ON THE INFINITUDE OF TWIN PRIME PAIRS AND THE GENERALIZED GOLDBACH'S CONJECTURE

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ABSTRACT

We prove that there are infinitely many twin prime pairs and further propose the generalized version of Goldbach's Conjecture

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INTRODUCTION

An Isolated prime P is such that neither P+2 nor P-2 is any prime. In the Infinitude of primes, if we strike out every twin prime pair, then the rest leftover prime numbers are called Isolated primes. We first establish a prerequisite result on the distribution of Isolated primes in the form of proportion which states that

Every prime number $k > 2$ such that $P_j + 4 \leq k \leq P_{j+2} - 4$ for $P_{j+2} - P_j \geq 10$ can neither be expressed as sum nor the difference of two distinct primes,

which implies that every such k is an Isolated prime. Throughout the proof of proposition, foothold of our arguments lies upon the existence of twin primes until we arrive at conclusion about its Infinitude. Further we propose the generalized version of Goldbach's Conjecture which enables us to cruise through all the divisors of all the positive even integers.

Proof of the Twin Prime Conjecture

Conjecture 1 (Twin Prime Conjecture). *There are infinitely many pairs of primes which differ by 2.*

The Twin Prime Conjecture was posed 174 years ago by De Polignac and since then it is evident that it has witnessed the most determined efforts by the greatest mathematicians of all times, and they have successfully made significant progress in past century. Our approach goes through the following proposition.

Proposition 1 (Isolated Primes). *Every prime number $k > 2$ such that $P_j + 4 \leq k \leq P_{j+2} - 4$ for $P_{j+2} - P_j \geq 10$ can neither be expressed as sum nor the difference of two distinct primes.*

Proof. Let $P_j < k < P_{j+2}$ be three consecutive primes. If (P_j, k) is any twin prime pair then $k - P_j = 2$ or $P_j + 2 = k$, also if (k, P_{j+2}) is any twin prime pair then $P_{j+2} - k = 2$ or $2 + k = P_{j+2}$ in either of the cases, it is always possible to express it as either sum or difference of two distinct primes, this implies that twin prime pairs are the only exception to the Proposition 1. Prime gaps are multiples of 2 and excluding the gap length of 2 we are left with gaps which are ≥ 4 which provides us sufficient condition to set $P_j + 4 \leq k$ and $k \leq P_{j+2} - 4$

For establishing gap between P_j and P_{j+2} consider $a_1 < a_2 < a_3... < a_n$ to be the positive integers between the successive twin prime pairs, where $a_1 > 4$ and $a_{n+1} - a_n \geq 6c$ where $a, c, n \in N$.

In general consider the following bound

$$a_n - 1 < a_n < a_n + 1 < (b_i)_1 < (b_i)_2 < ... < (b_i)_k < a_{n+1} - 1 < a_{n+1} < a_{n+1} + 1$$

Where the twin prime pairs are $(a_n \pm 1)$ and $(a_{n+1} \pm 1)$ respectively and $b, i, k \in N$. For each individual choice of k, b_i will take three distinct values due to the relation $a_{n+1} - a_n \geq 6c$ so for $k = 1$ we have $(b_i)_1 = \{(b_1)_1, (b_2)_1, (b_3)_1\}$ and so on.

Note that $a_n - 1$ and $a_{n+1} + 1$ are at extremes of bound and among twin prime pairs, which is meant to be excluded for the sake of proposition. In general the total number of elements including the elements at the extremes of bound $(a_n - 1)$ and $(a_{n+1} + 1)$ is $3(2c+1)$. Therefore we need minimum of 9 distinct elements in between the pairs (P_j, P_{j+2}) which implies that $P_{j+2} - P_j \geq 10$ and completes the proof of Proposition 1.

Remark. Proof of Proposition 1 is established upon the existence of twin primes, that is by far the most important aspect and whether it is finite or infinite would be its consequential result. However till here it does not yield any information about the Infinitude of twin primes, so we would not assume that the distribution of Isolated primes is independent of the distribution of twin primes and vice-versa. But investigating upon the aspects of finite twin prime is of great interest to the proposition, so we have the following arguments.

In the Infinitude of primes consider the total number of twin primes to be finite. Let (P_i, P_{i+1}) be the last twin prime pair. The immediate effect would be the vanishing of closed upper bound $k \leq P_{j+2} - 4$ since there are no more prime gaps of length 2, there will be no exceptions (Twin Primes) to the proposition which could break the succession of Isolated primes at some arbitrary length till infinity. It implies that every prime k such that

$$P_i + 4 \leq k < \infty$$

will be Isolated prime. Moreover in order to prove the significance of closed upper bound $k \leq P_{j+2} - 4$ which ensures that the succession of Isolated primes breaks at some point due to twin prime pairs Infinitely many times, we have to explore the remaining prime gaps, which is ≥ 4 respectively.

Let (P_r, P_{r+1}) and (P_t, P_{t+1}) be any two distinct pairs of cousin primes, with $P_r > P_t$ for $r > t$.
Consider the pair (P_r, P_{r+1}) happens to appear for the first time after the assumption of finite twin primes. So we have

$$P_{r+1} - P_r = 4 \textit{ and } P_{t+1} - P_t = 4$$

Let the positive integers in between the pairs of cousin primes be such

$P_r < \alpha_1 < \alpha_2 < \alpha_3 < P_{r+1}$ and $P_t < \beta_1 < \beta_2 < \beta_3 < P_{t+1}$ where $\beta_n - \alpha_n \geq 6l$, $l \in N$

Let the positive integers between P_{i+1} and P_r be such

$$P_i < P_i + 1 < P_{i+1} < d_1 < d_2 < d_3 \ldots < d_n < P_r$$

Note that $P_r - P_{i+1} \geq 6$,

else for $P_r - P_{i+1} = 4 = P_{r+1} - P_r$ one element of $\{P_{i+1}, P_r, P_{r+1}\}$ will must be a multiple of 3, which raises contradiction, it is possible only with $\{3, 7, 11\}$

For the sake of completeness, let the positive integers between P_{r+1} and P_t be such

$$P_{r+1} < f_1 < f_2 < f_3 \dots f_e < P_t$$

The prime gaps $P_r - P_{i+1} \geq 6$ and $P_{r+1} - P_r = P_{t+1} - P_t = 4$ is already well defined, now our priority shifts to define the prime gap $P_t - P_{r+1}$ for that same consider the following

$$P_r < \alpha_1 < \alpha_2 < \alpha_3 < P_{r+1} < f_1 < f_2 < f_3 \dots < f_e < P_t < \beta_1 < \beta_2 < \beta_3 < P_{t+1}$$

where $\alpha_3 = \alpha_2 + 1$, $P_{r+1} = \alpha_2 + 2$ and $\beta_1 = \beta_2 - 1$, $P_t = \beta_2 - 2$

$$\begin{aligned} Now\ P_t - P_{r+1} &= \beta_2 - 2 - (\alpha_2 + 2) \\ &= \beta_2 - 2 - \alpha_2 - 2 \\ &= \beta_2 - \alpha_2 - 4 \end{aligned}$$

Using the relation $\beta_n - \alpha_n \geq 6l$ for $n = 2$

$$P_t - P_{r+1} \geq 6l - 4$$

$$P_t - P_{r+1} \geq 2(3l - 2)\ for\ l \in N$$

Now the prime gap $P_t - P_{r+1}$ corresponds to lengths

$$\{2, 8, 14, 20, \dots 2(3l - 2)\}$$

with the minimum gap of 2, which is the desired contradiction to our assumption of finite twin primes.

At this very point, the pair (P_{r+1}, P_t) whenever differs by 2 it becomes an exception to the proposition and breaks the succession of Isolated primes which exists in the bound

$$P_{i+1} < d_1 < d_2 < d_3 \dots < d_n < P_r$$

Our arguments are consolidated by the conclusion that the closed upper bound $(k \leq P_{j+2} - 4)$ does not grow to infinity but rather takes any fixed value.
Here we gain advantage from the pre-established results because in either way we do not have to make this process happen infinitely many times.

The bound $P_j + 4 \leq k \leq P_{j+2} - 4$ itself propagates through the Infinitude of primes automatically resulting the above process to repeat infinitely many times. So it has infinitely many exceptions (Twin Primes). This completes the proof of Conjecture 1.

In conclusion there are infinitely many Twin Prime Pairs and there are infinitely many successive Isolated Primes of some arbitrary length.

GENERALIZED GOLDBACH'S CONJECTURE

Both the Twin Prime Conjecture and the Goldbach's Conjecture have deep connections with even natural numbers, specifically $2n = a + b$ partition form, $a, b \in N$.

To illustrate this exquisite relationship we have the following conjecture:

Conjecture 2. *For every $2n = a + b$, where $a, b, n \in N$, there always exists minimum one pair (a, b) such that both $2n + a$ and $2n + b$ are primes simultaneously, with $gcd(a, b) = 1$ and the values of $2n + a$ and $2n + b$ ranges over all the odd primes.*

As values of primes ranges over all the odd primes, we could expect distribution of twin primes too, so we have the following table illustrating $2n + a$, $2n + b = P$, $P + 2$ for prime P

n	$2n$	$a + b$	$P, P + 2$
2	4	$1 + 3$	$5, 7$
4	8	$3 + 5$	$11, 13$
6	12	$5 + 7$	$17, 19$
10	20	$9 + 11$	$29, 31$
14	28	$13 + 15$	$41, 43$
20	40	$19 + 21$	$59, 61$
24	48	$23 + 25$	$71, 73$
34	68	$33 + 35$	$101, 103$
36	72	$35 + 37$	$107, 109$
46	92	$45 + 47$	$137, 139$

Above is the illustration for first 10 twin prime pairs

Moreover $3n = P + 1$ and $a < n < b$ where a, n and b are three consecutive natural numbers.

Both n and $a + b$ are unique for each individual choice of Twin Prime pair $\{P, P + 2\}$.

Also we can replace $2n$ by all of its divisors, which in turn gives us the generalized version of the former one.

Conjecture 3. *For every $2n = a + b$, where $a, b, n \in N$, let d be the factor of $2n$ then there always exists minimum one pair (a, b) for every d such that both $d + a$ and $d + b$ are primes simultaneously with $gcd(a, b) = 1$ where the values of $d + a$ and $d + b$ rangers over all the primes.*

In the same spirit as Conjecture 2 and 3, we propose Goldbach's Conjecture and its generalized version.

Conjecture 4 (Goldbach's Conjecture). *For every $2n = a + b$ and for $n > 1$, where $a, b, n \in N$, there always exists minimum one pair (a, b) such that $2n - a$ and $2n - b$ are primes simultaneously, with $gcd(a, b) = 1$ and the values of $2n - a$ and $2n - b$ ranges over all the primes.*

Conjecture 5 (Generalized Goldbach's Conjecture). *For every $2n = a + b$ and for $n > 1$, where $a, b, n \in N$, let d be the factors of $2n$ then there always exists minimum one pair (a, b) for every d such that both $|d - a|$ and $|d - b|$ are primes simultaneously with $gcd(a, b) = 1$ where the values of $|d - a|$ and $|d - b|$ ranges over all the primes.*

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