ON ENTIRE FUNCTIONS-MINORANTS FOR SUBHARMONIC FUNCTIONS OUTSIDE OF A SMALL EXCEPTIONAL SET

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$\mathbb{N} := \{1, 2, \dots\}$ is the set of all *natural numbers,* and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. $\#S$ is *the cardinality of a set* S. We consider *the set* $\mathbb{R}$ *of real numbers* mainly as *the real axis* in *the complex plane* $\mathbb{C}$. By this means, $\mathbb{R}^+ := \{x \in \mathbb{R} \colon x \geq 0\}$ is *the positive semiaxis* in $\mathbb{C}$. Besides, $\mathbb{R}^+_* := \mathbb{R}^+ \setminus \{0\}$, $\mathbb{R}^+_{+\infty} := \mathbb{R}^+ \cup \{+\infty\}$, $\mathbb{R}_{\pm\infty} := \mathbb{R}^+_{+\infty} \cup (-\mathbb{R}^+_{+\infty})$.

Let $S \subset \mathbb{C}$. $\mathcal{B}(S)$ is is the class of all *Borel subsets* $B \subset S$, and $\mathcal{B}_{\mathrm{c}}(S) \subset \mathcal{B}(S)$ is the class of all *compact Borel subset in* S.

Let $S \in \mathcal{B}(\mathbb{C})$. $\mathrm{Meas}(S)$ is the class of all countably additive functions ν on $\mathcal{B}(S)$ with values in $\mathbb{R}_{\pm\infty}$ such that $\nu(K) \in \mathbb{R}$ for each $K \in \mathcal{B}_{\mathrm{c}}(S)$. Elements from $\mathrm{Meas}(S)$ are called *charges,* and $\mathrm{Meas}^+(S) \subset \mathrm{Meas}(S)$ is the subclass of positive charges called *measures.*

The classes $\mathrm{sbh}(S)$, $\mathrm{har}(S) := \mathrm{sbh}(S) \cap (-\mathrm{sbh}(S))$, $\mathrm{Hol}(S)$ consist of the restrictions to S of subharmonic, harmonic, holomorphic functions on open sets containing S resp., and $\mathrm{sbh}_*(S) := \{u \in \mathrm{sbh}(S) \colon u \not\equiv -\infty\}$, $\mathrm{Hol}_*(S) := \mathrm{Hol}(S) \setminus \{0\}$. *The Riesz measure of* $u \in \mathrm{sbh}(S)$ is the measure $\frac{1}{2\pi} \triangle u \in \mathrm{Meas}^+(S)$ where $\triangle$ is the Laplace operator acting in the sense of the theory of distributions [1], [2].

Given $z \in \mathbb{C}$ and $r \in \mathbb{R}^+$, $D(z, r) := \{z' \in \mathbb{C} \colon |z' - z| < r\}$ is *an open disk of radius* r *centered at* z*;* $D(r) := D(0, r)$; $\mathbb{D} := D(1)$ is *the unit disk.* Besides, $\overline{D}(z, r) := \{z' \in \mathbb{C} \colon |z' - z| \leq r\}$ is *a closed disk;* $\overline{D}(r) := \overline{D}(0, r)$; $\overline{\mathbb{D}} := \overline{D}(1)$, and $\partial\overline{D}(z, r)$ is *a circle of radius* r *centered at* z*;* $\partial\overline{D}(r) := \partial\overline{D}(0, r)$; $\partial\overline{\mathbb{D}} := \partial\overline{D}(1)$ is *the unit circle.*

Given $\nu \in \mathrm{Meas}(\mathbb{C})$, we denote by ν^+, $\nu^- := (-\nu)^+$ and $|\nu| := \nu^+ + \nu^-$ the *upper, lower,* and *total variations* of ν, and define *the counting function of* ν *at* $z \in \mathbb{C}$ as $\nu(z, r) := \nu\bigl(\overline{D}(z, r)\bigr)$, and *the radial counting function of* ν as $\nu^{\mathrm{rad}}(r) := \nu(0, r)$, $r \in \mathbb{R}^+$.

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For a function $v\colon \overline{D}(z,r)\to \mathbb{R}_{\pm\infty}$, we define [1, Definition 2.6.7], [2]

$$\mathsf{C}_v(z,r):=\frac{1}{2\pi}\int_0^{2\pi} v(z+re^{i\theta})\,\mathrm{d}\theta, \qquad \mathsf{C}_v(r):=\mathsf{C}_v(0,r), \tag{1C}$$

$$\mathsf{B}_v(z,r):=\frac{2}{r^2}\int_0^{r} \mathsf{C}_v(z,t)t\,\mathrm{d}t, \qquad \mathsf{B}_v(r):=\mathsf{B}_v(0,r), \tag{1B}$$

$$\mathsf{M}_v(z,r):=\sup_{z'\in\partial\overline{D}(z,r)} v(z'), \qquad \mathsf{M}_v(r):=\mathsf{M}_v(0,r), \tag{1M}$$

where

$$\mathsf{M}_v(z,r):=\sup_{z'\in\overline{D}(z,r)} v(z')$$

if $v\in \operatorname{sbh}\bigl(\overline{D}(z,r)\bigr)$ [1, Definition 2.6.7], [2].

Consider a function $d\colon \mathbb{C}\to\mathbb{R}^+$.

Given $S\subset\mathbb{C}$ and $r\colon\mathbb{C}\to\mathbb{R}$, we define

$$S^{\bullet d}:=\bigcup_{z\in S} D\bigl(z,d(z)\bigr)\subset\mathbb{C},$$

$$r^{\bullet d}\colon z\longmapsto \sup\Bigl\{r(z')\colon z'\in D\bigl(z,d(z)\bigr)\Bigr\}\in\mathbb{R}^+_{+\infty}, \quad z\in\mathbb{C},$$

and denote *the indicator function* of set S by

$$\mathbf{1}_S\colon z\longmapsto \begin{cases}1 & \text{if } z\in S\\ 0 & \text{if } z\notin S\end{cases}, \quad z\in\mathbb{C}.$$

Theorem 1 (cf. [3, Normal Points Lemma], [4, § 4. Normal points, Lemma])**.** *Let $r\colon\mathbb{C}\to\mathbb{R}^+$ be a Borel function such that*

$$d:=2\sup\{r(z)\colon z\in\mathbb{C}\}<+\infty, \tag{2}$$

and $\mu\in\operatorname{Meas}^+(\mathbb{C})$ be a measure with

$$E:=\left\{z\in\mathbb{C}\colon \int_0^{r(z)}\frac{\mu(z,t)}{t}\,\mathrm{d}t>1\right\}\subset\mathbb{C}. \tag{3}$$

Then there is a no-more-than countable set of disks $D(z_k,t_k)$, $k=1,2,\dots$, such that

$$\begin{gathered} z_k\in E,\quad t_k\leq r(z_k),\quad E\subset\bigcup_k D(z_k,t_k),\\ \sup_{z\in\mathbb{C}}\#\bigl\{k\colon z\in D(z_k,t_k)\bigr\}\leq 2020,\end{gathered} \tag{4}$$

i. e., the multiplicity of this covering $\{D(z_k,t_k)\}_{k=1,2,\dots}$ *of set* E *not larger than* 2020*, and for every* μ*-measurable subset* $S\subset E$*,*

$$\sum_{S\cap D(z_k,t_k)\neq\varnothing} t_k\leq 2020\int_{S^{\bullet d}} r^{\bullet r}\,\mathrm{d}\mu\leq 2020\int_{S^{\bullet d}} r^{\bullet d}\,\mathrm{d}\mu. \tag{5}$$

Proof. By definition (3), there is a number

$$t_z\in\bigl(0,r(z)\bigr)\quad\textit{such that}\quad 0<t_z<r(z)\mu(z,t_z)\quad\textit{for each } z\in E. \tag{6}$$

Thus, the system $\mathcal{D}=\bigl\{D(z,t_z)\bigr\}_{z\in E}$ of these disks has properties

$$E\subset\bigcup_{z\in E}D(z,t_z),\quad 0<t_z\leq r(z)\overset{(2)}{\leq} R. \tag{7}$$

By the Besicovitch covering theorem [5, 2.8.14]–[10, I.1, Remarks] in the Landkof version [11, Lemma 3.2], one can select some no-more-than counting subsystem in $\mathcal{D}$ of disks $D(z_k,t_k)\in\mathcal{D}$, $k=1,2,\dots$, $t_k:=t_{z_k}$, such that properties (4) are fulfilled.

Consider a μ-measurable subset $S\subset E$. In view of (6) it is easy to see that

$$\bigcup\Bigl\{D(z_k,t_k)\colon S\cap D(z_k,t_k)\neq\varnothing\Bigr\}\overset{(6),(2)}{\subset}\bigcup_{z\in S}D(z,d)=S^{\bullet d}. \tag{8}$$

Hence, in view of (6) and (4), we obtain

$$\begin{aligned}\sum_{S\cap D(z_k,t_k)\neq\varnothing} t_k &:= \sum_{S\cap D(z_k,t_k)\neq\varnothing} t_{z_k}\overset{(6)}{\leq}\sum_{S\cap D(z_k,t_k)\neq\varnothing} r(z_k)\mu(z,t_k)\\ &=\sum_{S\cap D(z_k,t_k)\neq\varnothing}\int_{D(z_k,t_k)} r(z_k)\,\mathrm{d}\mu(z)\overset{(6)}{\leq}\sum_{S\cap D(z_k,t_k)\neq\varnothing}\int_{D(z_k,t_k)} r^{\bullet r}\,\mathrm{d}\mu\\ &\overset{(8)}{=}\sum_{S\cap D(z_k,t_k)\neq\varnothing}\int_{S^{\bullet d}}\mathbf{1}_{D(z_k,t_k)}r^{\bullet r}\,\mathrm{d}\mu\\ &=\int_{S^{\bullet d}}\Biggl(\sum_{S\cap D(z_k,t_k)\neq\varnothing}\mathbf{1}_{D(z_k,t_k)}\Biggr)r^{\bullet r}\,\mathrm{d}\mu\overset{(4)}{\leq}2020\int_{S^{\bullet d}} r^{\bullet r}\,\mathrm{d}\mu\end{aligned}$$

that gives (5). □

Theorem 2 ([12, Corollary 2])**.** *Let* $w\in\operatorname{sbh}_*(\mathbb{C})$*,* $P\in\mathbb{R}^+$*, and*

$$p\colon z\longmapsto\frac{1}{(1+|z|)^P},\quad z\in\mathbb{C}. \tag{9}$$

There is an entire function $f\in\operatorname{Hol}_*(\mathbb{C})$ *such that*

$$\ln\bigl|f(z)\bigr|\leq\mathsf{B}_w\bigl(z,p(z)\bigr)\leq\mathsf{C}_w\bigl(z,p(z)\bigr)\quad\textit{for each } z\in\mathbb{C}. \tag{10}$$

A function $f\colon [a,+\infty)\to \mathbb{R}_{\pm\infty}$ is a function *of finite type* (with respect to an order $p\in\mathbb{R}^+$ near $+\infty$) iff (see [13, 2.1, (2.1t)])

$$\operatorname{type}_p[f]:=\operatorname{type}_p^{\infty}[f]:=\limsup_{r\to+\infty}\frac{f^+(r)}{r^p}<+\infty,\quad f^+:=\sup\{f,0\}.$$

A function $v\in\operatorname{sbh}(\mathbb{C})$ of *finite type* (with respect to an order $p\in\mathbb{R}^+$) iff $\operatorname{type}_p\bigl[v\bigr]\overset{(1M)}{:=}\operatorname{type}_p[\mathsf{M}_v]<+\infty$ [13, Remark 2.1].

The upper density $\operatorname{type}_1[\nu]$ of a charge $\nu\in\operatorname{Meas}$ is defined as $\operatorname{type}_1[\nu]:=\operatorname{type}_1\bigl[|\nu|^{\mathrm{rad}}\bigr]$.

The order of a function $f\colon [a,+\infty)\to\mathbb{R}_{\pm\infty}$ (near $+\infty$) is a value

$$\operatorname{ord}_\infty[f]:=\inf\bigl\{p\in\mathbb{R}^+\colon\ \operatorname{type}_p[f]<+\infty\bigr\}=\limsup_{r\to+\infty}\frac{\ln\bigl(1+f^+(r)\bigr)}{\ln r}\in\mathbb{R}^+_{+\infty}. \tag{11}$$

A charge $\nu\in\operatorname{Meas}(\mathbb{C})$ of finite order iff $\operatorname{ord}_\infty[\nu]:=\operatorname{ord}_\infty\bigl[|\nu|^{\mathrm{rad}}\bigr]<+\infty$. A function $v\in\operatorname{sbh}(\mathbb{C})$ of finite order iff $\operatorname{ord}_\infty[v]:=\operatorname{ord}_\infty[\mathsf{M}_v]<+\infty$. An trivial corollary of the Poisson – Jensen formula is

Theorem 3. *Let $w\in\operatorname{sbh}_*(\mathbb{C})$ with Riesz measure $\mu=\frac{1}{2\pi}\bigtriangleup w\in\operatorname{Meas}^+(\mathbb{C})$. Then we have $\operatorname{ord}_\infty[\mu]=\operatorname{ord}_\infty[\mathsf{C}_v]=\operatorname{ord}_\infty[\mathsf{B}_v]$, and*

$$\bigl[\operatorname{type}_p[\mu]<+\infty\bigr]\Longleftrightarrow\bigl[\operatorname{type}_p[\mathsf{C}_v]<+\infty\bigr]\Longleftrightarrow\bigl[\operatorname{type}_p[\mathsf{B}_v]<+\infty\bigr]$$

for each $p\in\mathbb{R}^+_*$.

Theorem 4. *Let $w\in\operatorname{sbh}_*(\mathbb{C})$ be a function with $\operatorname{ord}_\infty[\mathsf{C}_w]<+\infty$. Then for any $P\in\mathbb{R}^+$, there are $h\in\operatorname{Hol}_*(\mathbb{C})$ with $\operatorname{ord}_\infty\bigl[\ln|h|\bigr]\leq\operatorname{ord}_\infty[w]$ and $\operatorname{type}_q\bigl[\ln|h|\bigr]\leq\operatorname{type}_q[w]$ for each $q\in\mathbb{R}^+$, and a no-more-than countable set of disks $D(z_k,t_k)$, $k=1,2,\dots$, such that*

$$\Bigl\{z\in\mathbb{C}\colon\ \ln\bigl|h(z)\bigr|>w(z)\Bigr\}\subset\bigcup_k D(z_k,r_k), \tag{12I}$$

$$\sup_k t_k\leq 1,\qquad \sum_{|z_k|\geq R}t_k=O\Bigl(\frac{1}{R^P}\Bigr),\quad R\to+\infty. \tag{12E}$$

Proof. By Theorem 3, $\operatorname{ord}_\infty[\mu]=\operatorname{ord}_\infty[\mathsf{C}_w]<+\infty$ for $\mu:=\frac{1}{2\pi}\bigtriangleup w$.

Consider $P\in 1+\operatorname{ord}_\infty[\mu]+\mathbb{R}^+$ and an entire function f from Theorem 2 with (9)–(10). Then for $h:=e^{-1}f\in\operatorname{Hol}_*(\mathbb{C})$ we obtain

$$\ln\bigl|h(z)\bigr|\leq\mathsf{C}_w\bigl(z,p(z)\bigr)-1=w(z)+\int_0^{p(z)}\frac{\mu(z,t)}{t}\,\mathrm{d}t-1\quad\text{for each } z\in\mathbb{C}.$$

Hence, by Theorem 1 with $r \overset{(9)}{:=} p$ and $S \overset{(3)}{:=} E \setminus D(R)$, we have (12I) with properties (4)–(5)$\Longrightarrow$(12E). The relations between the orders and types of w and $\ln|h|$ are an obvious consequence of (12). □

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