

A PROOF OF SENDOV'S CONJECTURE

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ABSTRACT. In this paper we give a proof of Sendov's conjecture. We start by establishing the uniformly diminishing state of the mass of an expansion.

1. INTRODUCTION

The sendov conjecture is the assertion that any complex coefficient polynomial $P_n(z)$ of degree $n \geq 2$ with sufficiently small zeros must sit in the same unit disc with at least one of its critical point. That is to say, for each $|b_i| < 1$ such that $P_n(b_i) = 0$, then there exist some c_j such that

$$|b_i - c_j| < 1$$

where $P'_n(c_j) = 0$. There has been various successful attacks on variants of the conjecture most of which proceeded by the methods of complex variable and classical analysis, which is not surprising given the origin of the problem. Though it seems the state-of-art approach to the problem might not guarantee a solution, the results are noteworthy. In [1] the conjecture has been proved for polynomials of degree at most **six**. This was improved to polynomials of degree at most **seven** in [2] and polynomials of degree at most **eight** in [4]. An asymptotic version of the conjecture was also recently shown to hold [3]. In this paper, however, we adopt and follow an unconventional approach in resolving this conjecture. We start by developing some basic tools that allows us to obtain a general and a much stronger version of Sendov's conjecture. Consequently, we managed to prove the result

Theorem 1.1. *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$. Let $\mathcal{T} = \{b_1, b_2, \dots, b_n\}$ and $\mathcal{C} = \{c_1, c_2, \dots, c_{n-1}\}$ be the set of zeros and critical values of f , respectively. If $|b_i| < \delta \leq 1$, for $i = 1, 2, \dots, n$, then for each $b_i \in \mathcal{T}$, there exist some $c_j \in \mathcal{C}$ such that*

$$|b_i - c_j| < 1.$$

We also generalized this result which allows us to say something about the distribution of the zeros of any polynomial and the zeros of its higher order derivatives, as follows:

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Theorem 1.2. *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$. Let $\mathcal{T} = \{b_1, b_2, \dots, b_n\}$ be the set of zeros of f . If $|b_i| < \delta \leq 1$ for $i = 1, 2, \dots, n$, then for each $b_i \in \mathcal{T}$, there exist some c_j with $f^n(c_j) = 0$ such that*

$$|b_i - c_j| < 1$$

for all $1 \leq n \leq \deg(f) - 1$.

2. NOTATIONS

Through out this paper a tuple will always be denoted by \mathcal{S} or \mathcal{S}_j where $j \in \mathbb{N}$. Occasionally, we will use the tuple $\mathcal{S}_{\mathbb{C}}$ to denote a tuple of the base field and $\mathcal{S}_{\mathbb{C}[x]}$ a tuple of the polynomial ring $\mathbb{C}[x]$. We set $\mathcal{S}_0 = (0, 0, \dots, 0)$ and call it the null tuple and $\mathcal{S}_e = (1, 1, \dots, 1)$ the unit tuple. We denote the rank of an expansion on \mathcal{S} by $\mathcal{R}(\mathcal{S})$, the degree of an expansion on \mathcal{S} by $\deg(\mathcal{S})$ and the measure of an expansion on \mathcal{S} by $\mathcal{N}(\mathcal{S})$. Also we set $\mathcal{S}_a = (f_1(a), f_2(a), \dots, f_n(a))$, where $\mathcal{S} = (f_1, f_2, \dots, f_n)$.

3. PRELIMINARY DEFINITIONS AND TERMINOLOGIES

In this section we introduce the following language.

Definition 3.1. Let $\mathcal{S} = (f_1, f_2, \dots, f_n)$ such that each $f_i \in \mathbb{C}[x]$. By the derivative of \mathcal{S} denoted $\nabla(\mathcal{S})$, we mean

$$\nabla(\mathcal{S}) = \left(\frac{df_1}{dx}, \frac{df_2}{dx}, \dots, \frac{df_n}{dx} \right).$$

We denote the derivative of this tuple at a point $a \in \mathbb{R}$ to be

$$\nabla_a(\mathcal{S}) = \left(\frac{df_1(a)}{dx}, \frac{df_2(a)}{dx}, \dots, \frac{df_n(a)}{dx} \right).$$

Definition 3.2. Let $\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of $\mathbb{C}[x]$. Then by an expansion on $\{\mathcal{S}_i\}_{i=1}^{\infty}$, we mean the composite map

$$\gamma^{-1} \circ \beta \circ \gamma \circ \nabla : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty},$$

where

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Proposition 3.1. *An expansion $\gamma^{-1} \circ \beta \circ \gamma \circ \nabla : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty}$ is linear.*

4. THE RANK AND MEASURES OF AN EXPANSION

In this section we introduce the notion of the rank and measure of an expansion. We launch the following languages as follows:

Definition 4.1. Let $\mathcal{F} = \{\mathcal{S}_m\}_{m=1}^{\infty}$ be collection of tuples of $\mathbb{C}[x]$. Then the value of n such that the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) \neq \mathcal{S}_0$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}) = \mathcal{S}_0$ is called the degree of expansion and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S})$ is called the rank of an expansion, denoted $\mathcal{R}(\mathcal{S})$.

Theorem 4.2. Let $\mathcal{S}_i, \mathcal{S}_j \in \{\mathcal{S}_k\}_{k=1}^\infty$, a family of tuples of $\mathbb{C}[x]$. Let $\deg(\mathcal{S}_i) = \deg(\mathcal{S}_j)$. Then $\mathcal{R}(\mathcal{S}_i) = \mathcal{R}(\mathcal{S}_j)$ if and only if $\mathcal{S}_i - \mathcal{S}_j = (a_1, a_2, \dots, a_n)$ for each $a_i \in \mathbb{C}$.

Proof. Pick $\mathcal{S}_i, \mathcal{S}_j \in \{\mathcal{S}_k\}_{k=1}^\infty$ such that $\deg(\mathcal{S}_i) = \deg(\mathcal{S}_j)$. Suppose $\mathcal{S}_i - \mathcal{S}_j = (a_1, a_2, \dots, a_n)$, then by applying the n th expansion on both sides, we find that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_i - \mathcal{S}_j) = \mathcal{S}_0$. Since an expansion is linear, it follows that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_i) - (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j) = \mathcal{S}_0$. That is $\mathcal{R}(\mathcal{S}_i) = \mathcal{R}(\mathcal{S}_j) + \mathcal{S}_0 = \mathcal{R}(\mathcal{S}_j)$. Conversely, suppose $\mathcal{R}(\mathcal{S}_i) = \mathcal{R}(\mathcal{S}_j)$, then it follows that $\mathcal{R}(\mathcal{S}_i - \mathcal{S}_j) = \mathcal{S}_0$. To avoid a contradiction, we must allow the entries of \mathcal{S}_i and \mathcal{S}_j to differ by elements of \mathbb{C} . This completes the proof. \square

Definition 4.3. Let \mathcal{S} be a tuple of $\mathbb{C}[x]$, then by the measure of an expansion on \mathcal{S} , denoted $\mathcal{N}(\mathcal{S})$, we mean $\mathcal{N}(\mathcal{S}) = \|\mathcal{R}(\mathcal{S})\|$.

5. THE BOUNDARY POINTS AND MASS OF AN EXPANSION

In this section, we introduce the notion of the mass and boundary of an expansion on tuples of $\mathbb{C}[x]$.

Definition 5.1. Let $\{\mathcal{S}_j\}_{j=1}^\infty$ be a collection of tuples of $\mathbb{C}[x]$. By the boundary points of the n th expansion, denoted $\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j)]$, we mean the set

$$\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j)] := \{(a_1, a_2, \dots, a_n) : \text{Id}_i[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{a_i}^n(\mathcal{S}_j)] = 0\}.$$

Remark 5.2. To avoid writing the boundary of expansion in the form $\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j)]$, we choose to rather write $\mathcal{B}^n(\mathcal{S}_j)$. Also for the n th expansion on \mathcal{S}_j , we will choose to write \mathcal{S}_j^n . We will switch between these two notations occasionally without commenting too much about it.

Definition 5.3. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^\infty$ be a collection of tuples of $\mathbb{C}[x]$. Then by the mass of an expansion \mathcal{S}_j^n , denoted $\mathcal{H}(\mathcal{S}_j^n)$, we mean the finite sum

$$\mathcal{H}(\mathcal{S}_j^n) = \sum_{\mathcal{S}_k \in \mathcal{B}^n(\mathcal{S}_j)} \|\mathcal{S}_k\|,$$

where

$$\|\mathcal{S}_k\| = \sqrt{\sum_{i=1}^n |a_i|^2}.$$

for $\mathcal{S}_k = (a_1, a_2, \dots, a_n)$

6. THE SPEED, MOMENTUM, INDEX AND EMBEDDING OF AN EXPANSIONS

In this section we launch the notion of the speed and momentum of an expansion.

Definition 6.1. Let $\{\mathcal{S}_j\}_{j=1}^\infty$ be a collection of tuples of $\mathbb{C}[x]$. Then by the speed of an expansion on \mathcal{S} , denoted $\nu(\mathcal{S})$, we mean the expression

$$\nu(\mathcal{S}) = \frac{\mathcal{N}(\mathcal{S})}{\deg(\mathcal{S})}.$$

Definition 6.2. Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{C}[x]$. By the momentum of the n th expansion, denoted $\mathcal{M}(\mathcal{S}_j^n)$, we mean

$$\mathcal{M}(\mathcal{S}_j^n) := \nu(\mathcal{S}_j^n)\mathcal{H}(\mathcal{S}_j^n).$$

Definition 6.3. Let $\mathcal{P} = \{\mathcal{S}_j\}_{j=1}^n$ be a finite collection of tuples of $\mathbb{C}[x]$. Then by the index of the m th expansion of \mathcal{S}_k for $1 \leq k \leq n$, denoted $\mathcal{I}(\mathcal{S}_k^m)$, we mean

$$\mathcal{I}(\mathcal{S}_k^m) = \frac{\sum_{j=1}^m \mathcal{M}(\mathcal{S}_j^m)}{\mathcal{M}(\mathcal{S}_k^m)}.$$

Definition 6.4. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{C}[x]$. Let $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$, then we say the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_b)$ is an embedding of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_a)$ if

$$(6.1) \quad \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_b)] \subset \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_a)]$$

for some $n_1 > n_2$. Conversely, we say $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_a)$ is an extension of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_b)$.

Proposition 6.1. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of n tuples of $\mathbb{C}[x]$, and suppose $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$. If $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_a)$ is an embedding of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_b)$, then

$$\mathcal{H}(\mathcal{S}_a^{n_2}) < \mathcal{H}(\mathcal{S}_b^{n_1}).$$

Proof. Let $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$ and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_a)$ is an embedding of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_b)$, then it follows from definition 6.4

$$\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_b)] \subset \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_a)]$$

for some $n_1, n_2 \in \mathbb{N}$. The result follows from this fact by leveraging definition 5.3. \square

Remark 6.5. Next we establish an important inequality that relates the index of an expansion of a tuple to the largest size of the number of embedding of expansion, in the following result.

Lemma 6.6. Let $\mathcal{P} := \{\mathcal{S}_j\}_{j=1}^n$ and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_k}(\mathcal{S}_k)$ ($1 \leq k \leq n$) admits an embedding $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_j}(\mathcal{S}_j)$ for all $1 \leq j \leq n$. If $\nu(\mathcal{S}_k^{n_k}) \geq \nu(\mathcal{S}_j^{n_j})$ for all $1 \leq j \leq n$, then

$$\mathcal{I}(\mathcal{S}_k^{n_k}) < n.$$

Proof. Let $\mathcal{P} := \{\mathcal{S}_j\}_{j=1}^n$ and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_k}(\mathcal{S}_k)$ ($1 \leq k \leq n$) admits an embedding $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_j}(\mathcal{S}_j)$ for all $1 \leq j \leq n$. Then it follows that

$$\begin{aligned} \sum_{j=1}^n \mathcal{M}(\mathcal{S}_j^{n_j}) &= \mathcal{M}(\mathcal{S}_1^{n_1}) + \cdots + \mathcal{M}(\mathcal{S}_k^{n_k}) + \cdots + \mathcal{M}(\mathcal{S}_n^{n_n}) \\ &= \nu(\mathcal{S}_1^{n_1})\mathcal{H}(\mathcal{S}_1^{n_1}) + \cdots + \nu(\mathcal{S}_k^{n_k})\mathcal{H}(\mathcal{S}_k^{n_k}) + \cdots + \nu(\mathcal{S}_n^{n_n})\mathcal{H}(\mathcal{S}_n^{n_n}) \\ &\leq n\nu(\mathcal{S}_k^{n_k})\mathcal{H}(\mathcal{S}_k^{n_k}) \\ &= n\mathcal{M}(\mathcal{S}_k^{n_k}), \end{aligned}$$

and the inequality is established. \square

Theorem 6.7. *Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^\infty$ be a collection of tuples of $\mathbb{C}[x]$. Then for each $\mathcal{S} \in \mathcal{F}$*

$$\sum_{k=0}^{\deg(\mathcal{S})-1} \nu(\mathcal{S}^k) = \nu(\mathcal{S})\deg(\mathcal{S}) \log(\deg(\mathcal{S})) + \deg(\mathcal{S})\nu(\mathcal{S})\alpha + O(\nu(\mathcal{S})),$$

where $\alpha = 0.5772 \dots$, the euler-macheroni constant.

Proof. Clearly

$$\begin{aligned} \sum_{k=0}^{\deg(\mathcal{S})-1} \nu(\mathcal{S}^k) &= \nu(\mathcal{S}) + \nu(\mathcal{S}^1) + \cdots + \nu(\mathcal{S}^{\deg(\mathcal{S})-1}) \\ &= \frac{\mathcal{N}(\mathcal{S})}{\deg(\mathcal{S})} + \frac{\mathcal{N}(\mathcal{S}^1)}{\deg(\mathcal{S}^1)} + \cdots + \frac{\mathcal{N}(\mathcal{S}^{\deg(\mathcal{S})-1})}{\deg(\mathcal{S}^{\deg(\mathcal{S})-1})} \\ &= \mathcal{N}(\mathcal{S}) \left(\frac{1}{\deg(\mathcal{S})} + \frac{1}{\deg(\mathcal{S}^1)} + \cdots + \frac{1}{\deg(\mathcal{S}^{\deg(\mathcal{S})-1})} \right) \\ &= \mathcal{N}(\mathcal{S}) \left(\frac{1}{\deg(\mathcal{S})} + \frac{1}{\deg(\mathcal{S})-1} + \cdots + \frac{1}{2} + 1 \right) \\ &= \nu(\mathcal{S})\deg(\mathcal{S}) \sum_{m=1}^{\deg(\mathcal{S})} \frac{1}{m} \end{aligned}$$

thereby establishing the formula. \square

Remark 6.8. This formula, as it turns out, becomes extremely useful in establishing the diminishing state of the mass of an expansion. For the time being, we use this formula to prove that the mass of an expansion diminishes at some phase.

Theorem 6.9. *Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^\infty$ be a collection of tuples of $\mathbb{C}[x]$. Suppose $\mathcal{S} \in \mathcal{F}$, then*

$$\mathcal{H}(\mathcal{S}^n) > \mathcal{H}(\mathcal{S}^{n+1})$$

for some $0 \leq n \leq \deg(\mathcal{S}) - 2$.

Proof. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^\infty$ be a collection of tuples of $\mathbb{C}[x]$ and specify $\mathcal{S} \in \mathcal{F}$. Consider the finite collection $\mathcal{P} = \{\mathcal{S}^k\}_{k=0}^{\deg(\mathcal{S})-1}$. Suppose on the contrary that

$$\mathcal{H}(\mathcal{S}^n) \leq \mathcal{H}(\mathcal{S}^{n+1})$$

for all $0 \leq n \leq \deg(\mathcal{S}) - 2$. Then it follows by an application of Theorem 6.7 that

$$\begin{aligned}
\sum_{k=0}^{\deg(\mathcal{S})-1} \mathcal{M}(\mathcal{S}^k) &= \sum_{k=0}^{\deg(\mathcal{S})-1} \mathcal{H}(\mathcal{S}^k) \nu(\mathcal{S}^k) \\
&\leq \mathcal{H}(\mathcal{S}^{\deg(\mathcal{S})-1}) \sum_{k=0}^{\deg(\mathcal{S})-1} \nu(\mathcal{S}^k) \\
&\ll \mathcal{H}(\mathcal{S}^{\deg(\mathcal{S})-1}) \nu(\mathcal{S}) \deg(\mathcal{S}) \log(\deg(\mathcal{S})) \\
&\leq \mathcal{H}(\mathcal{S}^{\deg(\mathcal{S})-1}) \nu(\mathcal{S}^{\deg(\mathcal{S})-1}) \deg(\mathcal{S}) \log(\deg(\mathcal{S})) \\
&= \mathcal{M}(\mathcal{S}^{\deg(\mathcal{S})-1}) \deg(\mathcal{S}) \log(\deg(\mathcal{S})),
\end{aligned}$$

and it follows that the index of expansion $\mathcal{I}\left((\mathcal{S}^{\deg(\mathcal{S})-2})^1\right) \ll \deg(\mathcal{S}) \log(\deg(\mathcal{S}))$, thereby contradicting the upper bound in Lemma 6.6. \square

Next we conjecture a stronger version of Sendov's conjecture as follows:

Conjecture 6.1. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$ and let $\mathcal{S} = (f(x), f(x), \dots, f(x))$, where $f(x)$ has no repeated zeros. Suppose $\mathcal{H}(\mathcal{S}) < 1$, then for each $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S})]$

$$\|\mathcal{S}_0 - \mathcal{S}_j\| < 1$$

for all $\mathcal{S}_j \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S})]$ for all $1 \leq m \leq \deg(\mathcal{S}) - 1$.

7. REGULAR AND SUB-EXPANSIONS

In this section we introduce the notion of regularity of an expansion and sub-expansion of an expansion.

Definition 7.1. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{C}[x]$. Then for any $\mathcal{S}_k \in \mathcal{F}$, we say the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_k)$ is regular if $\mathcal{H}(\mathcal{S}_k^n) > \mathcal{H}(\mathcal{S}_k^{n+1})$ for some $0 \leq n \leq \deg(\mathcal{S}_k) - 2$.

Theorem 7.2. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be collection of tuples of $\mathbb{C}[x]$ and suppose the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_k)$ with $(n \leq \deg(\mathcal{S}_k) - 3)$ is regular for $\mathcal{S}_k \in \mathcal{F}$. If

$$\mathcal{H}(\mathcal{S}_k^n) < \delta$$

for $0 < \delta < 1$ sufficiently small, then for each $\mathcal{S}_l \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_k)]$, there exist some $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}_k)]$ such that

$$\|\mathcal{S}_l - \mathcal{S}_0\| < 1.$$

Proof. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be collection of tuples of $\mathbb{C}[x]$. Pick arbitrarily $\mathcal{S}_k \in \mathcal{F}$ and suppose the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_k)$ with $(n \leq \deg(\mathcal{S}_k) - 3)$ is regular. Suppose on the contrary that for each $\mathcal{S}_l \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_k)]$, then

$$\|\mathcal{S}_l - \mathcal{S}_0\| \geq 1$$

for all $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}_k)]$. Since $\mathcal{H}(\mathcal{S}_k^n) < \delta$ with $0 < \delta < 1$ sufficiently small and $n \leq \deg(\mathcal{S}_k) - 3$, it follows that $\mathcal{H}(\mathcal{S}_k^{n+1}) \geq 1$. Under the regularity condition, it must be that

$$1 > \delta > \mathcal{H}(\mathcal{S}_k^n) > \mathcal{H}(\mathcal{S}_k^{n+1}) \geq 1$$

which is absurd. This completes the proof of the theorem. \square

Definition 7.3. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$ be any two expansions with $m < n$, then we say $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a)$ is a sub-expansion of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$, denoted

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$$

if there exist some $j \geq 1$ such that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{m+j}(\mathcal{S}_b)$. We say the expansion is proper if $m + j = n$. We denote this proper expansion by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b).$$

Remark 7.4. Next we prove a result that indicates that the regularity condition on an expansion can be localized as well as extended through expansions.

Theorem 7.5. *Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$, a proper sub-expansion. Then $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a)$ is regular if and only if $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$ is regular.*

Proof. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$, a proper sub-expansion and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a)$ is regular. Then it follows that $\mathcal{H}(\mathcal{S}_a^m) > \mathcal{H}(\mathcal{S}_a^{m+1})$ for some $1 \leq m \leq \deg(\mathcal{S}_a) - 2$. Then by definition 7.3, It follows that there exist some $j \geq 1$ such that we can write $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{m+j}(\mathcal{S}_b)$. Since the expansion is proper, It follows that $m + j = n$ and we have

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b).$$

It follows that $\mathcal{H}(\mathcal{S}_a^m) = \mathcal{H}(\mathcal{S}_b^n)$. Since

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{m+1}(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}_b)$$

the regularity condition of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$ also follows. The converse on the other hand follows the same approach. \square

Proposition 7.1. *Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{C}[x]$. Then the set*

$$\mathcal{G} = \{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) : n \in \mathbb{N} \cup \{0\}, \mathcal{S} \in \mathcal{F}\}$$

is a group.

Proof. Clearly the set \mathcal{G} is non-empty, since any tuple $\mathcal{S} \in \mathcal{F}$ has a representation $\mathcal{S} = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S})$. The null tuple \mathcal{S}_0 is the neutral element of the set. Pick $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) \in \mathcal{G}$, then it turns out that

$$-(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) \in \mathcal{G}$$

is the inverse element, since an expansion is linear and for any tuple $\mathcal{S} \in \mathcal{F}$, then $-\mathcal{S} \in \mathcal{F}$. By the linearity of expansion, the set \mathcal{G} satisfies the associative property. This proves that \mathcal{G} is a group. \square

Remark 7.6. Next we prove a result that indicates that the structure of an expansion is preserved at each phase of expansion.

Theorem 7.7. *Let*

$$\mathcal{G} = \{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) : n \in \mathbb{N} \cup \{0\}, \mathcal{S} \in \mathcal{F}\}$$

and

$$\mathcal{G}' = \{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}) : n \in \mathbb{N} \cup \{0\}, \mathcal{S} \in \mathcal{F}\},$$

then $\mathcal{G} \simeq \mathcal{G}'$.

Proof. Consider the map

$$\lambda : \mathcal{G} \longrightarrow \mathcal{G}',$$

where

$$\mathcal{G} = \{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}) : n \in \mathbb{N} \cup \{0\}, \mathcal{S} \in \mathcal{F}\}$$

and

$$\mathcal{G}' = \{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}) : n \in \mathbb{N} \cup \{0\}, \mathcal{S} \in \mathcal{F}\},$$

with

$$\lambda[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S})] = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}).$$

We claim that the map is well-defined. For suppose $\mathcal{S}_1 = \mathcal{S}_2 + \mathcal{S}_{\mathbb{C}}$ with $\mathcal{S}_1 \neq \mathcal{S}_2$, then by appealing to Theorem 4.2

$$\begin{aligned} (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_1) &= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_2 + \mathcal{S}_{\mathbb{C}}) \\ &= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_2), \end{aligned}$$

and it follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}_2),$$

by applying an extra copy of expansion on both sides. This proves that the map is independent on the choice of representative of tuples of $\mathbb{C}[x]$ in the same equivalence class. We claim that the map is injective. Suppose

$$\lambda[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_1)] = \lambda[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_2)].$$

Then it follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n+1}(\mathcal{S}_2).$$

We can remove one copy of expansion on both sides and still preserve unicity of both elements upto equivalence. Thus it follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_2) + \mathcal{S}_{\mathbb{C}}.$$

This proves injectivity. Surjectivity follows by virtue of definition of the map. Finally we claim that the map λ so defined is a homomorphism. Consider the map

$$(7.1) \quad \lambda[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_1) + (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_2)].$$

Since expansion is linear, it follows that

$$(7.1) = \lambda[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_1)] + \lambda[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_2)].$$

Thus the map is an isomorphism. \square

Theorem 7.8. *Let $\mathcal{S} = (f_1(x), f_2(x), \dots, f_n(x))$ where each $f_i(x)$ has no repeated zeros. Then*

$$\mathcal{H}(\mathcal{S}^m) > \mathcal{H}(\mathcal{S}^{m+1})$$

for all $0 \leq m \leq \deg(\mathcal{S}) - 2$.

Proof. Let $\mathcal{S} = (f_1(x), f_2(x), \dots, f_n(x))$ where each $f_i(x)$ has no repeated zeros. Then by appealing to Theorem 6.9 there exist some $N_0 \geq 1$ such that $\mathcal{H}(\mathcal{S}^{N_0}) > \mathcal{H}(\mathcal{S}^{N_0+1})$. Then by applying Theorem 7.5 and Theorem 7.7, It follows that for all $N \geq N_0$, then we have

$$\mathcal{H}(\mathcal{S}^N) > \mathcal{H}(\mathcal{S}^{N+1}).$$

Again by appealing to Theorem 7.5 and Theorem 7.7, It follows that for all $0 < N \leq N_0$, then

$$\mathcal{H}(\mathcal{S}^{N-1}) > \mathcal{H}(\mathcal{S}^N).$$

Combining these two cases, we obtain the following decreasing sequence of the mass of expansion

$$\mathcal{H}(\mathcal{S}) = \mathcal{H}(\mathcal{S}^0) > \mathcal{H}(\mathcal{S}^1) > \dots > \mathcal{H}(\mathcal{S}^{\deg(\mathcal{S})-3}) > \mathcal{H}(\mathcal{S}^{\deg(\mathcal{S})-2}).$$

This proves the diminishing state of the mass of an expansion. \square

Theorem 7.9. *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$ for $n \geq 3$ and let $\mathcal{S} = (f(x), f(x), \dots, f(x))$, where $f(x)$ has no repeated zeros. Suppose $\mathcal{H}(\mathcal{S}) < 1$, then for each $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S})]$*

$$\|\mathcal{S}_0 - \mathcal{S}_j\| < 1$$

for all $\mathcal{S}_j \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(\mathcal{S})]$.

Proof. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$ for $n \geq 3$ and let $\mathcal{S} = (f(x), f(x), \dots, f(x))$. Suppose $\mathcal{H}(\mathcal{S}) < 1$, then by applying Theorem 7.8, It follows that

$$\mathcal{H}(\mathcal{S}^1) < \mathcal{H}(\mathcal{S}^0) = \mathcal{H}(\mathcal{S}) < 1.$$

The result follows from this fact. \square

8. PROOF OF SENDOV CONJECTURE

We are now ready to prove Sendov's conjecture. We assemble the tools we have developed thus far to solve the problem. We state our first theorem:

Theorem 8.1. *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$ with $n \geq 3$. Let $\mathcal{T} = \{b_1, b_2, \dots, b_n\}$ and $\mathcal{C} = \{c_1, c_2, \dots, c_{n-1}\}$ be the set of zeros and critical values of f , respectively. If $|b_i| < \delta \leq 1$ for $i = 1, 2, \dots, n$, then for each $b_i \in \mathcal{T}$, there exist some $c_j \in \mathcal{C}$ such that*

$$|b_i - c_j| < 1.$$

Proof. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$ with $n \geq 3$ and let $\mathcal{S} = (f(x), f(x), \dots, f(x))$. Let $\mathcal{T} = \{b_1, b_2, \dots, b_n\}$ and $\mathcal{C} = \{c_1, c_2, \dots, c_{n-1}\}$ be the set of zeros and critical values of f , respectively, with $|b_i| < \delta \leq 1$ for

$i = 1, 2, \dots, n$. Then we set $\mathcal{H}(\mathcal{S}^0) < 1$. Applying Theorem 7.9, It follows that for any $(b_{\tau(1)}, b_{\tau(2)}, \dots, b_{\tau(n)}) \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S})]$, then it must be the case that

$$\left\| (b_{\tau(1)}, b_{\tau(2)}, \dots, b_{\tau(n)}) - (c_{\alpha(1)}, c_{\alpha(2)}, \dots, c_{\alpha(n)}) \right\| < 1$$

for all $(c_{\alpha(1)}, c_{\alpha(2)}, \dots, c_{\alpha(n)}) \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(\mathcal{S})]$ where $\tau, \alpha : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$. Since each entry of $(b_{\tau(1)}, b_{\tau(2)}, \dots, b_{\tau(n)})$ is a zero of $f(x)$ and each entry of $(c_{\alpha(1)}, c_{\alpha(2)}, \dots, c_{\alpha(n)})$ is a critical value of $f(x)$ for all permutations $\alpha, \tau : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$, the result follows immediately. \square

9. EXTENSION OF SENDOV'S CONJECTURE AND FURTHER DISCUSSIONS

It turns out that the method we have adopted in this paper can also be extended to not only the critical values of an arbitrary polynomial but as well to the zeros of a general class of polynomials of the form $P_n^m(x)$ for $1 \leq m \leq \deg(P_n) - 1$. Since the mass of an expansion diminishes uniformly, we obtain a variant of Theorem 7.9:

Theorem 9.1. *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$ with $n \geq 3$ and let $\mathcal{S} = (f(x), f(x), \dots, f(x))$, where $f(x)$ has no repeated zeros. Suppose $\mathcal{H}(\mathcal{S}) < 1$, then for each $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S})]$*

$$\|\mathcal{S}_0 - \mathcal{S}_j\| < 1$$

for all $\mathcal{S}_j \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S})]$ for all $1 \leq m \leq \deg(\mathcal{S}) - 1$.

It follows from this result, an extension of the **sendov** conjecture:

Theorem 9.2. *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$ with $n \geq 3$. Let $\mathcal{T} = \{b_1, b_2, \dots, b_n\}$ be the set of zeros of f . If $|b_i| < \delta \leq 1$ for $i = 1, 2, \dots, n$, then for each $b_i \in \mathcal{T}$, there exist some c_j with $f^n(c_j) = 0$ such that*

$$|b_i - c_j| < 1$$

for all $1 \leq n \leq \deg(f) - 1$.

Even stronger than this is the assertion that

Theorem 9.3. *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$ and let $\mathcal{S} = (f(x), f(x), \dots, f(x))$, where $f(x)$ has no repeated zeros. Suppose $\mathcal{H}(\mathcal{S}) < \epsilon$, then for each $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S})]$*

$$\|\mathcal{S}_0 - \mathcal{S}_j\| < \epsilon$$

for all $\mathcal{S}_j \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S})]$ for all $1 \leq m \leq \deg(\mathcal{S}) - 1$.

The upshot of this is the result

Theorem 9.4. *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{C}[x]$ with $n \geq 3$. Let $\mathcal{T} = \{b_1, b_2, \dots, b_n\}$ be the set of zeros of f . If $|b_i| < \delta \leq \epsilon$ for $i = 1, 2, \dots, n$, then for each $b_i \in \mathcal{T}$, there exist some c_j with $f^n(c_j) = 0$ such that*

$$|b_i - c_j| < \epsilon$$

for all $1 \leq n \leq \deg(f) - 1$.

1.

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