ON IDEAL TOPOLOGICAL GROUPS

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(Received November 15, 2019)

ABSTRACT. In this paper, we introduce and study the class of ideal topological groups by using $\mathcal{I}$-open sets and $\mathcal{I}$-continuity.

1. Introduction

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [7] and Vaidyanathaswamy, [10]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star\colon \mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [10] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau\colon x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(\cdot)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $\star$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau, \mathcal{I})$ when there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. Recently, Hussain et. al. [4, 5] introduced and studied some new notions in topological groups. In this paper, we introduce and study the class of ideal topological groups by using $\mathcal{I}$-open sets and $\mathcal{I}$-continuity.

2000 Mathematics Subject Classification 54C10, 54C08, 54C05

2. Preliminaries

Throughout this paper $(G, \star, \tau, \mathcal{I})$, or simply G, will denote a group $(G, \star)$ endowed with a topology τ and ideal $\mathcal{I}$. The identity element of G is denoted by e, or e_G when it is necessary, the operation $\star \colon G \times G \to G$, $(x, y) \to x \star y$, is called the multiplication mapping and sometimes denoted by m, and the inverse mapping $i \colon G \to G$, $x \to x^{-1}$ is denoted by i. X and Y denote topological spaces on which no separation axioms are priori assumed. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and the interior of A in (X, τ), respectively. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-open [6] if $S \subset \mathrm{Int}(S^*)$. The complement of an $\mathcal{I}$-closed set is said to be an $\mathcal{I}$-open set. The $\mathcal{I}$-closure and the $\mathcal{I}$-interior, that can be defined in the same way as $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\mathcal{I}\,\mathrm{Cl}(A)$ and $\mathcal{I}\,\mathrm{Int}(A)$, respectively. The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}O(X)$ (resp. $\mathcal{I}C(X)$). The family of all $\mathcal{I}$-open (resp. $\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $\mathcal{I}O(X, x)$ (resp. $\mathcal{I}C(X, x)$).

Definition 2.1 ([1]). A subset M of an ideal topological space $(X, \tau, \mathcal{I})$ is called an $\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists an $\mathcal{I}$-open set S such that $x \in S \subset M$.

Definition 2.2 ([1]). A function $f \colon (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ is said to be:

(1) $\mathcal{I}$-continuous if $f^{-1}(V) \in \mathcal{I}O(X)$ for every $V \in \sigma$.
(2) $\mathcal{I}$-open if $f(U) \in \mathcal{I}O(Y)$ for every $U \in \mathcal{I}O(X)$.
(3) $\mathcal{I}$-closed if $f(U) \in \mathcal{I}C(Y)$ for every $U \in \mathcal{I}C(X)$.

Definition 2.3. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $U, V \subset X$. Then we say that the pair U, V is $\mathcal{I}$-separated if $\mathcal{I}\,\mathrm{Cl}(U) \cap V = \mathcal{I}\,\mathrm{Cl}(V) \cap U = \emptyset$. A set $S \subset X$ is $\mathcal{I}$-connected if there are no two nonempty $\mathcal{I}$-separated sets U and V such that $U \cup V = S$. The space X is $\mathcal{I}$-connected if it is an $\mathcal{I}$-connected subset of itself.

3. Properties of ideal topological groups

In this section, we introduce and study a new class of topological groups by using $\mathcal{I}$-open sets and $\mathcal{I}$-continuity.

Definition 3.1. A topologized group $(G, \star, \tau, \mathcal{I})$ is called an ideal topological group if for each $x, y \in G$ and each neighborhood W of $x \star y^{-1}$ in G there exist $\mathcal{I}$-open neighborhoods U of x and V of y such that $U \star V^{-1} \subset W$.

The following lemma will be used in the sequel.

Lemma 3.2. *If $(G, \star, \tau, \mathcal{I})$ is an ideal topological group, then*

(1) $A \in \mathcal{I}O(G)$ if, and only if $A^{-1} \in \mathcal{I}O(G)$;
(2) If $A \in \mathcal{I}O(G)$ and $B \subset G$, then $A \star B$ and $B \star A$ are both in $\mathcal{I}O(G)$.

Definition 3.3. A subset A of a group G is symmetric if $A = A^{-1}$.

Definition 3.4. A bijective function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ is said to be $\mathcal{I}$-homeomorphism if it is $\mathcal{I}$-continuous and $\mathcal{I}$-open.

The following simple result is of fundamental importance in what follows.

Theorem 3.5. *Let $(G, \star, \tau, \mathcal{I})$ be an ideal topological group. Then each left (right) translation $l_g : G \to G(r_g : G \to G)$ is an $\mathcal{I}$-homeomorphism.*

Proof. We prove the statement only for left translations. Of course, left translations are bijective mapping. We prove directly that for any $x \in G$, the translation l_x is $\mathcal{I}$-continuous. Let y be an arbitrary element in G and W an open neighbourhood of $l_x(y) = x \star y = x \star (y^{-1})^{-1}$. By definition of ideal topological groups, there are $\mathcal{I}$-open sets U and V containing x and y^{-1}, respectively, such that $U \star V^{-1} \subset W$. In particular, we have $x \star V^{-1} \subset W$. By Lemma 3.2 the set V^{-1} is an $\mathcal{I}$-open neighbourhood of y, so that the last inclusion actually says that l_x is $\mathcal{I}$-continuous at y. Since $y \in G$ was an arbitrary element in G, l_x is $\mathcal{I}$-continuous on G. We prove now that l_x is $\mathcal{I}$-open. Let A be an $\mathcal{I}$-open set in G. Then by Lemma 3.2, the set $l_x(A) = x \star A = \{x\} \star A$ is $\mathcal{I}$-open in G, which means that l_x is an $\mathcal{I}$-open mapping. ∎

Theorem 3.6. *Let $(G, \star, \tau, \mathcal{I})$ be an ideal topological group and let β_e be the base at identity element e of G. Then*

(1) for every $U \in \beta_e$, there exists $V \in \mathcal{I}O(G, e)$ such that $V^2 \subset U$.
(2) for every $U \in \beta_e$, there exists $V \in \mathcal{I}O(G, e)$ such that $V^{-1} \subset U$.
(3) for every $U \in \beta_e$, there exists $V \in \mathcal{I}O(G, e)$ such that $V \star x \subset U$.

Proof. (1). Let $U \in \beta_e$. This implies that $e \in U \subset G$ and $U_{e \star e^{-1}} = U$. Since $(G, \star, \tau, \mathcal{I})$ is an ideal topological group, there exists $V \in \mathcal{I}O(G, e)$ and by Lemma 3.2, $V^{-1} \in \mathcal{I}O(G, e)$ such that $V \star V \subset U$. Hence $V^2 \subset U$.
(2). Since $(G, \star, \tau, \mathcal{I})$ is an ideal topological group, for every $U \in \beta_e$ there exists

$V \in \mathcal{I}O(G, e)$ such that $i(V) = V^{-1} \in \mathcal{I}O(G, e)$.
(3). Since $(G, \star, \tau, \mathcal{I})$ is an ideal topological group, the left (right) translation $l_g : G \to G (r_g : G \to G)$ is an $\mathcal{I}$-homeomorphism. hence for each $U \in \beta_e$ containing x, there exists $V \in \mathcal{I}O(G, e)$ such that $r_x(V) = V \star x \subset U$. ∎

Corollary 3.7. *Let $(G, \star, \tau, \mathcal{I})$ be an ideal topological group and x be any element of G. Then for any local base β_e at $e \in G$, each of the families $\beta_x = \{x \star U : U \in \beta_e\}$ and $\{x \star U^{-1} : U \in \beta_e\}$ is an $\mathcal{I}$-open neighbourhood system at x.*

Definition 3.8. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-homogeneous if for all $x, y \in X$ there is an $\mathcal{I}$-homeomorphism f of the space X onto itself such that $f(x) = y$.

Corollary 3.9. *Every ideal topological group G is an $\mathcal{I}$-homogeneous space.*

Proof. Take any elements x and y in G and put $z = x^{-1} \star y$. Then l_z is an $\mathcal{I}$-homeomorphism of G and $l_z(x) = x \star z = x \star (x^{-1} \star y) = y$. ∎

Theorem 3.10. *Let $(G, \star, \tau, \mathcal{I})$ be an ideal topological group and H a subgroup of G. If H contains a nonempty $\mathcal{I}$-open set, then H is $\mathcal{I}$-open in G.*

Proof. Let U be a nonempty $\mathcal{I}$-open subset of G with $U \subset H$. For any $h \in H$ the set $l_h(U) = h \star U$ is $\mathcal{I}$-open in G and is a subset of H. Therefore, the subgroup $H = \bigcup_{h \in H} (h \star U)$ is $\mathcal{I}$-open in G as the union of $\mathcal{I}$-open sets. ∎

Theorem 3.11. *Every open subgroup H of an ideal topological group $(G, \star, \tau, \mathcal{I})$ is also an ideal topological group (called ideal topological subgroup of G).*

Proof. We have to show that for each $x, y \in H$ and each neighbourhood $W \subset H$ of $x \star y^{-1}$ there exist $\mathcal{I}$-open neighbourhoods $U \subset H$ of x and $V \subset H$ of y such that $U \star V^{-1} \subset W$. Since H is open in G, W is an open subset of G and since G is an ideal topological group there are $\mathcal{I}$-open neighbourhoods A of x and B of y such that $A \star B^{-1} \subset W$. The sets $U = A \cap H$ and $V = B \cap H$ are $\mathcal{I}$-open subsets of H because H is open. Also, $U \star V^{-1} \subset A \star B^{-1} \subset W$, which means that H is an ideal topological group. ∎

Theorem 3.12. *Let $(G, \star, \tau, \mathcal{I})$ be an ideal topological group. Then every open subgroup of G is $\mathcal{I}$-closed in G.*

Proof. Let H be an open subgroup of G. Then every left coset $x \star H$ of H is $\mathcal{I}$-open because l_x is an $\mathcal{I}$-open mapping. Thus, $Y = \bigcup_{h \in G \setminus H} x \star H$ is also $\mathcal{I}$-open as a union of $\mathcal{I}$-open sets. Then $H = G \setminus Y$ and so H is $\mathcal{I}$-closed. ∎

Definition 3.13. [2] A function $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ to be $\mathcal{I}$-irresolute if $f^{-1}(V)$ is $\mathcal{I}$-open in $(X, \tau, \mathcal{I})$ for every $\mathcal{J}$-open in $(Y, \sigma, \mathcal{J})$.

Theorem 3.14. *Let $f\colon G \to H$ be a homomorphism of ideal topological groups. If f is $\mathcal{I}$-irresolute at the neutral element e_G of G, then f is $\mathcal{I}$-irresolute (and thus $\mathcal{I}$-continuous) on G.*

Proof. Let $x \in G$. Suppose that W is an $\mathcal{I}$-open neighbourhood of $y = f(x)$ in H. Since the left translations in H are $\mathcal{I}$-continuous, there is an $\mathcal{I}$-open neighbourhood V of the neutral element e_H of H such that $L_y(V) = y \star V \subset W$. From $\mathcal{I}$-irresoluteness of f at e_G, it follows the existence of an $\mathcal{I}$-open set $U \subset G$ containing e_G such that $f(U) \subset V$. Since $l_x\colon G \to G$ is an $\mathcal{I}$-open mapping, the set $x \star U$ is an $\mathcal{I}$-open neighbourhood of x, and we have $f(x \star U) = f(x) \star f(U) = y \star f(U) \subset y \star V \subset W$: Hence f is $\mathcal{I}$-irresolute (and thus $\mathcal{I}$-continuous) at the point x of G, hence on G, because x is an arbitrary element in G. ∎

Definition 3.15. [8] An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-regular if for each closed set $F \subset X$ and each $x \in X \setminus F$, there are disjoint $H, W \in \mathcal{I}O(X)$ such that $F \subset H$ and $x \in W$.

Theorem 3.16. *Let $(G, \star, \tau, \mathcal{I})$ be an ideal topological group with base β_e at the identity element e such that for each $U \in \beta_e$ there is a symmetric $\mathcal{I}$-open neighbourhood V of e such that $V \star V \subset U$. Then G satisfies the axiom of $\mathcal{I}$-regularity at e.*

Proof. Let U be an open set containing the identity e. Then, by assumption, there is a symmetric $\mathcal{I}$-open neighbourhood V of e satisfying $V \star V \subset U$. We have to show that $\mathcal{I}\,\mathrm{Cl}(V) \subset U$. Let $x \in \mathcal{I}\,\mathrm{Cl}(V)$. The set $x \star V$ is an $\mathcal{I}$-open neighbourhood of x, which implies $x \star V \cap V \neq \emptyset$. Therefore, there are points $a, b \in V$ such that $b = x \star a$, that is, $x = b \star a^{-1} \in V \star V^{-1} = V \star V \subset U$. ∎

Theorem 3.17. *Let A and B be subsets of an ideal topological group G. Then:*

(1) $\mathcal{I}\,\mathrm{Cl}(A) \star \mathcal{I}\,\mathrm{Cl}(B) \subset \mathrm{Cl}(A \star B)$;
(2) $(\mathcal{I}\,\mathrm{Cl}(A))^{-1} \subset \mathrm{Cl}(A^{-1})$.

Proof. (1). Suppose that $x \in \mathcal{I}\,\mathrm{Cl}(A)$, $y \in \mathcal{I}\,\mathrm{Cl}(B)$. Let W be a neighbourhood of $x \star y$. Then there are $\mathcal{I}$-open neighbourhoods U and V of x and y such that

$U \star V \subset W$. Since $x \in \mathcal{I}\,\mathrm{Cl}(A)$, $y \in \mathcal{I}\,\mathrm{Cl}(B)$, there are $a \in A \cap U$ and $b \in B \cap V$. Then $a \star b \in (A \star B) \cap (U \star V) \subset (A \star B) \cap W$. This means $x \star y \in \mathrm{Cl}(A \star B)$, that is, we have $\mathcal{I}\,\mathrm{Cl}(A) \star \mathcal{I}\,\mathrm{Cl}(B) \subset \mathrm{Cl}(A \star B)$.
(2). Let $x \in (\mathcal{I}\,\mathrm{Cl}(A))^{-1}$ and U a neighbourhood of x. Since the inverse mapping is $\mathcal{I}$-open, the set U^{-1} is $\mathcal{I}$-open neighbourhood of x^{-1}. Since $x^{-1} \in \mathcal{I}\,\mathrm{Cl}(A)$, $U^{-1} \cap A \neq \emptyset$. Therefore, $U \cap A^{-1} \neq \emptyset$, that is, $x \in \mathrm{Cl}(A^{-1})$, and so $(\mathcal{I}\,\mathrm{Cl}(A))^{-1} \subset$ $\subset \mathrm{Cl}(A^{-1})$. ∎

Theorem 3.18. *If V is an $\mathcal{I}$-open neighbourhood of e in ideal topological group $(G, \tau, \star, \mathcal{I})$, then $V \subset \mathcal{I}\,\mathrm{Cl}(V) \subset V^2$.*

Proof. Since $s \star V^{-1}$ is an $\mathcal{I}$-open neighbourhood of s, it must intersects V. Thus there is $t \in V$ of the form $s \star v^{-1}$, where $v \in V$. But $s = t \star v \in V \star V = V^2$ and $\mathcal{I}\,\mathrm{Cl}(V) \subset V^2$. ∎

Theorem 3.19. *If $(G, \tau, \star, \mathcal{I})$ is an ideal topological group, then $\mathcal{I}\,\mathrm{Cl}(A) \subset A \star U$ holds for every subset A of G and every open neighbourhood U of e.*

Proof. Since $(G, \tau, \star, \mathcal{I})$ is an ideal topological group, for every open neighbourhood U of e, there exists $V \in \mathcal{I}O(G, e)$ such that $V^{-1} \subset U$. Let $x \in \mathcal{I}\,\mathrm{Cl}(A)$ and $x \star V$ is an $\mathcal{I}$-open neighbourhood of x. Then there exists $a \in A \cap x \star V$, that is, $a \in x \star V$. This implies that $a = a \star b^{-1} \in a \star V^{-1} \subset A \star U$. Hence $\mathcal{I}\,\mathrm{Cl}(A) \subset A \star U$. ∎

Theorem 3.20. *If $(G, \tau, \star, \mathcal{I})$ is an ideal topological group and β_e a base of the space $(G, \tau, \mathcal{I})$ at the neutral element e, then for every subset A of G, we have $\mathcal{I}\,\mathrm{Cl}(A) = \{A \star U : U \in \beta_e\}$.*

Proof. We only have to verify that if $x \notin \mathcal{I}\,\mathrm{Cl}(A)$, then there exists $U \in \beta_e$ such that $x \notin A \star U$. Since $x \notin A$, then by definition there exists an $\mathcal{I}$-open neighbourhood W of e such that $x \star W \cap A = \emptyset$. Take U in β_e satisfying the condition $U^{-1} \subset W$. Then $x \star U^{-1} \cap A = \emptyset$, that is $\{x\} \cap A \star U = \emptyset$. This implies that $x \notin A \star U$. ∎

Definition 3.21. [2] An ideal topological space $(X, \tau, \mathcal{I})$ is called $\mathcal{I}$-T_2 if for every two different points x, y of X, there exist disjoint $\mathcal{I}$-open sets U, V of X such that $x \in U$ and $y \in V$.

Theorem 3.22. *If $(G, \tau, \star, \mathcal{I})$ is an ideal topological group, then $(G, \tau, \mathcal{I})$ is $\mathcal{I}$-regular and $\mathcal{I}$-T_2 space.*

Proof. Suppose that $F \subset G$ is closed and $s \notin F$. Multiplication by s^{-1} allows us to assume that $s = e$. Since F is closed, $W = G \setminus F$ is an open neighbourhood of e. Then there exists $V \in \mathcal{I}O(G, e)$ such that $V^2 \subset W$. Hence $\mathcal{I}\,\mathrm{Cl}(V) \subset W$. Then $U = G \setminus \mathcal{I}\,\mathrm{Cl}(V)$ is an $\mathcal{I}$ neighbourhood containing F which is disjoint from V. This proves that $(G, \tau, \mathcal{I})$ is $\mathcal{I}$-regular. That is, $e \in V \in \mathcal{I}O(G)$ and $e \neq y \in F \subset U \in \mathcal{I}O(G)$ such that $V \cap U = \emptyset$. Hence G is $\mathcal{I}$-T_2. ∎

Definition 3.23. [9] An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-compact if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of X by open sets of X, there exists a finite subset Δ_0 of Δ such that $X \setminus \cup \{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{I}$.

Theorem 3.24. *Let $(G, \tau, \star, \mathcal{I})$ be an ideal topological group. If K is an $\mathcal{I}$-compact subset of G, and F an $\mathcal{I}$-closed subset of G. Then $F \star K$ and $K \star F$ are $\mathcal{I}$-closed subsets of G.*

Proof. If $F \star K = G$, we are done, so let $y \in G \setminus F \star K$. This means $F \cap y \star K^{-1} = \emptyset$. Since K is $\mathcal{I}$-compact, $y \star K^{-1}$ is $\mathcal{I}$-compact. Then there is an $\mathcal{I}$-open neighbourhood V of e such that $F \cap V \star y \star K^{-1} = \emptyset$. That is, $F \star K \cap V \star y = \emptyset$. Since $V \star y$ is $\mathcal{I}$-open neighbourhood of y contained in $G \setminus F \star K$, we have $F \star K$ is $\mathcal{I}$-closed and similar arguments for the proof of $K \star F$. ∎

Theorem 3.25. *A nonempty subgroup H of an ideal topological group G is $\mathcal{I}$-open if and only if its $\mathcal{I}$-interior is nonempty.*

Proof. Assume that $x \in \mathcal{I}\,\mathrm{Int}(H)$. Then by definition, there is an $\mathcal{I}$-open set V such that $x \in V \subset H$. For every $y \in H$, we have $y \star V \subset y \star H = H$. Since V is $\mathcal{I}$-open, so is $y \star V$, we conclude that $H = \cup\{y \star V : y \in H\}$ is an $\mathcal{I}$-open set. The converse is straightforward. ∎

Theorem 3.26. *If $U \in \mathcal{I}O(G)$, then the set $L = \bigcup_{n=1}^{\infty} U^n$ is an $\mathcal{I}$-open set in an ideal topological group $(G, \tau, \star, \mathcal{I})$.*

Proof. Since U is $\mathcal{I}$-open in an ideal topological group $(G, \tau, \star, \mathcal{I})$, then by Lemma 3.2, $U \star U = U^2 \in \mathcal{I}O(G)$, $U^2 \star U = U^3 \in \mathcal{I}O(G)$ and similarly U^4, $U^5, \ldots$ all are $\mathcal{I}$-open sets in G. Thus the set $L = \bigcup_{n=1}^{\infty} U^n$ being the union of $\mathcal{I}$-open sets is an $\mathcal{I}$-open set. ∎

Lemma 3.27. *If $(G, \tau, \star, \mathcal{I})$ is an ideal topological group, then the inverse map $i\colon G \to G$ defined by $i(x) = x^{-1}$ for all $x \in G$ is an $\mathcal{I}$-homeomorphism.*

Theorem 3.28. *If A is a subset of an ideal topological group $(G, \tau, \star, \mathcal{I})$, then $(\mathcal{I}\,\mathrm{Int}(A))^{-1} = \mathcal{I}\,\mathrm{Int}(A^{-1})$.*

Proof. Since the inverse mapping $i\colon G \to G$ is an $\mathcal{I}$-homeomorphism, $\mathcal{I}\,\mathrm{Int}(i(A)) = \mathcal{I}\,\mathrm{Int}(A^{-1}) = i(\mathcal{I}\,\mathrm{Int}(A)) = (\mathcal{I}\,\mathrm{Int}(A))^{-1}$. ∎

Definition 3.29. Suppose U is an $\mathcal{I}$-open neighbourhood of the neutral element e of an ideal topological group $(G, \tau, \star, \mathcal{I})$. A subset A of G is called U-$\mathcal{I}$-disjoint if $b \notin a \star U$ for any disjoint $a, b \in A$.

Definition 3.30. A collection Υ of subsets of a topological space $(G, \tau, \mathcal{I})$ is $\mathcal{I}$-discrete, provided each $x \in G$ has an $\mathcal{I}$-open neighbourhood that intersects at most one member of Υ.

Theorem 3.31. *Let U and V be $\mathcal{I}$-open neighbourhoods of the neutral element e in an ideal topological group $(G, \tau, \star, \mathcal{I})$ such that $V^4 \subset U$ and $V^{-1} = V$. If a subset A of G is $\mathcal{I}$-disjoint, then the family of $\mathcal{I}$-open sets $\{a \star V : a \in A\}$ is $\mathcal{I}$-discrete in G.*

Proof. It suffices to verify that, for every $x \in G$, an $\mathcal{I}$-open neighbourhood $x \star V$ of x intersects at most one element of the family $\{a \star V : a \in A\}$. Suppose to the contrary that, for some $x \in G$, there exists distinct elements $a, b \in A$ such that $x \star V \cap a \star V \neq \emptyset$ and $x \star V \cap b \star V \neq \emptyset$. Then $x^{-1} \star a \in V^2$ and $b^{-1} \star x \in V^2$, where $b^{-1} \star a = (b^{-1} \star x)(x^{-1} \star a) \in V^4 \subset U$. This implies that $a \in b \star U$. This contradicts the assumption that A is $\mathcal{I}$-disjoint. ∎

4. On $\mathcal{I}$-connectedness in ideal topological groups

In this section, we continue the study of ideal topological groups, then we will present some results on $\mathcal{I}$-connectedness in the presence of ideal topological groups.

Theorem 4.1. *Let $(G, \tau, \star, \mathcal{I})$ be an ideal topological group. Then every $\mathcal{I}$-open subgroup of G is $\mathcal{I}$-closed in G.*

Proof. Let H be an $\mathcal{I}$-open subgroup of G. Then every left coset $x \star H$ of H is $\mathcal{I}$-open. Thus $Y = \bigcup_{x \in G \setminus H} x \star H$ is also $\mathcal{I}$-open as a union of $\mathcal{I}$-open sets. Hence $H = G \setminus Y$ is $\mathcal{I}$-closed. ∎

Theorem 4.2. *Let U be any symmetric $\mathcal{I}$-open neighbourhood of e in an ideal topological group $(G, \tau, \star, \mathcal{I})$. Then the set $L = \bigcup_{n=1}^{\infty} U^n$ is an $\mathcal{I}$-open and an $\mathcal{I}$-closed subgroup of G.*

Proof. First we prove that $L = \bigcup_{n=1}^{\infty} U^n$ is a subgroup of G. Let $x, y \in L$. If $x = u^k$, $y = u^l$, $x \star y = u^k \star u^l = u^{k+l} \in L$, $x^{-1} = (u^k)^{-1} = (u^{-1})^k = u^k \in L$. This implies that L is a subgroup of G and $L = \bigcup_{n=1}^{\infty} U^n$ is an $\mathcal{I}$-open in G. Hence $L = \bigcup_{n=1}^{\infty} U^n$ is $\mathcal{I}$-closed in G. ∎

Definition 4.3. Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then a point $x \in A$ is said to be an $\mathcal{I}$-isolated point of A if there exists an $\mathcal{I}$-open set containing x which does not contain any point of A different from x.

Theorem 4.4. *A subgroup H of an ideal toplogical group G is $\mathcal{I}$-discrete if and only if it has an $\mathcal{I}$-isolated point.*

Proof. Suppose that $x \in H$ and x is $\mathcal{I}$-isolated in the relative topology of $H \subset G$. That is, there is an $\mathcal{I}$-open neighbourhood U of e in G such that $(x \cdot U) \cap H = \{x\}$. Then for arbitrary $y \in H$, we have $(y \cdot U) \cap H = (y \cdot U) \cap \{y \cdot \cdot x^{-1} \cdot H\} = y \cdot x^{-1} \cdot ((x \cdot U) \cap H)) = \{y\}$. Thus every point of H is $\mathcal{I}$-isolated, so that H is indeed $\mathcal{I}$-discrete. If H is $\mathcal{I}$-discrete, then by definition, all of its points are $\mathcal{I}$-isolated. ∎

Theorem 4.5. *For any neighbourhood U of identity e in an ideal topological group, there exists a symmetric $\mathcal{I}$-open neighbourhood V of e such that $V \subset U$.*

Proof. Since U is a neighbourhood of e and the inverse function is $\mathcal{I}$-continuous, there exists an open neighbourhood W of e such that $W \subset U$ and W^{-1} is $\mathcal{I}$-open neighbourhood of e. Let $V = W \cap W^{-1} \neq \emptyset$. Since V is the intersection of open and $\mathcal{I}$-open sets, V is $\mathcal{I}$-open and clearly $V = V^{-1}$. ∎

Theorem 4.6. *Let $(G, \tau, \star, \mathcal{I})$ be an ideal topological group, C the $\mathcal{I}$-component of e, and U any neighbourhood of e. Then $C \subset \bigcup_{n=1}^{\infty} U^n$, in particular, if G is $\mathcal{I}$-connected, then $G = \bigcup_{n=1}^{\infty} U^n$.*

Proof. Let V be the symmetric $\mathcal{I}$-open neighbourhood of e such that $V \subset U$. Therefore $L = \bigcup_{n=1}^{\infty} U^n$ is $\mathcal{I}$-open as well as $\mathcal{I}$-closed subgroup of G. Since C is $\mathcal{I}$-connected component of e, we have $C \subset \bigcup_{n=1}^{\infty} V^n \subset \bigcup_{n=1}^{\infty} U^n$. Now if G is $\mathcal{I}$-connected, then $G = \bigcup_{n=1}^{\infty} U^n$. ∎

Theorem 4.7. *Let* $(G, \tau, \star, \mathcal{I})$ *be an* $\mathcal{I}$*-connected ideal topological group and* H *a subgrop which contains any* $\mathcal{I}$*-neighbourhood. Then* $H = G$*. In particular, an* $\mathcal{I}$*-open subgroup of* G *equals* G*.*

Proof. Since H contains any $\mathcal{I}$-neighbourhood, the $\mathcal{I}$-interior of H is non-empty. By H is $\mathcal{I}$-open and $\mathcal{I}$-closed. Since G is $\mathcal{I}$-connected, $G = H$. ∎

Definition 4.8. A ideal topological group with respect to $\mathcal{I}$-continuity is a group G endowed with a topology such that for each $a \in G$, the translations $l_a, r_a : G \to G$, $l_a(x) = a \cdot x$, $r_a(x) = x \cdot a$ are $\mathcal{I}$-continuous, and such that the inverse mapping $i : G \to G$, $i(x) = x^{-1}$ is $\mathcal{I}$-continuous.

Theorem 4.9. *Let* $(G, \tau, \star, \mathcal{I})$ *be an ideal topological group with respect to* $\mathcal{I}$*-continuity and the* $\mathcal{I}$*-component* $\mathcal{I}C(e)$ *of identity* e *be open. Then*

1. *for all* $x \in \mathcal{I}C(e)$, $l_{x^{-1}}$ $(r_{x^{-1}})$ *is also open, then* $\mathcal{I}C(e)$ *is subgroup.*
2. *if all translations are also open, then* $\mathcal{I}C(e)$ *is a normal subgroup.*

Theorem 4.10. *Let* G *be a Hausdorff ideal topological group with respect to* $\mathcal{I}$*-continuity such that left translations are continuous (*$\mathcal{I}$*-continuous), right translations are* $\mathcal{I}$*-continuous (continuous) and inverse mapping is* $\mathcal{I}$*-continuous. For any subset* M *of* G*, the subgroup* $C_G(M) = \{g \in G : mg = gm\}$ *is* $\mathcal{I}$*-closed in* G*. In particular, the centre of* G *is* $\mathcal{I}$*-closed.*

Corollary 4.11. *Let* G *be a Hausdorff ideal topological group such that left translations are continuous (*$\mathcal{I}$*-continuous), right translations are* $\mathcal{I}$*-continuous (continuous) and inverse mapping is* $\mathcal{I}$*-continuous. For any subset* M *of* G*, the subgroup* $C_G(M) = \{g \in G : mg = gm\}$ *is* $\mathcal{I}$*-closed in* G*. In particular, the centre of* G *is* $\mathcal{I}$*-closed.*

Theorem 4.12. *Let* G *be an* $\mathcal{I}$*-connected ideal topological group and* e *its identity element. If* U *is any* $\mathcal{I}$*-open neighborhood of* e *then* G *is generated by* U*.*

Proof. Let U be an $\mathcal{I}$-open neighborhood of e. For each $n \in \mathbb{N}$, we denote by U^n the set of elements of the form $u_1 \ldots u_n$, where each $u_i \in U$. Let $W = \bigcup_{n=1}^{\infty} U^n$. Since each U^n is $\mathcal{I}$-open, we have that W is an $\mathcal{I}$-open set. We now see that it is also $\mathcal{I}$-closed. Let G be an element of $\mathcal{I}$-closure W. That is, $g \in \mathcal{I}\,\mathrm{Cl}(W)$. Since gU^{-1} is an $\mathcal{I}$-open neighborhood of g, it must intersect W. Thus, let $h \in W \cap gU^{-1}$. Since $h \in gU^{-1}$, then $h = gu^{-1}$ for some elements $u \in U$. Since $h \in W$, then $h \in U^n$ for some $n \in \mathbb{N}$, that is, $h = u_1 \ldots u_n$ with each $u_i \in U$. We then have $g = u_1 \ldots u_n.u$, that is, $g \in U^{n+1} \subset W$. Hence W is $\mathcal{I}$-closed. Since

G is $\mathcal{I}$-connected and W is $\mathcal{I}$-open and $\mathcal{I}$-closed, we must have $W = G$. This means that G is generated by U. ∎

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