$g^{\star}bp$-CONTINUOUS, ALMOST $g^{\star}bp$-CONTINUOUS AND WEAKLY $g^{\star}bp$-CONTINUOUS FUNCTIONS

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Abstract. In this paper we introduce new types of functions called $g^{\star}bp$-continuous function, almost $g^{\star}bp$-continuous function, and weakly $g^{\star}bp$-continuous function in topological spaces and study some of their basic properties and relations among them.

1. Introduction

Biswas [8], Husain [17], Levine [23], Noiri and Ahmed [36] and Tong [41] have introduced and investigated many types of continuity such as simple, almost, weak, semi, quasi, α, strong semi, semi-weak , weak almost, A- and B-continuity. Balachandran, Sundaram and Maki [5] have introduced and studied generalized continuous function in topological spaces. Mashour and Deeb [30] have introduced pre-continuous and weak pre continuous mappings. EL Etik [15] also introduced the concept of gb-continuous function by utilizing b-open sets. Omari and Noorani [37] introduced and studied the concept of generalized b-closed sets and gb-continuous function in topological spaces. Vidhya and Parimelazhgana [43] introduced and studied the properties of $g^{\star}b$-closed sets, $g^{\star}b$-continuous and $g^{\star}b$-irresolute in topological spaces.

The aim of this paper is to introduce and study new types of functions called $g^{\star}bp$ (almost $g^{\star}bp$ and weakly $g^{\star}bp$)-continuous functions.

Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represents the non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, $Cl(A)$ and $Int(A)$ represents the closure of A and Interior of A respectively. A subset A is said to be preopen [30] (resp., α-open [32], semi open [24], regular open[45]) set if $A \subseteq IntCl(A)$ (resp., $A \subseteq IntClInt(A)$, $A \subseteq ClInt(A)$, $A = IntCl(A)$). The complement of a preopen set is called preclosed. The intersection of all preclosed [6] (resp., semi closed) sets containing A is called the preclosure (resp. semi closure) of A and is denoted by $pCl(A)$ (resp., $sClA$). The preinterior of A is defined by the union of all preopen sets contained in A and is denoted by $pInt(A)$. It is clear that A is a preopen set if and only if $A = pInt(A)$ and A is preclosed if $A = pCl(A)$. The family of all preopen sets of X is denoted by $PO(X)$ and the family of all preclosed sets of X

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containing x is denoted by $PC(X, x)$.

2. Preliminaries

In this section we recall some definitions and results which are used in the next sections.

Definition 2.1. A subset A of a topological space (X, τ) is called

(1) b-open set [2], if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ and b-closed set if $Cl(Int(A)) \cup Int(Cl(A)) \subseteq A$.
(2) generalized closed set (briefly g-closed)[23] ($g^\star$-closed [42]), if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open(g-open) in X.
(3) pg-closed [28], if $pCl(A) \subseteq U$ whenever $A \subseteq U$ and U is preopen in X.
(4) gb-closed [37], and ($g^\star b$-closed [43]) if $bCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open(g-open) in X. The complement of a gb-closed ($g^\star b$-closed) set is called gb-open ($g^\star b$-open) respectively.
(5) $p\delta$-open set [18], if for each $x \in A$, there exists a preopen set U in X such that $x \in U \subseteq pIntpCl(U) \subseteq A$.
(6) pre-regular p-open [19] (resp., pre-regular p-closed [20]) if $A = pIntpCl(A)$(resp., $A = pClpInt(A)$).

Remark 2.2. It is worth to mention that the notion of pre-regular p-open is called regular preopen in [11]. S. Jafari investigated the fundamental properties of pre-regular p-open sets in [20]. M. Caldas et al. [9] introduced and investigated some weak separation axioms via pre-regular p-open sets. In this paper we use the notions of regular preopen and regular preclosed sets instead of pre-regular p-open and pre-regular p-closed sets.

Definition 2.3. Let (X, τ) and (Y, σ) be two topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(1) g-continuous [5] (b-continuous [15], gb-continuous [37], $g^\star b$-continuous [43], and pre-continuous [30]) if $f^{-1}(A)$ is g-closed (b-closed,gb-closed, $g^\star b$-closed, and pre-closed) in X for every closed set A in Y.
(2) preirresolute [38] if $f^{-1}(A) \in PO(X)$ for each $A \in PO(Y)$.
(3) $g^\star b$-irresolute [43], if the inverse image of every $g^\star b$-closed set in Y is $g^\star b$-closed in X.
(4) weakly continuous [26] (resp., weakly precontinuous [30], and weakly α-continuous [34])If for each $x \in X$ and each open set A of Y containing $f(x)$, there exists an open (resp., preopen and α-open) set U of X containing x such that $f(U) \subseteq Cl(A)$.
(5) complete continuous [3], if the inverse image of each open set of Y is regular open in X.
(6) almost continuous [40] (resp., almost α-continuous [35], R-map [10]) if the inverse image of each regular open subset of Y is open (resp., α-open, regular open)in X.

(7) δ-continuous [33], if for each $x \in X$ and each open set A of Y containing $f(x)$, there exists an open set U of X containing x such that $f(Int(Cl(U))) \subseteq Int(Cl(A))$.

Lemma 2.4. [18] *For any subset A of a topological space X, the following statements are true:*

(1) *A is regular open $\Rightarrow$ A is regular preopen $\Rightarrow$ A is $p\delta$-open $\Rightarrow$ A is preopen.*
(2) *$pIntpCl(A)$ is regular preopen set.*

Lemma 2.5. [21] *Let A be a subset of a space (X, τ). Then $A \in PO(X, \tau)$ if and only if $sCl(A) = IntCl(A)$.*

Theorem 2.6. [25] *Let $f : X \to Y$ be a function and $\{B_\alpha : \alpha \in \Delta\}$ be an indexed family of subsets of Y. Then the induced function $f^{-1} : Y \to X$ has the following properties:*

(1) $f^{-1}(\cup(\{B_\alpha : \alpha \in \Delta\})) = \cup(f^{-1}(\{B_\alpha : \alpha \in \Delta\}))$.
(2) $f^{-1}(\cap(\{B_\alpha : \alpha \in \Delta\})) = \cap(f^{-1}(\{B_\alpha : \alpha \in \Delta\}))$.

Definition 2.7. [4] A space X is said to be

(1) Pre-T_0 if and only if to each pair of distinct points x, y in X, there exists a preopen set containing one of the points but not the other.
(2) Pre-T_1 if and only if to each pair of distinct points x, y of X, there exists a pair of preopen sets one containing x but not y and other containing y but not x.
(3) Pre-T_2 if and only if to each pair of distinct points x, y of X, there exists a pair of disjoint preopen sets one containing x and the other containing y.

Professor M. Ganster in 2003, in a private conversation with the third author showed that every topological space is pre-T_0.

Definition 2.8. A topological space (X, τ) is said to be:

(1) submaximal [21], if the closure of every dense subset of X is open.
(2) extremally disconnected [27], if the closure of every open set of X is open in X.
(3) locally indiscrete [13], if every open set of X is closed in X.
(4) pre-$T_{\frac{1}{2}}$ [28], space if every pg-closed set is preclosed.
(5) r-T_1 [14], if for each pair of distinct points x and y of X, there exists regular open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$.

Definition 2.9. A space X is called:

(1) preregular [7](resp., p-regular [31]) if for each preclosed (resp., closed) set F and each point $x \notin F$, there exists disjoint preopen sets U and V such that $x \in U$ and $F \subseteq V$.

(2) Almost regular [39], if for any regular closed set F of X and any point $x \in X \setminus F$, there exists disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.
(3) semi-regular [39], if for any open set U of X and each point $x \in U$ there exists a regular open set V of X such that $x \in V \subseteq U$.
(4) almost p-regular [29], if for each $A \in RC(X)$ and each point $x \in X \setminus A$ there exists preopen sets U, V such that $x \in U$ and $U \cap V = \phi$.
(5) strongly s-regular [16], if for each closed set A and any point $x \in (X \setminus A)$, there exists a $F \in RC(X)$ such that $x \in F$ and $F \cap A = \phi$.

Theorem 2.10. [13] *If $R \in RO(X)$ and $P \in PO(X)$, then $R \cap P \in RO(P)$.*

Lemma 2.11. [21] *A space X is submaximal if and only if every preopen set is open.*

Theorem 2.12. [1] *Let (Y, τ_Y) be subspace of a space (X, τ). If $A \in PO(X, \tau)$ and $A \subseteq Y$, then $A \in PO(Y, \tau_Y)$.*

Theorem 2.13. [44] *Let A be a subset of a topological space (X, τ), if $A \in \tau$, then $Cl_\theta(A) = Cl(A)$.*

Definition 2.14. [22] A space X is said to be:

(1) $g^\star b$-T_0 if for each pair of distinct points x, y in X, there exists a $g^\star b$-open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.
(2) $g^\star b$-T_1 if for each pair of distinct points x, y in X, there exist two $g^\star b$-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.
(3) $g^\star b$-T_2 if for each distinct points x, y in X, there exist two disjoint $g^\star b$-open sets U and V containing x and y respectively.

3. $g^\star bp$-continuous function

In this section, we Introduce the concept of $g^\star bp$-continuous function in topological spaces.

Definition 3.1. Let (X, τ) and (Y, σ) be two topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is called $g^\star bp$-continuous at a point $x \in X$ if for each preopen set A in Y containing $f(x)$, there exists a $g^\star b$-open set U of X containing x such that $f(U) \subseteq A$.

Proposition 3.2. *For a function $f : (X, \tau) \to (Y, \sigma)$ the following are equivalent.*

(1) *f is $g^\star bp$-continuous.*
(2) *$f^{-1}(A)$ is $g^\star b$-open in X, for each preopen set A in Y.*
(3) *$f^{-1}(B)$ is $g^\star b$-closed in X, for each preclosed set B in Y.*

Proof. (1) $\Rightarrow$ (2). Let A be any preopen set of Y, we have to show that $f^{-1}(A)$ is $g^\star b$-open in X. Let $x \in f^{-1}(A)$. Then $f(x) \in A$. By(1), there exsits a $g^\star b$-open set U in X containing x such that $f(U) \subseteq A$ which implies

that $x \in U \subseteq f^{-1}(A)$. Therefore, $f^{-1}(A)$ is $g^{\star}b$-open in X.
(2) $\Rightarrow$ (3). Let B be preclosed set of Y. Then $Y \setminus B$ is preopen set of Y. By(2), $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is $g^{\star}b$-open set in X and hence $f^{-1}(B)$ is $g^{\star}b$-closed in X.
(3) $\Rightarrow$ (1). Let A be any preopen set of Y. Then $(Y \setminus A)$ is preclosed in Y. By(3), $f^{-1}(Y \setminus A)$ is $g^{\star}b$-closed set in X. But $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$. Thus $X \setminus f^{-1}(A)$ is $g^{\star}b$-closed in X so $f^{-1}(A)$ is $g^{\star}b$-open in X. Therefore, we obtain $f(f^{-1}(A)) \subseteq A$, hence f is $g^{\star}bp$-continuous. □

Proposition 3.3. *If a function $f : (X, \tau) \to (Y, \sigma)$ is $g^{\star}bp$-continuous, then it is $g^{\star}b$-continuous.*

Proof. Let A be any open set in Y, then its preopen set in Y. Since f is $g^{\star}bp$-continuous, then $f^{-1}(A)$ is $g^{\star}b$-open set in X. Hence f is $g^{\star}b$-continuous. □

The converse of Proposition 3.3 need not be true in general as it is shown in the following example.

Example 3.4. Let $X = Y = \{a, b, c\}$, and $\tau = \{\phi, \{b\}, \{a, b\}, X\}$, $\sigma = \{\phi, \{c\}, \{a, b\}, Y\}$, and a function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = c$, $f(b) = b$, $f(c) = a$ f is $g^{\star}b$-continuous but not $g^{\star}bp$-continuous, since for the preclosed set $B = \{a, b\}$ in Y, $f^{-1}(B) = \{b, c\}$ is not $g^{\star}b$-closed in X.

Note: If Y is submaximal, then by Lemma 2.11 we have $PO(X) = \tau$. Hence, every $g^{\star}b$-continuous function is $g^{\star}bp$-continuous.

Proposition 3.5. *If a function $f : (X, \tau) \to (Y, \sigma)$ is $g^{\star}b$-irresolute, then it is $g^{\star}bp$-continuous but not conversely.*

Proof. Let A be preclosed set in Y, then it is $g^{\star}b$-closed in Y. Since f is $g^{\star}b$-irresolute, then $f^{-1}(A)$ is $g^{\star}b$-closed in X. Hence it is $g^{\star}bp$-continuous. □

The converse of Proposition 3.5 is not true in general.

Example 3.6. Let $X = Y = \{a, b, c\}$ and let τ=$\{\phi,\ \{a\},\ \{a, b\},\ X\}$, $\sigma = \{\phi, \{c\}, \{a, c\}, Y\}$. The identity function $f : (X, \tau) \to (Y, \sigma)$ is $g^{\star}bp$-continuous but not $g^{\star}b$-irresolute because $B = \{a, b\}$ is $g^{\star}b$-closed set in Y and $f^{-1}(B) = \{a, b\}$ is not $g^{\star}b$-closed in X.

Proposition 3.7. *Let $X = R_1 \cup R_2$, where R_1 and R_2 are $g^{\star}b$-closed set in X. Let $f : R_1 \to Y$ and $g : R_2 \to Y$ be $g^{\star}bp$-continuous. If $f(x) = g(x)$ for each $x \in R_1 \cap R_2$. Then $h : R_1 \cup R_2 \to Y$ such that*

$$h(x) = \begin{cases} f(x) & \text{if } x \in R_1 \\ g(x) & \text{if } x \in R_2 \end{cases}$$

is $g^{\star}bp$-continuous.

Proof. Let A be any preopen set in Y. Clearly $h^{-1}(A) = f^{-1}(A) \cup g^{-1}(A)$. Since f is $g^{\star}bp$-continuous, then $f^{-1}(A)$ is $g^{\star}b$-open in R_1. But R_1 is $g^{\star}b$-open in X. Then by Theorem 3.30 [43], $f^{-1}(A)$ is $g^{\star}b$-open in X. Similarly,

$g^{-1}(A)$ is $g^\star b$-open in R_2 and hence a $g^\star b$-open in X. Since a union of two $g^\star b$-open sets is $g^\star b$-open. Therefore, $h^{-1}(A) = f^{-1}(A) \cup g^{-1}(A)$ is $g^\star b$-open in X. Hence h is $g^\star bp$-continuous. □

Theorem 3.8. *For a function $f : (X, \tau) \to (Y, \sigma)$ the following are equivalent:*

(1) *f is $g^\star bp$-continuous.*
(2) *$f(g^\star bCl(B)) \subseteq pCl(f(B))$, for every subset B of X.*
(3) *$g^\star bCl(f^{-1}(A)) \subseteq f^{-1}(pCl(A))$, for each subset A of Y.*
(4) *$f^{-1}(pInt(A)) \subseteq g^\star bInt(f^{-1}(A))$, for each subset A of Y.*
(5) *$pInt(f(B)) \subseteq f(g^\star bInt(B))$, for each subset B of X.*

Proof. (1) ⇒ (2). Let B be any subset of X .Then $f(B) \subseteq pCl(f(B))$ and $pClf(B)$ is preclosed in Y. Hence $B \subseteq f^{-1}(pClf(B))$, since f is $g^\star bp$-continuous. By Proposition 3.2, $f^{-1}(pClf(B))$ is $g^\star b$-closed set in X. Therefore, $g^\star bCl(B) \subseteq f^{-1}(pCl(f(B)))$. Hence $f(g^\star bCl(B)) \subseteq (pCl(f(B)))$.
(2) ⇒ (3). Let A be any subset of Y, then $f^{-1}(A)$ is a subset of X. By (2) we have $f(g^\star bClf^{-1}(A)) \subseteq pCl(f(f^{-1}(A))) = pCl(A)$. It follow that $g^\star b(Clf^{-1}(A)) \subseteq f^{-1}(pCl(A))$.
(3) ⇒ (4). Let A be any subset of Y. Then apply(3) to $(Y \setminus A)$ we obtain $g^\star bCl(f^{-1}(Y \setminus A)) \subseteq f^{-1}(pCl(Y \setminus A)) \Leftrightarrow g^\star bCl(X \setminus f^{-1}(A)) \subseteq f^{-1}(Y \setminus pInt(A)) \Leftrightarrow X \setminus g^\star bInt(f^{-1}(A)) \subseteq X \setminus f^{-1}(pInt(A)) \Leftrightarrow f^{-1}(pInt(A) \subseteq g^\star bInt(f^{-1}(A))$.
(4) ⇒ (5). Let B be any subset of X, Then $f(B)$ is a subset of Y. By(4), we have $f^{-1}(pInt(f(A))) \subseteq g^\star bInt(f^{-1}(f(A))) = g^\star bInt(A)$. Therefore, $pInt(f(A)) \subseteq f(g^\star bInt(A))$.
(5) ⇒ (1). Let $x \in X$ and let A be any preopen set of Y containing $f(x)$. Then $x \in f^{-1}(A)$ and $f^{-1}(A)$ is a subset of X. By(5), we have $pInt(f(f^{-1}(A))) \subseteq f(g^\star bInt(f^{-1}(A)))$. Then $pInt(A) \subseteq f(g^\star bInt(f^{-1}(A)))$, since A is preopen, then $A \subseteq f(g^\star bInt(f^{-1}(A)))$ implies that $f^{-1}(A) \subseteq g^\star bInt(f^{-1}(A))$. Therefore $f^{-1}(A)$ is $g^\star b$-open in X containing x and clearly $f(f^{-1}(A)) \subseteq A$. Hence f is $g^\star bp$-continuous. □

Proposition 3.9. *Let $f : X \to Y$ be $g^\star bp$-continuous and $Y \subseteq Z$. If Y is preclosed subset of a topological space Z, then $f : X \to Z$ is $g^\star bp$-continuous.*

Proof. Let F be any preclosed set in Z. Then $F \cap Y$ is preclosed in Z, by Theorem 2.22 [1], $F \cap Y$ is preclosed in Y. Since f is $g^\star bp$-continuous, so $f^{-1}(F \cap Y)$ is $g^\star b$-closed in X but $f(x) \in Y$ for each $x \in X$, and thus $f^{-1}(F) = f^{-1}(F \cap Y)$ is $g^\star b$-closed subset of X. Therefore, by Proposition 3.2 $f : X \to Z$ is $g^\star bp$-continuous. □

Theorem 3.10. *If $f : (X, \tau) \to (Y, \sigma)$ is $g^\star bp$-continuous and A is $g^\star b$-closed set in X, then $f|A : A \to Y$ is $g^\star bp$-continuous.*

Proof. Let B be preclosed set in Y, since f is $g^\star bp$-continuous, then $f^{-1}(B)$ is $g^\star b$-closed in X. Since $(f|A)^{-1}(B)$=$f^{-1}(B) \cap A$, so Since $(f|A)^{-1}(B)$ is $g^\star b$-closed in X because the intersection of two $g^\star b$-closed sets is $g^\star b$-closed.

Hence by Theorem 3.30 [43], $(f|A)^{-1}(B)$ is $g^\star b$-closed set in A. Therefore $f|A$ is $g^\star bp$-continuous. □

Theorem 3.11. *If $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\delta)$ be any two functions ,then $g \circ f : (X,\tau) \to (Z,\delta)$ is $g^\star bp$-continuous if g is preirresolute function and f is $g^\star bp$-continuous.*

Proof. Let A be any preclosed set in Z. Since g is preirresolute function, then $g^{-1}(A)$ is preclosed in Y. Since f is $g^\star bp$-continuous, then $f^{-1}(g^{-1}(A))$ is $g^\star b$-closed in X. Hence $g \circ f$ is $g^\star bp$-continuous. □

Proposition 3.12. *If a function $f : X \to Y$ is $g^\star b$-continuous and Y is p-regular, then f is $g^\star bp$-continuous.*

Proof. Let $x \in X$ and A be any preopen set of Y containing $f(x)$. Since Y is p-regular then there exists an open set G of Y such that $f(x) \in G \subseteq A$, since f is $g^\star b$-continuous, then there exists a $g^\star b$-open set U of X containing x such that $f(U) \subseteq G \subseteq A$. Therefore, f is $g^\star bp$-continuous. □

Theorem 3.13. *If $f : X \to Y$ is a $g^\star bp$-continuous injection and Y is pre-T_1, then X is $g^\star b$-T_1.*

Proof. Assume that Y is pre-T_1. For any distinct points x and y in X, there exist preopen sets A and W such that $f(x) \in A$, $f(y) \notin A$, $f(x) \notin W$ and $f(y) \in W$. Since f is $g^\star bp$-continuous, so there exist $g^\star b$-open sets G and H such that $x \in G$, $y \in H$, $f(G) \subseteq A$ and $f(H) \subseteq W$. Thus we obtain $y \notin G$, $x \notin H$. This shows that X is $g^\star b$-T_1. □

Theorem 3.14. *If $f : X \to Y$ is $g^\star bp$-continuous injection and Y is pre-T_2 then X is $g^\star b$-T_2.*

Proof. For any pair of distinct points x and y in X, there exist disjoint preopen sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is $g^\star bp$-continuous, there exist $g^\star b$-open sets G and H in X containing x and y, respectively, such that $f(G) \subseteq U$ and $f(H) \subseteq V$. Since U and V are disjoint, we have $U \cap V = \phi$, hence $G \cap H = \phi$. This shows that X is $g^\star b$-T_2. □

4. Almost $g^\star bp$-continuous function

Definition 4.1. A function $f : (X,\tau) \to (Y,\sigma)$ is called almost $g^\star bp$-continuous at a point $x \in X$ if for each preopen set A of Y containing $f(x)$, there exists a $g^\star b$-open set U of X containing x such that $f(U) \subseteq IntClA$. If f is almost $g^\star bp$-continuous at every point of X, then it is called almost $g^\star bp$-continuous.

Definition 4.2. A function $f : X \to Y$ is said to be almost $g^\star bp$-open if $f(U) \subseteq IntCl(f(U))$ for every $g^\star b$-open set U in X.

Theorem 4.3. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:*

(1) *f is almost $g^\star bp$-continuous,*
(2) *For each $x \in X$ and each preopen set A of Y containing $f(x)$, there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq sCl(A)$.*
(3) *For each $x \in X$ and each regular open set A of Y containing $f(x)$, there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq A$.*
(4) *For each $x \in X$ and each δ-open set A of Y containing $f(x)$, there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq A$.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and A be any preopen set of Y containing $f(x)$. By (1) there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq IntCl(A)$. Since A is preopen by Lemma 2.5, $f(U) \subseteq sCl(A)$.
(2) $\Rightarrow$ (3). Let $x \in X$ and A be any regular open set of Y containing $f(x)$, then A is preopen set in Y. By (2), there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq sCl(A)$, then by Lemma 2.5, $f(U) \subseteq IntCl(A)$. Since A is regular open, then $f(U) \subseteq A$.
(3) $\Rightarrow$ (4). Let $x \in X$ and let A be any δ-open set of Y containing $f(x)$. Then for each $f(x) \in A$, there exists an open set G containing $f(x)$ such that $G \subseteq IntCl(G) \subseteq A$. Since $IntCl(G)$ is regular open set of Y containing $f(x)$. By(3), there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq IntCl(G) \subseteq A$.
(4) $\Rightarrow$ (1). Let $x \in X$ and A be any preopen set of Y containing $f(x)$, then $IntCl(A)$ is δ-open set of Y containing $f(x)$. By(4), there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq IntCl(A)$. Therefore, f is almost $g^\star bp$-continuous. □

Theorem 4.4. *A function $f : X \to Y$ is almost $g^\star bp$-continuous if and only if for each $x \in X$ and each regular open set A containing $f(x)$, there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq A$.*

Proof. For every $x \in X$ and let A be any regular open set containing $f(x)$, then A is preopen set containing $f(x)$. Since f is almost $g^\star bp$-continuous, then there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq IntCl(A) = A$. Conversely. Obvious. □

Theorem 4.5. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) *f almost $g^\star bp$-continuous.*
(2) *$f^{-1}(IntCl(A))$ is $g^\star b$-open set in X, for each preopen set A in Y.*
(3) *$f^{-1}(ClInt(F))$ is $g^\star b$-closed set in X, for each preclosed set F in Y.*
(4) *$f^{-1}(F)$ is $g^\star b$-closed set in X, for each regular closed set F in Y.*
(5) *$f^{-1}(A)$ is $g^\star b$-open set in X , for each regular open set A in Y.*

Proof. (1) $\Rightarrow$ (2). Let A be any preopen set in Y. We have to show that $f^{-1}(IntCl(A))$ is $g^\star b$-open set in X. Let $x \in f^{-1}(IntCl(A))$. Then $f(x) \in IntCl(A)$ and $IntCl(A)$ is regular open set in Y. Since f is almost $g^\star bp$-continuous. By Theorem 4.3, there exists a $g^\star b$-open set U of X containing x such that $f(U) \subseteq IntCl(A)$. Which implies that $x \in U \subseteq f^{-1}(IntCl(A))$.

Therefore, $f^{-1}(IntCl(A))$ is $g^{\star}b$-open set in X.
(2) $\Rightarrow$ (3). Let F be any preclosed set of Y. Then $Y \setminus F$ is preopen set of Y. By (2), $f^{-1}(IntCl(Y \setminus F))$ is $g^{\star}b$-open set in X and $f^{-1}(IntCl(Y \setminus F)) = f^{-1}(Int(Y \setminus Int(F))) = f^{-1}(Y \setminus ClInt(F)) = X \setminus f^{-1}(ClInt(F))$ is $g^{\star}b$-open set in X and hence $f^{-1}(ClInt(F))$ is $g^{\star}b$-closed set in X.
(3) $\Rightarrow$ (4). Let F be any regular closed set of Y. Then F is preclosed set of Y. By (3). $f^{-1}(ClInt(F))$ is $g^{\star}b$-closed set in X since F is regular closed set, then $f^{-1}(ClInt(F)) = f^{-1}(F)$. Therefore $f^{-1}(F)$ is $g^{\star}b$-closed set in X.
(4) $\Rightarrow$ (5). Let A be any regular open set of Y. Then $Y \setminus A$ is regular closed set of Y and by (4), we have $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ is $g^{\star}b$-closed set in X and hence $f^{-1}(A)$ is $g^{\star}b$-open set in X.
(5) $\Rightarrow$ (1). Let $x \in X$ and let A be any regular open set of Y containing $f(x)$, so $x \in f^{-1}(A)$. By (5), we have $f^{-1}(A)$ is $g^{\star}b$-open set in X. Therefore we obtain $f(f^{-1}(A)) \subseteq A$. Hence by Theorem 4.3, f is almost $g^{\star}bp$-continuous. □

Proposition 4.6. *If a function $f : (X, \tau) \to (Y, \sigma)$ is $g^{\star}bp$-continuous, then it is almost $g^{\star}bp$-continuous.*

Proof. Let A be any regular open set in Y, so A is preopen. Since f is $g^{\star}bp$-continuous, then $f^{-1}(A)$ is $g^{\star}b$-open in X. Hence by Theorem 4.5, f is almost $g^{\star}bp$-continuous. □

The converse of Propostion 4.6 is not true in general as it is shown by the following example.

Example 4.7. Consider $X = Y = \{a, b, c\}$ with the topology τ=$\{\phi$, $\{c\}$, $\{a, c\}$, $\{b, c\}$, $X\}$, $\sigma = \{\phi, \{a\}, Y\}$. The identity function $f : (X, \tau) \to (Y, \sigma)$ is almost $g^{\star}bp$-continuous but not $g^{\star}bp$-continuous since the subset $B = \{b, c\}$ is preclosed in Y and $f^{-1}(B) = \{b, c\}$ is not $g^{\star}b$-closed in X.

Proposition 4.8. *If a function $f : X \to Y$ is almost α-continuous, then f is almost $g^{\star}bp$-continuous.*

Proof. Let A be any regular open set in Y. Since f is almost α-continuous, then $f^{-1}(A)$ is α-open set in X, hence by Theorem 3.8 [43], $f^{-1}(A)$ is $g^{\star}b$-open in X. Therefore, f is almost $g^{\star}bp$-continuous. □

Theorem 4.9. *If a function $f : X \to Y$ is δ-continuous, then f is almost $g^{\star}bp$-continuous.*

Proof. Let $x \in X$ and A be any preopen set in Y, then $A \subseteq IntCl(A)$. Since f is δ-continuous, there exists an open set U of X containig x such that $f(IntCl(U)) \subseteq IntCl(IntCl(A))$, then $f(IntCl(U)) \subseteq IntCl(A)$. Since $IntCl(U)$ is regular open set, so by Lemma 2.4, $IntCl(U)$ is preopen and by Theorem 3.12 [43], $IntCl(U)$ is $g^{\star}b$-open set of X. Therefore, f is almost $g^{\star}bp$-continuous. □

Theorem 4.10. *If $f : X \to Y$ is an almost $g^\star bp$-continuous function and Y is locally indescrete, then f is $g^\star b$-continuous.*

Proof. Let $x \in X$ and let A be any open set of Y, hence A is preopen in Y. Since f is almost $g^\star bp$-continuous, there exists a $g^\star b$-open set U in X containing x such that $f(U) \subseteq IntCl(A) \subseteq Cl(A) = A$ and hence f is $g^\star b$-continuous. □

Theorem 4.11. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

1. *f is almost $g^\star bp$-continuous.*
2. *$f(g^\star bCl(A)) \subseteq Cl_\delta(f(A))$ for every subset A of X.*
3. *$g^\star bCl(f^{-1}(B)) \subseteq f^{-1}(Cl_\delta(B))$ for every subset B of Y.*
4. *$f^{-1}(B)$ is $g^\star b$-closed for every δ-closed set B of Y.*
5. *$f^{-1}(A)$ is $g^\star b$-open for every δ-open set A of Y.*

Proof. (1) $\Rightarrow$ (2). Let A be a subset of X, since $Cl_\delta(f(A))$ is δ-closed in Y and it is equal to $\cap\{F_\alpha : F_\alpha$ is regular closed in Y, $\alpha \in \Delta\}$ where Δ is an index set. By Theorem 2.6, we have $A \subseteq f^{-1}(Cl_\delta(f(A))) = f^{-1}(\cap\{F_\delta : \alpha \in \Delta\}) = \cap\{f^{-1}(F_\alpha) : \alpha \in \Delta\}$. By(1), $f^{-1}(Cl_\delta f(A))$ is $g^\star b$-closed in X. Hence $g^\star bCl(A) \subseteq f^{-1}(Cl_\delta(f(A)))$. This shows that $f(g^\star bCl(A)) \subseteq Cl_\delta(f(A))$.
(2) $\Rightarrow$ (3). Taking $A = f^{-1}(B)$ in(2), then we have
$f(g^\star bCl(f^{-1}(B))) \subseteq Cl_\delta(f(f^{-1}(B))) \subseteq Cl_\delta(B)$ and hence $g^\star bCl(f^{-1}(B)) \subseteq f^{-1}(Cl_\delta(B))$.
(3) $\Rightarrow$ (4). Let F be δ-closed set of Y, then $g^\star bCl(f^{-1}(F)) \subseteq f^{-1}(F))$ so $f^{-1}(F)$ is $g^\star b$-closed.
(4) $\Rightarrow$ (5). Let A be δ-open set of Y, then $Y \setminus A$ is δ-closed in Y. By(4), we have $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ is $g^\star b$-closed in X. Hence $f^{-1}(A)$ is $g^\star b$-open in X.
(5) $\Rightarrow$ (1). Let A be any regular open set of Y. Since A is δ-open in Y then $f^{-1}(A)$ is $g^\star b$-open and hence from $f(f^{-1}(A) \subseteq A = IntCl(A)$. Thus f is almost $g^\star bp$-continuous.

□

Theorem 4.12. *If $f : X \to Y$ is almost $g^\star bp$-continuous function, then we have $f^{-1}(A) \subseteq g^\star bInt(f^{-1}(IntCl(A)))$ for every preopen set A in Y.*

Proof. Let A be any preopen set in Y, then $A \subseteq IntCl(A)$. Since $IntCl(A)$ is regular open set in Y, and Since f is almost $g^\star bp$-continuous function, so by Theorem 4.5, $f^{-1}(intCl(A))$ is $g^\star b$-open set in X. Hence $f^{-1}(A) \subseteq f^{-1}(intCl(A)) = g^\star bInt(f^{-1}(IntCl(A)))$. □

Corollary 4.13. *If $f : X \to Y$ is almost $g^\star bp$-continuous function, then we have $f^{-1}(A) \subseteq g^\star bInt(f^{-1}(sCl(A)))$, for every preopen set A in Y.*

Proof. Follows from Lemma 2.5 and Theorem 4.12. □

Corollary 4.14. *If $f : X \to Y$ is almost $g^\star bp$-continuous function, then we have $g^\star bCl(f^{-1}(ClInt(E))) \subseteq f^{-1}(E)$, for every preclosed set E in Y.*

Proof. Let E be any preclosed set in Y, so $Y \setminus E$ is preopen. By Theorem 4.12, $f^{-1}(Y \setminus E) \subseteq g^{\star}bInt(f^{-1}(IntCl(Y \setminus E)))$ this implies that $X \setminus f^{-1}(E) \subseteq g^{\star}bInt(f^{-1}(Y \setminus ClInt(E)))$, then $X \setminus f^{-1}(E) \subseteq g^{\star}bInt(X \setminus f^{-1}(ClInt(E)))$. It follows that $X \setminus f^{-1}(E) \subseteq X \setminus g^{\star}bCl(f^{-1}(ClInt(E)))$. Hence $g^{\star}bCl(f^{-1}(ClInt(E))) \subseteq f^{-1}(E)$. □

Corollary 4.15. *If $f : X \to Y$ is almost $g^{\star}bp$-continuous function, then we have $g^{\star}bCl(f^{-1}(sInt(E))) \subseteq f^{-1}(E)$, for every preclosed set E in Y.*

Proof. Follows from Lemma2.5 and Corollary 4.14. □

Theorem 4.16. *Let $f : X \to Y$ be an almost $g^{\star}bp$-continuous. If Y is preopen set in Z, then $f : X \to Z$ is almost $g^{\star}bp$-continuous.*

Proof. Let A be any regular open set of Z. Since Y is preopen, then by Theorem 2.10, $A \cap Y$ is regular open set in Y. Since f is almost $g^{\star}bp$-continuous then $f^{-1}(A \cap Y)$ is $g^{\star}b$-open set in X. But $f(x) \in Y$ for each $x \in X$. Thus $f^{-1}(A) = f^{-1}(A \cap Y)$ is a $g^{\star}b$-open set in X. Therefore f is almost $g^{\star}bp$-continuous. □

Theorem 4.17. *If $f : X \to Y$ be a $g^{\star}b$-irresolute and $g : Y \to Z$ is an almost $g^{\star}bp$-continuous function, then $g \circ f : X \to Z$ is almost $g^{\star}bp$-continuous function.*

Proof. Let A be any preopen set in Z. Since g is almost $g^{\star}bp$-continuous function, then $g^{-1}(A)$ is $g^{\star}b$-open set in Y. Since f is $g^{\star}b$-irresolute, then by Theorem 4.2 [43], $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$ is $g^{\star}b$-open set in X. Hence $g \circ f$ is almost $g^{\star}bp$-continuous function. □

Theorem 4.18. *If $f : X \to Y$ be almost $g^{\star}bp$-continuous and $g : Y \to Z$ is completely continuous function and Z is submaximal, then $g \circ f : X \to Z$ is $g^{\star}bp$-continuous function.*

Proof. Let A be any preopen set in Z since Z is submaximal then A is open in Z, since g is completely continuous, then $g^{-1}(A)$ is regular open in Y. Since f is almost $g^{\star}bp$-continuous then $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is $g^{\star}b$-open in X. Hence $g \circ f$ is $g^{\star}bp$-continuous. □

Theorem 4.19. *If $f : X \to Y$ be almost $g^{\star}bp$-continuous and $g : Y \to Z$ is R-map, then $g \circ f : X \to Z$ is almost $g^{\star}bp$-continuous.*

Proof. Let A be any regular open set in Z. Since g is R-map then $g^{-1}(A)$ is regular open in Y. Since f is almost $g^{\star}bp$-continuous $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is almost $g^{\star}bp$-continuous. □

Theorem 4.20. *If $f : X \to Y$ is an almost $g^{\star}bp$-continuous function and A is $g^{\star}b$-closed set of X, then the restriction function $f|A : A \to Y$ is almost $g^{\star}bp$-continuous function.*

Proof. Let B be any regular closed set of Y. Since f is almost $g^{\star}bp$-continuous function, then by Theorem 4.5, $f^{-1}(B)$ is $g^{\star}b$-closed set in X

and $(f|A)^{-1}(B) = A \cap f^{-1}(B)$. Since A is $g^\star b$-closed, so by Theorem 3.30 [43], $A \cap f^{-1}(B)$ is $g^\star b$-closed set in A. Hence $f|A$ is almost $g^\star bp$-continuous function. □

Theorem 4.21. *Let $f : X \to Y$ be a function and $x \in X$. If A is both $g^\star b$-closed and g-open set and the restriction $f|A$ is almost $g^\star bp$-continuous function then f is almost $g^\star bp$-continuous.*

Proof. Suppose that B is any regular closed set in Y containing $f(x)$. Since $f|A$ is almost $g^\star bp$-continuous, there exists a $g^\star b$-closed set G of A containing x such that $f(G) = (f|A)(G) \subseteq B$. Since A is both $g^\star b$-closed and g-open set in X, it follows from Theorem 3.31[43], that G is $g^\star b$-closed in X. This shows that f is almost $g^\star bp$-continuous. □

Theorem 4.22. *If $f : X \to Y$ is an almost $g^\star bp$-continuous injection and Y is r-T_1, then X is $g^\star b - T_1$.*

Proof. Assume that Y is r-T_1, then for any distinct points x and y in X, there exist regular open sets A and W such that $f(x) \in A$, $f(y) \notin A$, $f(x) \notin W$ and $f(y) \in W$. Since f is almost $g^\star bp$-continuous there exist $g^\star b$-open sets G and H such that $x \in G$, $y \in H$, $f(G) \subseteq A$ and $f(H) \subseteq W$. Thus we obtain $y \notin G$, $x \notin H$. This shows that X is $g^\star b - T_1$. □

Theorem 4.23. *If $f : X \to Y$ is almost $g^\star bp$-continuous injection and Y is pre-T_2 then X is $g^\star b - T_2$.*

Proof. For any pair of distinct points x and y in X, there exist disjoint preopen sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is almost $g^\star bp$-continuous, there exists $g^\star b$-open sets G and H in X containing x and y, respectively, such that $f(G) \subseteq IntCl(U)$ and $f(H) \subseteq IntCl(V)$. Since U and V are disjoint, we have $IntCl(U) \cap IntCl(V) = \phi$, hence $G \cap H = \phi$. This shows that X is $g^\star b - T_2$. □

5. Weakly $g^\star bp$-continuous function

Definition 5.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called weakly $g^\star bp$-continuous at a point $x \in X$ if for each preopen set A of Y containing $f(x)$, there exists a $g^\star b$-open set U of X containing x such that $f(U) \subseteq ClA$. If f is weakly $g^\star bp$-continuous at every point of X , then it is called weakly $g^\star bp$-continuous.

Theorem 5.2. *Let $f : X \to Y$ be a function. If $f^{-1}(ClA)$ is $g^\star b$-open set in X for each preopen set A in Y, then f is weakly $g^\star bp$-continuous.*

Proof. Let $x \in X$ and A be any preopen set of Y containing $f(x)$. Then $x \in f^{-1}(A) \subseteq f^{-1}(ClA)$. By hypothesis, we have $f^{-1}(ClA)$ is $g^\star b$-open set in X containing x. Therefore, we obtain $f(f^{-1}(ClA)) \subseteq ClA$. Hence f is weakly $g^\star bp$-continuous. □

It is obvious that if the function f is almost $g^{\star}bp$-continuous, then it is weakly $g^{\star}bp$-continuous. However, the converse is not true in general as it shown in the following example.

Example 5.3. Consider $X = Y = \{a, b, c, d\}$ with the topology $\tau = \sigma = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$, with identity function $f : (X, \tau) \to (Y, \sigma)$ f is weakly $g^{\star}bp$-continuous but not almost $g^{\star}b$-continuous since for a preopen set $B = \{a, b\}$ in Y $f^{-1}(IntClB) = \{a, b\}$ which is not $g^{\star}b$-open in X.

Theorem 5.4. *If $f : X \to Y$ is weakly $g^{\star}bp$-continuous function and Y is almost p-regular, then f is almost $g^{\star}bp$-continuous.*

Proof. Let $x \in X$ and let A be preopen set of Y. By the almost p-regularity of Y there exists a regular open set G of Y such that $f(x) \in G \subseteq Cl(G) \subseteq IntCl(A)$. Since f is weakly $g^{\star}bp$-continuous, there exists a $g^{\star}b$-open set U in X such that $f(U) \subseteq Cl(G) \subseteq IntCl(A)$. Therefore f is almost $g^{\star}bp$-continuous. □

Theorem 5.5. *If $f : X \to Y$ is almost $g^{\star}bp$-open and weakly $g^{\star}bp$-continuous function, then f is almost $g^{\star}bp$-continuous function.*

Proof. Let $x \in X$ and A be preopen set of Y containing $f(x)$. Since f is weakly $g^{\star}bp$-continuous, then there exists a $g^{\star}b$-open set U in X containing x such that $f(U) \subseteq Cl(A)$. since f is almost $g^{\star}bp$-open function, then $f(U) \subseteq IntCl(f(U)) \subseteq IntCl(A)$. Hence f is almost $g^{\star}bp$-continuous. □

Corollary 5.6. *Let $f : X \to Y$ be a function. If $f^{-1}(IntF)$ is $g^{\star}b$-closed set in X for each preclosed set F in Y, then f is weakly $g^{\star}bp$-continuous.*

Theorem 5.7. *Let $f : X \to Y$ be a function. If for each $x \in X$ and each regular closed set R of Y containing $f(x)$, there exists a $g^{\star}b$-open set U in X containing x such that $f(U) \subseteq R$, then f is weakly $g^{\star}bp$-continuous.*

Proof. Let $x \in X$ and A be any preopen set of Y containing $f(x)$. Then put $R = Cl(A)$ which is a regular closed set of Y containing $f(x)$. By hypothesis, there exists a $g^{\star}b$-open set U in X containing x such that $f(U) \subseteq R$. Hence f is weakly $g^{\star}bp$-continuous. □

Theorem 5.8. *Let $f : X \to Y$ be a function. If the inverse image of each regular closed set of Y is a $g^{\star}b$-open set in X, then f is weakly $g^{\star}bp$-continuous.*

Proof. Let A be any preopen set of Y. Then $Cl(A)$ is a regular closed set in Y. By hypothesis, we have $f^{-1}(Cl(A))$ is a $g^{\star}b$-open set in X. Therefore by theorem5.2, f is weakly $g^{\star}bp$-continuous. □

Corollary 5.9. *Let $f : X \to Y$ be a function. If the inverse image of each regular open set of Y is a $g^{\star}b$-closed set in X, then f is weakly $g^{\star}bp$-continuous.*

Proof. Follows from Theorem 5.8 □

Theorem 5.10. *Let $f : X \to Y$ be weakly $g^{\star}bp$-continuous function, if A is $g^{\star}b$-closed subset of X, then the restriction $f|A : A \to Y$ is weakly $g^{\star}bp$-continuous in the subspace A.*

Proof. Let $x \in A$ and B be a preclosed set of Y containing $f(x)$. Since f is weakly $g^{\star}bp$-continuous, so by Corollary 5.9, $f^{-1}(IntB)$ is $g^{\star}b$-closed set in X, and $(f|A)^{-1}(IntB) = A \cap f^{-1}(IntB)$ is $g^{\star}b$-closed in X. Hence, by Theorem 3.30 [43], $(f|A)^{-1}(IntB)$ is $g^{\star}b$-closed in A. Therefore, $f|A$ is weakly $g^{\star}bp$-continuous. □

Theorem 5.11. *Let $f : X \to Y$ be weakly $g^{\star}bp$-continuous function and for each $x \in X$. If Y is any subset of Z containing $f(x)$, then $f : X \to Z$ is weakly $g^{\star}bp$-continuous.*

Proof. Let $x \in X$ and A be any preopen set of Z containing $f(x)$. Then $A \cap Y$ is preopen in Y containing $f(x)$. Since $f : X \to Y$ is weakly $g^{\star}bp$-continuous, there exists a $g^{\star}b$-open set U of X containing x such that $f(U) \subseteq Cl(A \cap Y)$ and hence $f(U) \subseteq ClA$. Therefore, $f : X \to Z$ is weakly $g^{\star}bp$-continuous. □

Theorem 5.12. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition function $g \circ f : X \to Z$ is weakly $g^{\star}bp$-continuous if f is $g^{\star}b$-irresolute and g is weakly $g^{\star}bp$-continuous.*

Proof. Let $x \in X$ and A be preopen set of Z containing $g(f(x))$. Since g is weakly $g^{\star}bp$-continuous, there exists a $g^{\star}b$-open set U of Y containing $f(x)$ such that $g(U) \subseteq ClA$. It is clear that $g^{-1}(ClA)$ is $g^{\star}b$-open set of Y containing $f(x)$. Since f is $g^{\star}b$-continuous, then $f^{-1}(g^{-1}(ClA)) = (g \circ f)^{-1}(ClA)$ is $g^{\star}b$-open set in X containing x and Clearly $(g \circ f)(g \circ f)^{-1}(ClA) \subseteq ClA$. Hence $(g \circ f)$ is weakly $g^{\star}bp$-continuous. □

Theorem 5.13. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) *f is weakly $g^{\star}bp$-continuous.*
(2) *$g^{\star}bClf^{-1}(IntpClB) \subseteq f^{-1}(pClB)$,for each $B \subseteq Y$.*
(3) *$f^{-1}(pIntB) \subseteq g^{\star}bIntf^{-1}(ClpIntB)$,for each $B \subseteq Y$.*
(4) *$f^{-1}(pIntpClA) \subseteq g^{\star}bIntf^{-1}(ClA)$,for each preopen set A of Y.*
(5) *$f^{-1}(A) \subseteq g^{\star}bIntf^{-1}(ClA)$,for each regular preopen set A of Y.*
(6) *$g^{\star}bClf^{-1}(IntF) \subseteq f^{-1}(F)$,for each regular preclosed set F of Y.*
(7) *$g^{\star}bClf^{-1}(IntF) \subseteq f^{-1}(ClIntF)$,for each preclosed set F of Y.*
(8) *$g^{\star}bClf^{-1}(A) \subseteq f^{-1}(ClA)$,for each preopen set A of Y.*
(9) *$f^{-1}(IntF) \subseteq g^{\star}bIntf^{-1}(F)$,for each preclosed set F of Y.*

Proof. (1) ⇒ (2). Let B be any subset of Y. Assume that $x \notin f^{-1}(pClB)$. Then $f(x) \notin pClB$ and there exists a preopen set A containing $f(x)$ such that $A \cap B = \phi$, hence $A \cap IntpClB = \phi$,then $A \subseteq (IntpClB)^c$ then

$ClA \cap IntpClB = \phi$ By(1), there exists a $g^{\star}b$-open set U of X containing x such that $f(U) \subseteq ClA$. Therefore, we have $f(U) \cap IntpClB = \phi$ which implies $U \cap f^{-1}(IntpClB) = \phi$ and hence $x \notin g^{\star}bClf^{-1}(IntpClB)$. Therefore, we obtain $g^{\star}bClf^{-1}(IntpClB) \subseteq f^{-1}(pClB)$.
(2) $\Rightarrow$ (3). Let B be any subset of Y. Then apply(2) to $Y \setminus B$, we obtain $g^{\star}bClf^{-1}(IntpCl(Y \setminus B)) \subseteq f^{-1}(pCl(Y \setminus B)) \Rightarrow g^{\star}bClf^{-1}(Int(Y \setminus pIntB)) \subseteq f^{-1}(Y \setminus pIntB) \Rightarrow g^{\star}bClf^{-1}(Y \setminus ClpIntB) \subseteq f^{-1}(Y \setminus pIntB) \Rightarrow g^{\star}bCl(X \setminus f^{-1}(ClpIntB) \subseteq X \setminus f^{-1}(pIntB) \Rightarrow X \setminus g^{\star}bInt(f^{-1}(ClpIntB)) \subseteq X \setminus f^{-1}(pIntB) \Rightarrow f^{-1}(pIntB) \subseteq g^{\star}bInt(f^{-1}(ClpIntB))$. Therefore, we obtain $f^{-1}(pIntB) \subseteq g^{\star}bInt(f^{-1}(ClpIntB))$.
(3) $\Rightarrow$ (4). Let A be any preopen set of Y. Then apply(3) to $pClA$, we obtain $f^{-1}(pIntpClA) \subseteq g^{\star}bInt(f^{-1}(ClpIntpClA)) \subseteq g^{\star}bInt(f^{-1}(ClIntClA)) = g^{\star}bIntf^{-1}(ClA)$. Therefore we obtain $f^{-1}(pIntpClA) \subseteq g^{\star}bIntf^{-1}(ClA)$.
(4) $\Rightarrow$ (5). Let A be any regular preopen set of Y. Then A is preopen set of Y. By(4), we have $f^{-1}(A) = f^{-1}(pIntpClA) \subseteq g^{\star}bIntf^{-1}(ClA)$. Therefore we obtain $f^{-1}(A) \subseteq g^{\star}bIntf^{-1}(ClA)$.
(5) $\Rightarrow$ (6). Let F be any regular preclosed set of Y. Then $Y \setminus F$ is a regular preopen set of Y. By(5), we have $f^{-1}(Y \setminus F) \subseteq g^{\star}bIntf^{-1}(Cl(Y \setminus F)) \Rightarrow X \setminus f^{-1}(F) \subseteq g^{\star}bIntf^{-1}(Y \setminus IntF) \Rightarrow X \setminus f^{-1}(F) \subseteq g^{\star}bInt(X \setminus f^{-1}(IntF)) \Rightarrow X \setminus f^{-1}(F) \subseteq X \setminus g^{\star}bClf^{-1}(IntF) \Rightarrow g^{\star}bClf^{-1}(IntF) \subseteq f^{-1}(F)$. Hence $g^{\star}bClf^{-1}(IntF) \subseteq f^{-1}(F)$.
(6) $\Rightarrow$ (7). Let F be any preclosed set of Y. Then $pClpIntF$ is regular preclosed set of Y. By(6), we have $g^{\star}bClf^{-1}(IntpClpIntF) = g^{\star}bClf^{-1}(IntF) \subseteq f^{-1}(pClpIntF)$. Therefore we obtain $g^{\star}bClf^{-1}(IntF) \subseteq f^{-1}(pClpIntF)$.
(7) $\Rightarrow$ (8). Let A be any preopen set of Y. Then by(7) we have $g^{\star}bClf^{-1}(A) \subseteq g^{\star}bClf^{-1}(IntClA) \subseteq f^{-1}(pClpIntClA) \subseteq f^{-1}(ClIntClA) = f^{-1}(ClA)$. Therefore,$g^{\star}bClf^{-1}(A) \subseteq f^{-1}(ClA)$.
(8) $\Rightarrow$ (9). Let F be any preclosed set of Y. Then $Y \setminus F$ is preopen set of Y. By(8), we have $g^{\star}bClf^{-1}(Y \setminus F) \subseteq f^{-1}(Cl(Y \setminus F)) \Rightarrow g^{\star}bCl(X \setminus f^{-1}(F) \subseteq f^{-1}(Y \setminus IntF) \Rightarrow X \setminus g^{\star}bIntf^{-1}(F) \subseteq X \setminus f^{-1}(IntF) \Rightarrow f^{-1}(IntF) \subseteq g^{\star}bIntf^{-1}(F)$. Therefor $f^{-1}(IntF) \subseteq g^{\star}bIntf^{-1}(F)$.
(9) $\Rightarrow$ (1). Let $x \in X$ and let A be any preopen set in Y containing $f(x)$. Then $x \in f^{-1}(A)$ and ClA is a closed set, hence preclosed, in Y. By (9), we have $x \in f^{-1}(A) \subseteq f^{-1}(IntClA) \subseteq g^{\star}bIntf^{-1}(ClA)$. If we put $U = g^{\star}bIntf^{-1}(ClA)$, then we obtain that $x \in U$ and $f(U) \subseteq ClA$. Therefore,, f is weakly $g^{\star}bp$-continuous.

□

Theorem 5.14. *The followings are equivalent for a function $f : X \to Y$.*

(1) *f is weakly $g^{\star}bp$-continuous.*
(2) *$f(g^{\star}bCl(A)) \subseteq Cl_{\theta}(f(A))$ for each subset A of X.*
(3) *$g^{\star}bCl(f^{-1}(B)) \subseteq f^{-1}(Cl_{\theta}(B))$ for each subset B of Y.*
(4) *$g^{\star}bCl(f^{-1}(Int(Cl_{\theta}(B)))) \subseteq f^{-1}(Cl_{\theta}(B))$ for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2). Let A be any subset of X. Suppose that $f(g^{\star}bCl(A)) \nsubseteq Cl_{\theta}(f(A))$. Then there exists $y \in f(g^{\star}bCl(A))$ such that $y \notin Cl_{\theta}(f(A))$,

then there exists an open set G in Y containing y such that $ClG \cap f(A) = \phi$. If $f^{-1}(y) = \phi$, then there is nothing to prove. Suppose that x be any arbitrary point of $f^{-1}(y)$, so $f(x) \in G$. Since G is open then its preopen set in Y, by(1), there exists a $g^{\star}b$-open set U of X containing x such that $f(U) \subseteq Cl(G)$. Therefore, we have $f(U) \cap f(A) = \phi$. Then $x \notin g^{\star}bCl(A)$. Hence $y \notin f(g^{\star}bCl(A))$ which is a contradiction. Then $f(g^{\star}bCl(A)) \subseteq Cl_{\theta}(f(A))$.
(2) $\Rightarrow$ (3). Let B be any subset of Y. Set $A = f^{-1}(B)$ in (2) then we have $f(g^{\star}bCl(f^{-1}(B))) \subseteq Cl_{\theta}(B)$ and $g^{\star}bCl(f^{-1}(B)) \subseteq f^{-1}(Cl_{\theta}(B))$.
(3) $\Rightarrow$ (4). Let B be any subset of Y. Since $Cl_{\theta}(B)$ is closed in Y hence is preclosed in Y. We have $g^{\star}bCl(f^{-1}(Int(Cl_{\theta}(B)))) \subseteq f^{-1}(Cl_{\theta}(Int(Cl_{\theta}(B)))) \subseteq f^{-1}(Cl(Int(Cl_{\theta}(B)))) \subseteq f^{-1}(Cl_{\theta}(B))$.
(4) $\Rightarrow$ (1). Let G be any preopen set of Y, then $G \subseteq IntCl(G)$. Apply(4) to $IntCl(G)$, we get $g^{\star}bClf^{-1}(IntCl_{\theta}(IntCl(G))) \subseteq f^{-1}(Cl_{\theta}(IntCl(G)))$. By Theorem 2.13, we have $g^{\star}bClf^{-1}(IntCl(G)) \subseteq f^{-1}(Cl(IntCl(G)))$. So, we get, $g^{\star}bCl(f^{-1}(G)) \subseteq g^{\star}bClf^{-1}(IntCl(G)) \subseteq f^{-1}(Cl(IntCl(G))) \subseteq f^{-1}(ClG)$. Hence, by Theorem 5.13, f is weakly $g^{\star}bp$-continuous. □

Corollary 5.15. *If a function $f : X \to Y$ is weakly $g^{\star}bp$-continuous, then $f^{-1}(A)$ is $g^{\star}b$-closed in X for every θ-closed set A in Y.*

Proof. If A is θ-closed, so by Theorem 5.14, we obtain that $g^{\star}bCl(f^{-1}(A)) \subseteq f^{-1}(Cl_{\theta}A) = f^{-1}(A)$. Therefore, $f^{-1}(A)$ is $g^{\star}b$-closed. □

Corollary 5.16. *Let $f : X \to Y$ be any function. If $f^{-1}(Cl_{\theta}(B))$ is $g^{\star}b$-closed in X for every subset B of Y, then $f : X \to Y$ is weakly $g^{\star}bp$-continuous .*

Proof. Since $f^{-1}(Cl_{\theta}(B))$ is $g^{\star}b$-closed in X, we have $g^{\star}bCl(f^{-1}(B)) \subseteq g^{\star}bClf^{-1}(Cl_{\theta}(B)) = f^{-1}(Cl_{\theta}(B))$. Therefore, by Theorem 5.14, f is weakly $g^{\star}bp$-continuous. □

Theorem 5.17. *A function $f : X \to Y$ is is weakly $g^{\star}bp$-continuous if and only if $f^{-1}(A) \subseteq g^{\star}bIntf^{-1}(Cl(A))$ for each preopen set A in Y.*

Proof. **Necessity**. Let f be weakly $g^{\star}bp$-continuous and A be any preopen set of Y, then $A \subseteq IntCl(A)$. Therefore, by Theorem 5.13, we get $f^{-1}(A) \subseteq f^{-1}(IntCl(A)) \subseteq g^{\star}bIntf^{-1}(Cl(A))$. Hence, $f^{-1}(A) \subseteq g^{\star}bIntf^{-1}(Cl(A))$.
Sufficiency. Let A be any regular preopen set of Y, then A is preopen set in Y. By hypothesis, we have $f^{-1}(A) \subseteq g^{\star}bIntf^{-1}(Cl(A))$. Therefore, by Theorem 5.13, f is weakly $g^{\star}bp$-continuous. □

Corollary 5.18. *A function $f : X \to Y$ is is weakly $g^{\star}bp$-continuous if and only if $g^{\star}bClf^{-1}(Int(F)) \subseteq f^{-1}(F)$ for each preopen set F in Y.*

Theorem 5.19. *If $f : X \to Y$ is a weakly $g^{\star}bp$-continuous function and Y is extremally disconnected space, then f is almost $g^{\star}bp$-continuous.*

Proof. Let $x \in X$ and let A be any preopen set of Y containing $f(x)$. Since f is weakly $g^{\star}bp$-continuous, there exists a $g^{\star}b$-open set U of X containing x

such that $f(U) \subseteq Cl(A)$. Since Y is extremally disconnected, then $f(U) \subseteq IntCl(A)$. Therefore, f is almost $g^{\star}bp$-continuous. □

Theorem 5.20. *If $f : X \to Y$ is weakly $g^{\star}bp$-continuous injection and Y is pre-T_1 then X is $g^{\star}b - T_1$.*

Proof. Assume that Y is pre-T_1. For any distinct points x and y in X, there exists preopen set A and W such that $f(x) \in A$, $f(y) \notin A$, $f(x) \notin W$ and $f(y) \in W$. Since f is weakly $g^{\star}bp$-continuous, there exists a $g^{\star}b$-open sets G and H in X containing x and y respectively, such that $f(G) \subseteq Cl(U)$, $f(H) \subseteq Cl(A)$, $f(H) \subseteq Cl(W)$ since A and W are disjoint then $Cl(A)$ and $Cl(W)$ are disjoint. Thus we obtain $y \notin G$, $x \notin H$. This show that X is $g^{\star}b - T_1$. □

Theorem 5.21. *If $f : X \to Y$ is weakly $g^{\star}bp$-continuous injection and Y is pre-T_2, then X is $g^{\star}b - T_2$.*

Proof. For any pair of distinct points x and y in X, there exist disjoint preopen sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is weakly $g^{\star}bp$-continuous, there exist $g^{\star}b$-open sets G and H in X containing x and y, respectively, such that $f(G) \subseteq Cl(U)$ and $f(H) \subseteq Cl(V)$. Since U and V are disjoint, we have $Cl(U) \cap Cl(V) = \phi$, hence $G \cap H = \phi$. This shows that X is $g^{\star}b - T_2$. □

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