),Eq.(7.2.19) and Eq.(7.2.20) submanifold $\cup_{\theta\in\Theta_\eta}\Xi_\eta(\theta)$ presented the singular boundary of distributional spacetime with Colombeau generalized metric (7.2.26).

Remark 7.2.9. Let $\tilde{\tilde{\Xi}}_\eta(\theta)$ be a submanifold given by equations (i) $\phi = \text{const}$ and (ii) $\cos^2\theta \leq \eta$, then Colombeau generalized metric (7.2.26) restricted on submanifold $\cup_{\theta\in\Theta_\eta}\tilde{\tilde{\Xi}}_\eta(\theta)$ reads

$$(ds_{\varepsilon}^{+2})_{\varepsilon} = -\chi(\theta,\eta)[(\Xi_{\varepsilon}(r_{\varepsilon},\theta,\eta))_{\varepsilon}]dt^2 + \left[\left(\frac{\rho_{\varepsilon}^2}{\Delta_{a,\varepsilon}}\right)_{\varepsilon}\right][(dr_{\varepsilon}^2)_{\varepsilon}] + [(\rho_{\varepsilon}^2)_{\varepsilon}]d\theta^2. \quad (7.2.27)$$

Note that Colombeau generalized metric (7.2.27) nondegenerate on outer ergosurfaces

$(r_{\varepsilon})_{\varepsilon} = r_{E_+}(\theta)$, see Pic.7.3.

From Eq.(7.2.27) and and Eq.(A.1)-Eq.(A.2) one obtains that main singular part $\mathbf{sing}[(\mathbf{R}^{a>m}(r_{\varepsilon},\varepsilon))_{\varepsilon}]$ of the Colombeau generalized curvature scalar $(\mathbf{R}^{a>m}(r_{\varepsilon},\varepsilon))_{\varepsilon}$ corresponding to the metric (7.2.27) (mod nonsingular multiplier) reads

$$\mathbf{sing}[(\mathbf{R}^{a>m}(r_{\varepsilon},\varepsilon))_{\varepsilon}] =_{\widetilde{\mathbb{R}}} \left(\frac{\chi(\theta,\eta)\varepsilon^2}{\left[(r_{\varepsilon} - r_{E_+}(\theta))^2 + \varepsilon^2\right]^2}\right)_{\varepsilon}. \quad (7.2.28)$$

Remark 7.2.10.(**I**) Let $\left(\mathbf{R}^{\mu\nu(a>m)}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\mu\nu}^{(a>m)}(r_{\varepsilon},\varepsilon)\right)_{\varepsilon}$ be Colombeau generalized quadratic scalar $(\mathbf{R}^{\mu\nu}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\mu\nu}(r_{\varepsilon},\varepsilon))_{\varepsilon}$ corresponding to the metric (7.2.27) with $a > m$. From Eq.(7.2.27) and Eq.(A.1)-Eq.(A.2) one obtains that main singular part

$$\mathbf{sing}\left[\left(\mathbf{R}^{\mu\nu(a>m)}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\mu\nu}^{(a>m)}(r_{\varepsilon},\varepsilon)\right)_{\varepsilon}\right]$$

of the Colombeau generalized quadratic scalar $\left(\mathbf{R}^{\mu\nu(a>m)}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\mu\nu}^{(a>m)}(r_{\varepsilon},\varepsilon)\right)_{\varepsilon}$ reads

$$\mathbf{sing}\left[\left(\mathbf{R}^{\mu\nu(a>m)}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\mu\nu}^{(a>m)}(r_{\varepsilon},\varepsilon)\right)_{\varepsilon}\right] =_{\widetilde{\mathbb{R}}} \left(\frac{\chi(\theta,\eta)\varepsilon^4}{4(r_{E_+}(\theta))^4\left[\varepsilon^2 + (r_{\varepsilon} - 2m)^2\right]^3}\right)_{\varepsilon}. \quad (7.2.29)$$

(**II**) Let $\left(\mathbf{R}^{\rho\sigma\mu\nu(a>m)}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}^{(a>m)}(r_{\varepsilon},\varepsilon)\right)_{\varepsilon}$ be Colombeau generalized quadratic scalar $(\mathbf{R}^{\rho\sigma\mu\nu}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r_{\varepsilon},\varepsilon))_{\varepsilon}$ corresponding to the metric (7.2.27) with $a > m$. From Eq.(7.2.27) and Eq.(A.1)-Eq.(A.2) one obtains that main singular part

$$\mathbf{sing}\left[\left(\mathbf{R}^{\rho\sigma\mu\nu(a>m)}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}^{(a>m)}(r_{\varepsilon},\varepsilon)\right)_{\varepsilon}\right]$$

of the Colombeau generalized quadratic scalar $\left(\mathbf{R}^{\rho\sigma\mu\nu(a>m)}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}^{(a>m)}(r_{\varepsilon},\varepsilon)\right)_{\varepsilon}$ reads

$$\mathbf{sing}\left[\left(\mathbf{R}^{\rho\sigma\mu\nu(a>m)}(r_{\varepsilon},\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}^{(a>m)}(r_{\varepsilon},\varepsilon)\right)_{\varepsilon}\right] =_{\widetilde{\mathbb{R}}} \left(\frac{\chi(\theta,\eta)\varepsilon^4}{4(r_{E_+}(\theta))^4\left[\varepsilon^2 + (r_{\varepsilon} - 2m)^2\right]^3}\right)_{\varepsilon}. \quad (7.2.30)$$

8. Conclusions and remarks.

This paper dealing with an extension of the Einstein field equations using apparatus of contemporary generalization of the classical Lorentzian geometry named in literature Colombeau distributional geometry,see for example [1]-[2],[5]-[7] and [14]-[15].The regularizations of singularities present in some solutions of the Einstein equations is an important part of this approach. Any singularities present in some solutions of the Einstein equations recognized only in the sense of Colombeau generalized functions [1]-[2] and not classically.

In this paper essentially new class Colombeau solutions to Einstein fild equations is obtained. We have shown that a succesfull approach for dealing with curvature tensor valued distribution is to first impose admisible the nondegeneracy conditions on the metric tensor, and then take its derivatives in the sense of classical distributions in space

$\mathcal{S}'_{2m}(\mathbb{R}^3)$.

The distributional meaning is then equivalent to the junction condition formalism. Afterwards, through appropiate limiting procedures, it is then possible to obtain well behaved distributional tensors with support on submanifolds of $d \leq 3$, as we have shown for the energy-momentum tensors associated with the Schwarzschild spacetimes. The above procedure provides us with what is expected on physical grounds. However, it should be mentioned that the use of new supergeneralized functions (supergeneralized Colombeau algebras $\widetilde{\mathcal{G}}(\mathbb{R}^3,\Sigma)$). in order to obtain superdistributional curvatures, may renders a more rigorous setting for discussing situations like the ones considered in this paper.

The vacuum energy density of free scalar quantum field Φ with a distributional background spacetime also is considered.It has been widely believed that, except in very extreme situations, the influence of gravity on quantum fields should amount to just small, sub-dominant contributions. Here we argue that this belief is false by showing that there exist well-behaved spacetime evolutions where the vacuum energy density of free quantum fields is forced, by the very same background distributional spacetime such BHs, to become dominant over any classical energy density component. This semiclassical gravity effect finds its roots in the singular behavior of quantum fields on curved spacetimes. In particular we obtain that the vacuum fluctuations $\langle\Phi^2\rangle$ has a singular behavior on BHs horizon r_+: $\langle\Phi^2(r)\rangle \sim |r-r_+|^{-2}$.We argue that this vacuum dominance may bear importent astrophysical implications.

9.Acknowledgments

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Appendix A.

Expressions for the Colombeau quantities $(\mathbf{R}(\{\},(\epsilon)))_\epsilon$, $(\mathbf{R}^{\mu\nu}(\{\},(\epsilon))\mathbf{R}_{\mu\nu}(\{\},(\epsilon)))_\epsilon$ and $\left(\mathbf{R}^{\rho\sigma\mu\nu}(\{\},(\epsilon))\mathbf{R}_{\rho\sigma\mu\nu}(\{\},(\epsilon))\right)_\epsilon$ in terms of $(A_\epsilon)_\epsilon,(B_\epsilon)_\epsilon,(C_\epsilon)_\epsilon$ and $(D_\epsilon)_\epsilon, \epsilon \in (0,1]$:

Let us introduce now Colombeau generalized metric which has the form

$$\begin{cases} (ds^2)_\epsilon = -(A_\epsilon(r)(dx^0)^2)_\epsilon - 2(D_\epsilon(r)dx^0dr)_\epsilon + ((B_\epsilon(r)+C_\epsilon(r))(dr)^2)_\epsilon \\ \qquad +(B_\epsilon(r)r^2[(d\theta)^2+\sin^2\theta(d\phi)^2])_\epsilon \; r = \mathbf{cl}[(r_\epsilon)_\epsilon] \in \widetilde{\mathbb{R}}. \end{cases} \quad (A.1)$$

The Colombeau scalars $(\mathbf{R}(\epsilon))_\epsilon,(\mathbf{R}^{\mu\nu}(\epsilon)\mathbf{R}_{\mu\nu}(\epsilon))_\epsilon$ and $(R^{\rho\sigma\mu\nu}(\epsilon)R_{\rho\sigma\mu\nu}(\epsilon))_\epsilon$, in terms of Colombeau generalized functions $(A_\epsilon(r))_\epsilon,(B_\epsilon(r))_\epsilon,(C_\epsilon(r))_\epsilon,(D_\epsilon(r))_\epsilon$ is expressed as

$$
\begin{aligned}
(\mathbf{R}(\epsilon))_\epsilon &= \left(\frac{A_\epsilon}{\Delta_\epsilon}\left[\frac{2}{r}\left(-2\frac{A'_\epsilon}{A_\epsilon} - 3\frac{B'_\epsilon}{B_\epsilon} + \frac{\Delta'_\epsilon}{\Delta_\epsilon}\right) + \frac{2}{r^2}\frac{A_\epsilon C_\epsilon + D^2_\epsilon}{A_\epsilon B_\epsilon} - \frac{A''_\epsilon}{A_\epsilon} - 2\frac{B''_\epsilon}{B_\epsilon} \right.\right. \\
&\left.\left. + \frac{1}{2}\left(\frac{B'_\epsilon}{B_\epsilon}\right)^2 - 2\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} + \left(\frac{1}{2}\frac{A'_\epsilon}{A_\epsilon} + \frac{B'_\epsilon}{B_\epsilon}\right)\frac{\Delta'_\epsilon}{\Delta_\epsilon} \right]\right)_\epsilon, \\
(\mathbf{R}^{\mu\nu}(\epsilon)\mathbf{R}_{\mu\nu}(\epsilon))_\epsilon &= \left(\frac{A^2_\epsilon}{\Delta^2_\epsilon}\left(\frac{1}{2}\frac{A''_\epsilon}{A_\epsilon} - \frac{1}{4}\frac{A'_\epsilon\Delta'_\epsilon}{A_\epsilon\Delta_\epsilon} + \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} + \frac{1}{r}\frac{A'_\epsilon}{A_\epsilon}\right)^2\right)_\epsilon + \\
&+2\left(\frac{A^2_\epsilon}{\Delta^2_\epsilon}\left[\frac{1}{r}\left(\frac{1}{2}\frac{\Delta'_\epsilon}{\Delta_\epsilon} - \frac{A'_\epsilon}{A_\epsilon} - 2\frac{B'_\epsilon}{B_\epsilon}\right) + \frac{1}{r^2}\frac{A_\epsilon C_\epsilon + D^2_\epsilon}{A_\epsilon B_\epsilon} - \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} - \right.\right. \\
&\left.\left. -\frac{1}{2}\frac{B''_\epsilon}{B_\epsilon} + \frac{1}{4}\frac{B'_\epsilon\Delta'_\epsilon}{B_\epsilon\Delta_\epsilon} \right]^2\right)_\epsilon + \\
&\left(\frac{A^2_\epsilon}{\Delta^2_\epsilon}\left[\frac{1}{2}\frac{A''_\epsilon}{A_\epsilon} - \frac{1}{4}\frac{A'_\epsilon\Delta'_\epsilon}{A_\epsilon\Delta_\epsilon} + \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} + \frac{B''_\epsilon}{B_\epsilon} - \frac{1}{2}\left(\frac{B'_\epsilon}{B_\epsilon}\right)^2 \right.\right. \\
&\left.\left. -\frac{1}{2}\frac{B'_\epsilon\Delta'_\epsilon}{B_\epsilon\Delta_\epsilon} + \frac{1}{r}\left(\frac{A'_\epsilon}{A_\epsilon} - \frac{\Delta'_\epsilon}{\Delta_\epsilon} + 2\frac{B'_\epsilon}{B_\epsilon}\right) \right]^2\right)_\epsilon, \qquad (A.2) \\
(\mathbf{R}^{\rho\sigma\mu\nu}(\epsilon)\mathbf{R}_{\rho\sigma\mu\nu}(\epsilon))_\epsilon &= \\
&\left(\frac{A^2_\epsilon}{\Delta^2_\epsilon}\left(\frac{A''_\epsilon}{A_\epsilon} - \frac{1}{2}\frac{A'_\epsilon\Delta'_\epsilon}{A_\epsilon\Delta_\epsilon}\right)^2 + 2\frac{A^2_\epsilon}{\Delta^2_\epsilon}\left(\frac{1}{r}\frac{A'_\epsilon}{A_\epsilon} + \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon}\right)^2 \right. \\
&+4\frac{A^2_\epsilon}{\Delta^2_\epsilon}\left[\frac{1}{r}\frac{B'_\epsilon}{B_\epsilon} - \frac{1}{r^2}\frac{A_\epsilon C_\epsilon + D^2_\epsilon}{A_\epsilon B_\epsilon} + \frac{1}{4}\left(\frac{B'_\epsilon}{B_\epsilon}\right)^2 \right]^2 + \\
&+2\frac{A^2}{\Delta^2_\epsilon}\left[\frac{1}{r}\left(\frac{A'_\epsilon}{A_\epsilon} + 2\frac{B'_\epsilon}{B_\epsilon} - \frac{\Delta'_\epsilon}{\Delta_\epsilon}\right) + \frac{1}{2}\frac{A'_\epsilon B'_\epsilon}{A_\epsilon B_\epsilon} + \frac{B''_\epsilon}{B_\epsilon} \right. \\
&\left.\left. -\frac{1}{2}\left(\frac{B'_\epsilon}{B_\epsilon}\right)^2 - \frac{1}{2}\frac{B'_\epsilon\Delta'_\epsilon}{B_\epsilon\Delta_\epsilon} \right]^2 \right)_\epsilon .
\end{aligned}
$$

Here

$$
(\Delta_\epsilon)_\epsilon = (A_\epsilon(r)(B_\epsilon(r) + C_\epsilon(r)))_\epsilon + (D^2_\epsilon(r))_\epsilon. \qquad (A.3)
$$

Assume that

$$
(\Delta_\epsilon(r))_\epsilon = 1, (B_\epsilon(r))_\epsilon = 1, (D_\epsilon(r))_\epsilon = 0. \qquad (A.4)
$$

From Eq.(A.2)-Eq.(A.4) one obtains

$$
\begin{aligned}
(\mathbf{R}(\epsilon))_\epsilon &= \left(-\frac{4A'_\epsilon}{r} + \frac{2A_\epsilon C_\epsilon}{r^2} - A''_\epsilon \right)_\epsilon, \\
(\mathbf{R}^{\mu\nu}(\epsilon)\mathbf{R}_{\mu\nu}(\epsilon))_\epsilon &= 2\left(\left(\frac{1}{2}A''_\epsilon + \frac{1}{r}A'_\epsilon\right)^2\right)_\epsilon + 2\left(\left[-\frac{A'_\epsilon}{r} + \frac{A_\epsilon C_\epsilon}{r^2}\right]^2\right)_\epsilon, \qquad (A.5) \\
(\mathbf{R}^{\rho\sigma\mu\nu}(\epsilon)\mathbf{R}_{\rho\sigma\mu\nu}(\epsilon))_\epsilon &= \left((A''_\epsilon)^2 + 4\left(\frac{A'_\epsilon}{r}\right)^2 + 4\frac{(A_\epsilon C_\epsilon)^2}{r^4} \right)_\epsilon .
\end{aligned}
$$

We choose now

$$
B_\varepsilon(r_\varepsilon) = 1, C_\varepsilon(r_\varepsilon) = -1 + A^{-1}_\varepsilon(r_\varepsilon), D_\varepsilon(r_\varepsilon) = 0, \qquad (A.6)
$$

and rewrite Colombeau generalized object corresponding to Schwarzschild metric above horizon in the following form

$$(ds_\varepsilon^2)_\varepsilon = -(A_\varepsilon(r_\varepsilon)dt^2)_\varepsilon + (A_\varepsilon^{-1}(r_\varepsilon)dr_\varepsilon^2)_\varepsilon + r_\varepsilon^2 d\Omega^2, \qquad (A.7)$$

where $A_\varepsilon(r)$

$$A_\varepsilon(r_\varepsilon) = -r_\varepsilon^{-1}\sqrt{(r_\varepsilon - 2m)^2 + \varepsilon^2}\,, r_\varepsilon \geq 2m. \qquad (A.8)$$

By differentiation we obtain

$$\begin{gathered}\Delta_\varepsilon = A_\varepsilon(B_\varepsilon + C_\varepsilon) = 1, \Delta'_\varepsilon = 0,\\ A'_\varepsilon(r_\varepsilon) = \frac{-2m(r_\varepsilon - 2m)}{r_\varepsilon^2\sqrt{(r_\varepsilon - 2m)^2 + \varepsilon^2}},\\ A''_\varepsilon(r) = \frac{2m(-16m^3 + 24m^2 r_\varepsilon - 12mr_\varepsilon^2 - 4m\varepsilon^2 + 2r_\varepsilon^3 + r_\varepsilon\varepsilon^2)}{r_\varepsilon^3\left[(r_\varepsilon - 2m)^2 + \varepsilon^2\right]^{3/2}} = \\ \frac{4m(r_\varepsilon - 2m)^3 + (r_\varepsilon - 4m)\varepsilon^2}{r_\varepsilon^3\left[(r_\varepsilon - 2m)^2 + \varepsilon^2\right]^{3/2}}.\end{gathered} \qquad (A.9)$$

From Eqs.(A.2)-(A.5) and Eq.(A.9) we obtain

$$\begin{gathered}(\mathbf{R}(r,\varepsilon))_\varepsilon = \left(\frac{A_\varepsilon}{\Delta_\varepsilon}\left[\frac{2}{r_\varepsilon}\left(-2\frac{A'_\varepsilon}{A_\varepsilon} - 3\frac{B'_\varepsilon}{B_\varepsilon} + \frac{\Delta'_\varepsilon}{\Delta_\varepsilon}\right) + \frac{2}{r_\varepsilon^2}\frac{A_\varepsilon C_\varepsilon + D_\varepsilon^2}{A_\varepsilon B_\varepsilon} - \frac{A''_\varepsilon}{A_\varepsilon} - 2\frac{B''_\varepsilon}{B_\varepsilon}\right.\right.\\ \left.\left. + \frac{1}{2}\left(\frac{B'_\varepsilon}{B_\varepsilon}\right)^2 - 2\frac{A'_\varepsilon B'_\varepsilon}{A_\varepsilon B_\varepsilon} + \left(\frac{1}{2}\frac{A'_\varepsilon}{A_\varepsilon} + \frac{B'_\varepsilon}{B_\varepsilon}\right)\frac{\Delta'_\varepsilon}{\Delta_\varepsilon}\right]\right)_\varepsilon = \\ \left(A_\varepsilon\left[\frac{2}{r_\varepsilon}\left(-2\frac{A'_\varepsilon}{A_\varepsilon}\right) - \frac{2A_\varepsilon(1 - A_\varepsilon^{-1})}{r_\varepsilon^2} - \frac{A''_\varepsilon}{A_\varepsilon}\right]\right)_\varepsilon = \\ \left(-\frac{4A'_\varepsilon}{r_\varepsilon} - \frac{2A_\varepsilon}{r_\varepsilon^2} + \frac{2}{r_\varepsilon^2} - A''_\varepsilon\right)_\varepsilon = \\ \left(\frac{8m(r_\varepsilon - 2m)}{r_\varepsilon^3\sqrt{(r_\varepsilon - 2m)^2 + \varepsilon^2}}\right)_\varepsilon + 2r_\varepsilon^{-3}\left(\sqrt{(r_\varepsilon - 2m)^2 + \varepsilon^2}\right)_\varepsilon - \frac{2}{(r_\varepsilon^2)_\varepsilon} - \\ -\left(\frac{2m(-16m^3 + 24m^2 r_\varepsilon - 12mr_\varepsilon^2 - 4m\varepsilon^2 + 2r_\varepsilon^3 + r_\varepsilon\varepsilon^2)}{r_\varepsilon^3\left[(r_\varepsilon - 2m)^2 + \varepsilon^2\right]^{3/2}}\right)_\varepsilon.\end{gathered} \qquad (A.10)$$

Finally from Eq.(A.10) one obtains the following expression for the distributional Colombeau scalar $(\mathbf{R}(r,\varepsilon))_\varepsilon$

$$\begin{gathered}(\mathbf{R}(r_\varepsilon,\varepsilon))_\varepsilon = \left(\frac{8m(r_\varepsilon - 2m)}{r_\varepsilon^3\sqrt{(r - 2m)^2 + \varepsilon^2}}\right)_\varepsilon + \\ 2[(r_\varepsilon^{-3})_\varepsilon]\left(\sqrt{(r_\varepsilon - 2m)^2 + \varepsilon^2}\right)_\varepsilon - \frac{2}{(r_\varepsilon^2)_\varepsilon} - \left(\frac{4m(r_\varepsilon - 2m)^3 + (r_\varepsilon - 4m)\varepsilon^2}{r_\varepsilon^3\left[(r_\varepsilon - 2m)^2 + \varepsilon^2\right]^{3/2}}\right)_\varepsilon.\end{gathered} \qquad (A.11)$$

Remark A.1. Note that from Eq.(A.11) follows that: if $\mathbf{st}((r_\varepsilon)_\varepsilon) \neq 0$, i.e. $(r_\varepsilon)_\varepsilon \not\approx_{\widetilde{\mathbb{R}}} 2m$ then $(r_\varepsilon)_\varepsilon \not\approx_{\widetilde{\mathbb{R}}} 2m \Rightarrow (\mathbf{R}(r_\varepsilon,\varepsilon))_\varepsilon \sim (\varepsilon^2)_\varepsilon \approx_{\widetilde{\mathbb{R}}} 0$.
We assume now that $\mathbf{cl}[(r_\varepsilon)_\varepsilon] \approx_{\widetilde{\mathbb{R}}} 2m$ and therefore from Eq.(A.11) we obtain

$$(\mathbf{R}(r_\varepsilon,\varepsilon))_\varepsilon \approx_{\widetilde{\mathbb{R}}} \left(\frac{4m^2\varepsilon^2}{8m^3\left[(r_\varepsilon-2m)^2+\varepsilon^2\right]^{3/2}}\right)_\varepsilon. \quad (A.12)$$

Remark A.2. Note that from Eq.(A.12) at horizon $r =_{\widetilde{\mathbb{R}}} 2m$ follows that:

$$(\mathbf{R}(r,\varepsilon))_\varepsilon = \left(\frac{4m^2\varepsilon^2}{8m^3[\varepsilon^2]^{3/2}}\right)_\varepsilon = (4m)^{-1}(\varepsilon^{-1})_\varepsilon \approx_{\widetilde{\mathbb{R}}} \infty. \quad (A.13)$$

Remark A.3. Note that from Eq.(A.11) follows that:

$$w\text{-}\lim_{\varepsilon\to 0} \mathbf{R}(r,\varepsilon) \sim \delta(r-2m). \quad (A.14)$$

Remark A.4. Let $[(r_\varepsilon-2m)_\varepsilon] \approx_{\widetilde{\mathbb{R}}} 0,$ then from Eq.(A.13) we obtain

$$[(\mathbf{R}(r_\varepsilon,\varepsilon))_\varepsilon] \approx_{\widetilde{\mathbb{R}}} \left[\left(\frac{\varepsilon^2}{2m\left[(r_\varepsilon-2m)^2+\varepsilon^2\right]^{3/2}}\right)_\varepsilon\right]. \quad (A.15)$$

From Eqs.(A.2) and Eq.(A.9) we obtain

$$\begin{gathered}(\mathbf{R}^{\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon = \\ +2\left(\left[\frac{1}{r_\varepsilon}A'_\varepsilon + \frac{-A_\varepsilon+1}{r_\varepsilon^2}\right]^2\right)_\varepsilon + 2\left(\left[\frac{1}{2}A''_\varepsilon + \frac{1}{r_\varepsilon}A'_\varepsilon\right]^2\right)_\varepsilon = \\ 2\left(\left[\left(\frac{1}{r_\varepsilon^3}\sqrt{\varepsilon^2+(r_\varepsilon-2m)^2} + \frac{1}{r_\varepsilon^2}\right)_\varepsilon - 2\frac{m}{r_\varepsilon^3}\frac{r_\varepsilon-2m}{\sqrt{\varepsilon^2+(r_\varepsilon-2m)^2}}\right]^2\right)_\varepsilon + \\ 2\left(\left[\frac{4m(r_\varepsilon-2m)^3+(r_\varepsilon-4m)\varepsilon^2}{r_\varepsilon^3\left[(r_\varepsilon-2m)^2+\varepsilon^2\right]^{3/2}} \right.\right. \\ \left.\left. -2\frac{m}{r_\varepsilon^3}\frac{r_\varepsilon-2m}{\sqrt{\varepsilon^2+(r_\varepsilon-2m)^2}}\right]^2\right)_\varepsilon .\end{gathered} \quad (A.16)$$

Remark A.5. Note that from Eq.(A.16) follows that:if $\mathbf{st}((r_\varepsilon)_\varepsilon) \neq 0,$i.e.$(r_\varepsilon)_\varepsilon \not\approx_{\widetilde{\mathbb{R}}} 2m$ then

$$\begin{gathered} r \not\approx_{\widetilde{\mathbb{R}}} 2m \Rightarrow (\mathbf{R}^{\mu\nu}(r,\varepsilon)\mathbf{R}_{\mu\nu}(r,\varepsilon))_\varepsilon \approx_{\widetilde{\mathbb{R}}} K(r), \\ K(r) = 12\frac{r_s^2}{r^6}, r_s = 2m. \end{gathered} \quad (A.17)$$

We assume now that $(r_\varepsilon)_\varepsilon \approx_{\widetilde{\mathbb{R}}} 2m$ and therefore from Eq.(A.16) we obtain

$$(\mathbf{R}^{\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon \approx_{\widetilde{\mathbb{R}}} K(r_s) + \left(\frac{\varepsilon^4}{4m^4\left[\varepsilon^2+(r_\varepsilon-2m)^2\right]^3}\right)_\varepsilon \quad (A.18)$$

Remark A.6. Note that from Eq.(A.18) at horizon $r =_{\widetilde{\mathbb{R}}} 2m$ follows that:

$$(\mathbf{R}^{\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon = \left(\frac{1}{4m^4\varepsilon^2}\right)_\varepsilon \approx_{\widetilde{\mathbb{R}}} \infty, \quad (A.19)$$

Remark A.7. Let $[(r_\varepsilon-2m)_\varepsilon] \approx_{\widetilde{\mathbb{R}}} 0,$ then from Eq.(A13) and Eq.(A.12) we obtain

$$[(\mathbf{R}^{\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon] \approx_{\widetilde{\mathbb{R}}} K(r_s) + \left[\left(\frac{\varepsilon^4}{4m^4\left(\varepsilon^2+(r_\varepsilon-2m)^2\right)^3}\right)_\varepsilon\right] \qquad (A.20)$$

From Eqs.(A.2) and Eq.(A.3) we obtain

$$\begin{gathered}(\mathbf{R}^{\rho\sigma\mu\nu}(r,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r,\varepsilon))_\varepsilon = \\ \left((A''_\varepsilon)^2 + 2\left(\frac{A'_\varepsilon}{r}\right)^2\right)_\varepsilon + 4\left(\left[\frac{1}{r^2}(1-A_\varepsilon)\right]^2\right)_\varepsilon + 2\left(\left[\frac{A'_\varepsilon}{r}\right]^2\right)_\varepsilon = \\ \left(A''^2_\varepsilon + 4\frac{A'^2_\varepsilon}{r^2}\right)_\varepsilon + 4\left(\frac{1}{r^4}(1-A_\varepsilon)^2\right)_\varepsilon = \\ \left(\left[\frac{4m(r-2m)^3+(r-4m)\varepsilon^2}{r^3\left[(r-2m)^2+\varepsilon^2\right]^{3/2}}\right]^2\right)_\varepsilon - \frac{8m^2(r-2m)^2}{r^6\left[(r-2m)^2+\varepsilon^2\right]} + \\ \frac{4}{r^4}\left(1+r^{-1}\sqrt{(r-2m)^2+\varepsilon^2}\right)^2.\end{gathered} \qquad (A.21)$$

Remark A.8. Note that from Eq.(C.15) follows that:

$$r \not\approx_{\widetilde{\mathbb{R}}} 2m \Rightarrow (\mathbf{R}^{\rho\sigma\mu\nu}(r,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r,\varepsilon))_\varepsilon \approx_{\widetilde{\mathbb{R}}} K(r), \qquad (A.22)$$

see Definition 1.5.2.(i).

We assume now that $(r_\varepsilon)_\varepsilon \approx_{\widetilde{\mathbb{R}}} 2m$ and therefore from Eq.(C.10) we obtain

$$(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon \approx_{\widetilde{\mathbb{R}}} K(r_s) + \left(\frac{\varepsilon^4}{4m^4\left[\varepsilon^2+(r_\varepsilon-2m)^2\right]^3}\right)_\varepsilon. \qquad (A.23)$$

Remark A.9. Let $[(r_\varepsilon - 2m)_\varepsilon] \approx_{\widetilde{\mathbb{R}}} 0$, then from Eq.(A.13) and Eq.(A.12) we obtain

$$[(\mathbf{R}^{\rho\sigma\mu\nu}(r,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r,\varepsilon))_\varepsilon] = K(r_s) + \left(\frac{\varepsilon^4}{4m^4\left[\varepsilon^2+(r_\varepsilon-2m)^2\right]^3}\right)_\varepsilon. \qquad (A.24)$$

Remark A.10. Note that from Eq.(A.15) at horizon $r = 2m$ follows that:

$$(\mathbf{R}^{\rho\sigma\mu\nu}(r,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r,\varepsilon))_\varepsilon \approx_{\widetilde{\mathbb{R}}} \infty, \qquad (A.25)$$

see Definition 1.5.2.(ii).

Remark A.11. We assume now there exist an fundamental generalized lengh $(l_\varepsilon)_\varepsilon$

$$\begin{gathered}(l_\varepsilon)_{\varepsilon\in(0,\eta]} = a(\varepsilon)_{\varepsilon\in(0,\eta]}, \eta \ll 1, \\ (l_\varepsilon)_{\varepsilon\in(\eta,1]} = a,\end{gathered} \qquad (A.26)$$

such that $|(r_\varepsilon - \overline{\rho})_\varepsilon| \geq (l_\varepsilon)_\varepsilon = a(\varepsilon)_\varepsilon$ It meant there exist a thickness $th_{hor} = (l_\varepsilon)_\varepsilon$ of BH horizon. We introduce a norm $\|th_{hor}\|$ of a thickness th_{hor} by formula

$$\|th_{hor}\| = \sup_{\varepsilon\in(0,\eta]}|l_\varepsilon| = \eta, \qquad (A.27)$$

By using (A.20) we get the estimate

$$(\mathbf{R}^{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon)\mathbf{R}_{\rho\sigma\mu\nu}(r_\varepsilon,\varepsilon))_\varepsilon \approx_{\widetilde{\mathbb{R}}} K(r_s) + \left(\frac{\varepsilon^4}{4m^4\left[(r_\varepsilon - 2m)^2 + \varepsilon^2\right]^3}\right)_{\varepsilon\in(0,\eta]} =$$
$$K(r_s) + \left(\frac{1}{4m^4\left[(r_\varepsilon - 2m)^2 + \varepsilon^2\right]}\right)_{\varepsilon\in(0,\eta]} \times \left(\frac{\varepsilon^2}{\left[(r_\varepsilon - 2m)^2 + \varepsilon^2\right]}\right)_{\varepsilon\in(0,\eta]} \times$$
$$\times\left(\frac{\varepsilon^2}{\left[(r_\varepsilon - 2m)^2 + \varepsilon^2\right]}\right)_{\varepsilon\in(0,\eta]} \leq \qquad (A.28)$$
$$K(r_s) + \frac{1}{4m^4[a^2+1]^2}\left(\frac{1}{\left[(r_\varepsilon - 2m)^2 + \varepsilon^2\right]}\right)_{\varepsilon\in(0,\eta]} \leq$$
$$K(r_s) + \frac{1}{4m^4[a^2+1]^2}\left(\frac{1}{(r-2m)^2}\right)_{r-2m\in(0,\eta]}.$$

.

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