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OPERATION APPROACH OF g^*-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

S. JAFARI[1], A. SELVAKUMAR[2], M. PARIMALA[3]

Abstract. In this article we introduce $(\mathcal{I},\gamma)$-g^*-closed sets in topological spaces and also introduce γg^*-$T_{\mathcal{I}}$-spaces and investigate some of their properties.

1. Introduction

The concept of generalized closed sets and semi-open sets in topological space was introduced by Levine [8], [9]. Further, M.K.R.S. Veerakumar [14] introduced and investigate the notion of g^*-closed sets in topological spcaes. Julian Dontchev et. al. [4] introduced the notion of the generalized closed sets in ideal topological space (i.e. $\mathcal{I}$-g-closed sets). In this paper we introduce $(\mathcal{I},\gamma)$-g^*-closed sets and γg^*-$T_{\mathcal{I}}$ space and discussed some of their applications.

An ideal $\mathcal{I}$ on a topological space (X,τ) is a non-empty collection of subsets of X satisfying the following two properties:

(1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$
(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$

For a subset $A \subset X$, $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau(x)\}$ is called the local function of A with respect to $\mathcal{I}$ and τ. Recall that $A \subset (X,\tau,\mathcal{I})$ is called τ^*-closed [6] if $A^* \subset A$. It is well known that $\mathrm{cl}^*(A) = A \cup A^*$ [13] defines a Kuratowski closure operator for a topology $\tau^*(I)$, finer than τ.

An operation γ [7,11] on the topology τ on a given topological space (X,τ) is a function from the topology itself into the power set $P(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. The following operators are examples of the operation γ: the closure operator γ_{cl} defined by $\gamma(U) = \mathrm{cl}\,(U)$, the interior-closure operator γ_{ic} defined by $\gamma(U) = \mathrm{int}(\mathrm{cl}(\mathrm{U}))$ and the identity operator γ_{id} defined by $\gamma(U) = U$. Another example of the operation γ is the γ_f-operator defined by $U^{\gamma_f} = (FrU)^c = X/FrU$ [12] where FrU denotes frontier of U and $(FrU)^c$ denotes complement of frontier of U. Two operators γ_1 and γ_2 are called mutually dual [12] if $U^{\gamma_1} \cap U^{\gamma_2} = U$ for each $U \in \tau$. For example the identity operator is mutually dual to any other operator, while the γ_f-operator is mutually dual to the closure operator [12].

Definition 1.1. A subset A of a space (X,τ) is called

(a) an α-open set [10] if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$.
(b) a semi-open set [9] if $A \subset \mathrm{cl}(\mathrm{int}(A))$.

Definition 1.2. A subset A of a space $(X,\tau,\mathcal{I},\gamma)$ is called

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(a) a generalized closed (briefly g-closed) set [8] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ).
(b) an $(\mathcal{I}, \gamma)$-generalized closed (briefly $(\mathcal{I}, \gamma)$-g-closed) set [4] if $A^* \subset U^\gamma$ whenever $A \subset U$ and U is open in (X, τ).
(c) an $(\mathcal{I}, \gamma)$-generalized semi-closed (briefly $(\mathcal{I}, \gamma)$-gs-closed) set [2] if $A^* \subset U^\gamma$ whenever $A \subset U$ and U is semi-open in (X, τ).
(d) an $(\mathcal{I}, \gamma)$-generalized α-closed (briefly $(\mathcal{I}, \gamma)$-$g\alpha$-closed) set [1] if $A^* \subset U^\gamma$ whenever $A \subset U$ and U is α-open in (X, τ).

We denote the family of all $(\mathcal{I}, \gamma)$-g^*-closed subsets of a space $(X, \tau, \mathcal{I}, \gamma)$ by $IG^*(X)$ and simply write $\mathcal{I}$-g^*-closed in case when γ is an identity operator. Throughout this paper the operator γ is defined as $\gamma : \tau^g \to P(X)$, where τ^g denotes the set of all g-open sets of (X, τ).

2. Basic properties of $(\mathcal{I}, \gamma)$- g^*-closed sets

Definition 2.1. A subset A of a topological space (X, τ) is called $(\mathcal{I}, \gamma)$-g^*-closed if $A^* \subset U^\gamma$, whenever $A \subset U$ and U is g-open in (X, τ).

Example 2.1. Let $X = \{a, b, c, d, e\}$, $\tau = \{X, \phi, \{c\}, \{a, b\}, \{a, b, c\}\}$, $\mathcal{I} = \{\phi\}$ and $U^\gamma = \mathrm{int}(\mathrm{cl}(U))$. Here $(\mathcal{I}, \gamma)$-g^*-closed sets are $X, \phi, \{a, c\}, \{b, c\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, c, e\}, \{b, c, d\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, c, d, e\}, \{a, b, d, e\}, \{a, b, c, e\}, \{b, c, d, e\}$.

Theorem 2.2. (a) *Every g^*-closed set is $\mathcal{I}$-g^*-closed.*
(b) *Every $(\mathcal{I}, \gamma)$-g^*-closed set is $(\mathcal{I}, \gamma)$-g-closed.*
(c) *Every $(\mathcal{I}, \gamma)$-g^*-closed set is $(\mathcal{I}, \gamma)$-gs-closed and hence $(\mathcal{I}, \gamma)$-$g\alpha$-closed.*
The converse of the above theorem need not be true by the following example.

Example 2.3. (a) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Here $A = \{a\}$ is $\mathcal{I}$-g^*-closed but not g^*-closed.
(b) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$ and γ = identity. Here $A = \{a, b\}$ is $(\mathcal{I}, \gamma)$-g-closed but not $(\mathcal{I}, \gamma)$-g^*-closed.
(c) Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$, $\mathcal{I} = \{\phi, \{a\}\}$ and γ =identity. Here $A = \{a, b\}$ is $(\mathcal{I}, \gamma)$-gs-closed and $(\mathcal{I}, \gamma)$-$g\alpha$-closed but not $(\mathcal{I}, \gamma)$-g^*-closed.

Theorem 2.4. *If A is $\mathcal{I}$-g^*-closed and g-open, then A is τ^*-closed.*

Proof. Since A is $\mathcal{I}$-g^*-closed, then $A^* \subset U$, U is g-open. It is given that A is g-open implies $A^* \subset A$. Hence A is τ^*-closed. □

Lemma 2.5. *[5, Theorem II3] Let $(A_i)_{i \in I}$ be a locally finite family of sets in $(X, \tau, \mathcal{I})$. Then $\cup_{i \in I} A_i^*(\mathcal{I}) = (\cup_{i \in I} A_i)^*(\mathcal{I})$.*

Theorem 2.6. *Let $(X, \tau, \mathcal{I}, \gamma)$ be a topological space.*
(a) *If $(A_i)_{i \in I}$ is a locally finite family of sets and each $A_i \in IG^*(X)$, then $\cup_{i \in I} A_i \in IG^*(X)$.*
(b) *Finite intersection of $(\mathcal{I}, \gamma)$-g^*-closed sets need not be $(\mathcal{I}, \gamma)$-g^*-closed.*

Proof. (a) Let $\cup_{i \in I} A_i \subset U$, where U is g-open. Since $A_i \in IG^*(X)$ for each $i \in I$, then $A_i^* \subset U^\gamma$. Hence $\cup_{i \in I} A_i^* \subset U^\gamma$. By Lemma 2.6, $(\cup_{i \in I} A_i)^* = \cup_{i \in I} A_i^*$, then $(\cup_{i \in I} A_i)^* \subset U^\gamma$. Hence $\cup_{i \in I} A_i \in IG^*(x)$.
(b) Let $X = \{a, b, c, d, e\}$, $\tau = \{X, \phi, \{c\}, \{a, b\}, \{a, b, c\}\}$, $\mathcal{I} = \{\phi\}$ and $\gamma = \gamma_{ic}$. Set $A = \{a, c\}$ and $B = \{b, c\}$. Clearly $A, B \in IG^*(X)$ but $A \cap B = \{c\} \notin IG^*(X)$. □

Lemma 2.7. *[4] If A and B are subsets of $(X, \tau, \mathcal{I})$, then $(A \cap B)^*(\mathcal{I}) \subset A^*(\mathcal{I}) \cap B^*(\mathcal{I})$.*

Theorem 2.8. *Let $(X, \tau, \mathcal{I}, \gamma_{id})$ be a topological space. If $A \subset X$ is $\mathcal{I}$-g^*-closed and B is closed and τ^*-closed, then $A \cap B$ is $\mathcal{I}$-g^*-closed.*

Proof. Let $U \in \tau^g$ be such that $A \cap B \subset U$. Then $A \subset U \cap (X \backslash B)$. Since A is $\mathcal{I}$-g^*-closed, then $A^* \subset U \cap (X \backslash B)$. Hence $B \cap A^* \subset U \cap B \subset U$, but we know that $B^* \subset B$. Therefore $(A \cap B)^* \subset A^* \cap B^* \subset A^* \cap B \subset U$, by Lemma 2.7. Hence $A \cap B$ is $\mathcal{I}$-g^*-closed. □

Theorem 2.9. *Let A be a subset of $(X, \tau, \mathcal{I}, \gamma_{id})$. Then, A is $\mathcal{I}$-g^*-closed if and only if $A^* - A$ does not contain any non-empty closed subset.*

Proof. (Necessity) Assume that F is a closed subset of $A^* - A$. Note that clearly $A \subset X - F$, where A is $\mathcal{I}$-g^*-closed and $X - F \in \tau$. Then $A^* \subset X - F$, that is $F \subset X - A^*$. Since due to our assumption $F \subset A^*$, $F \subset (X - A^*) \cap A^* = \phi$.

(Sufficiency) Let U be an open subset and hence g-open subset containing A. Since A^* is closed [6, Theorem 2.3(c)] and $A^* \cap (X - U) \subset A^* - A$ holds, then $A^* \cap (X - U)$ is a closed set contained in $A^* - A$. By assumption, $A^* \cap (X - U) = \phi$ and hence $A^* \subset U$. □

A subset S of a space $(X, \tau, \mathcal{I}, \gamma)$ is a topological space with an ideal $\mathcal{I}_s = \{I \in \mathcal{I} : I \subset S\} = \{I \cap S : I \in \mathcal{I}\}$ on S [3].

Lemma 2.10. *[4] Let $(X, \tau, \mathcal{I})$ be a topological space and $A \subset S \subset X$. Then, $A^*(\mathcal{I}_S, \tau/S) = A^*(\mathcal{I}, \tau) \cap S$ holds.*

Theorem 2.11. *Let $A \subset S \subset (X, \tau, \mathcal{I}, \gamma_{id})$. If A is $\mathcal{I}_S$-g^*-closed in $(S, \tau/S, \mathcal{I}_S, \gamma_{id})$ and S is closed in (X, τ), then A is $\mathcal{I}$-g^*-closed in $(X, \tau, \mathcal{I}, \gamma_{id})$.*

Proof. Let $A \subset U$, where $U \in \tau^g$. Let $x \notin U$. We consider the following two cases.
Case(i) $x \in S$. By assumption, $A^*(\mathcal{I}_S, \tau/S) \subset U \cap S \subset U$. We show that $A^*(\mathcal{I}) \subset A^*(\mathcal{I}_S, \tau/S)$. Let $x \notin A^*(\mathcal{I}_S, \tau/S)$. Since $x \in S$, then for some open subset V_S of $(S, \tau/S)$ containing x, we have $V_S \cap A \in \mathcal{I}_S$, since $V_S = V \cap S$ for some $V \in \mathcal{I}$, then $(S \cap V) \cap A \in \mathcal{I}_S \subset \mathcal{I}$, that is $V \cap A \in \mathcal{I}$ for some $V \in \tau$ containing x. This shows that $x \notin A^*(\mathcal{I})$. Hence $A^*(\mathcal{I}) \subset U$.
Case(ii) $x \notin S$. Then X/S is an open neighbourhood of x disjoint from A. Hence $x \notin A^*(\mathcal{I})$. Consequently $A^*(\mathcal{I}) \subset U$.

Both cases we show that the local function of A with respect to $\mathcal{I}$ and τ is in U. Hence A is $\mathcal{I}$-g^*-closed in $(X, \tau, \mathcal{I}, \gamma_{id})$. □

Theorem 2.12. *Let $(X, \tau, \mathcal{I}, \gamma_{id})$ be a topological space and $A \subset S \subset X$. If A is $\mathcal{I}_S$-g^*-closed in $(S, \tau/S, \mathcal{I}_S, \gamma_{id})$ and S is $\mathcal{I}$-g^*-closed in X, then A is $\mathcal{I}$-g^*-closed in X.*

Proof. Let $A \subset U$ and $U \in \tau^g$. By assumption and Lemma 2.10, $A^*(\mathcal{I}, \tau) \cap S \subset U \cap S$. Then we have $S \subset U \cup (X/A^*(\mathcal{I}, \tau))$. Since $X/A^*(\mathcal{I}, \tau) \in \tau^g$, then $A^*(\mathcal{I}, \tau) \subset S^*(\mathcal{I}, \tau) \subset U \cup (X/A^*(\mathcal{I}, \tau))$. Therefore, we have that $A^*(\mathcal{I}, \tau) \subset U$ and hence A is $\mathcal{I}$-g^*-closed in X. □

Corollary 2.13. *Let $(X, \tau, \mathcal{I}, \gamma_{id})$ be a topological space and A and F subsets of X. If A is $\mathcal{I}$-g^*-closed and F is closed in (X, τ), then $A \cap F$ is $\mathcal{I}$-g^*-closed.*

Proof. Since $A \cap F$ is closed in $A, \tau/A$, then $A \cap F$ is $\mathcal{I}_A$-g^*-closed in $(A, \tau/A, \mathcal{I}_A)$. By Theorem 2.13, $A \cap F$ is $\mathcal{I}$-g^*-closed. □

Theorem 2.14. *Let $A \subset S \subset (X, \tau, \mathcal{I}, \gamma)$. If $A \in IG^*(X)$ and $S \in \tau^g$, then $A \in IG^*(S)$.*

Proof. Let U be a g-open subset of $(S, \tau/S)$ such that $A \subset U$. Since $S \in \tau^g$, then $U \in \tau^g$. Then $A^*(\mathcal{I}) \subset U^\gamma$, since $A \in IG^*(X)$. By Lemma 2.10, we have $A^*(\mathcal{I}_S, \tau/S) \subset U^{\gamma/S}$, where $U^{\gamma/S}$ means the image of the operation $\gamma/S : \tau^g/S \to P(S)$, defined by $(\gamma/S)(U) = \gamma(U) \cap S$ for each $U \in \tau^g/S$. Hence $A \in IG^*(S)$. □

Theorem 2.15. *If the set $A \subset (X, \tau, \mathcal{I})$ is both $(\mathcal{I}, \gamma_1)$-g^*-closed and $(\mathcal{I}, \gamma_2)$-g^*-closed, then it is $\mathcal{I}$-g^*-closed, granted the operators γ_1 and γ_2 are mutually dual.*

Proof. Let $A \subset U$, where $U \in \tau^g$. Since $A^* \subset U^{\gamma_1}$ and $A^* \subset U^{\gamma_2}$, then $A^* \subset U^{\gamma_1} \cap U^{\gamma_2} = U$, since γ_1 and γ_2 are mutually dual. Hence A is $\mathcal{I}$-g^*-closed. □

Theorem 2.16. *Every set $A \subset (X, \tau, \mathcal{I})$ is $(\mathcal{I}, \gamma_{cl})$-$g^*$-closed.*

Proof. Let $A \subset U$, U is g-open. We know that $A \cup A^* = \text{cl}^*(A) \subset \text{cl}(A) \subset \text{cl}(U)$. This implies that $A^* \subset \text{cl}(U)$. Hence A is (I, γ_{cl})-g^*-closed. □

Corollary 2.17. *For a set $A \subset (X, \tau, \mathcal{I}, \gamma)$, the following conditions are equivalent.*

(a) *A is $(\mathcal{I}, \gamma_f)$-g^*-closed.*
(b) *A is $\mathcal{I}$-g^*-closed.*

Proof. **(a) ⇒ (b)** By Theorem 2.16, A is $(\mathcal{I}, \gamma_{cl})$-$g^*$-closed. Since γ_f and γ_{cl} are mutually dual due to [12], then $\gamma_f(U) \cap \gamma_{cl}(U) = U$. This implies that $A^* \subset U$, that is, A is $\mathcal{I}$-g^*-closed.
(b) ⇒ (a) Let $A \subset U$, U is g-open. Since A is $\mathcal{I}$-g^*-closed, $A^* \subset U$. But we know that $U \subset U^{\gamma_f}$, we have $A^* \subset U^{\gamma_f}$, this implies that A is $(\mathcal{I}, \gamma_f)$-g^*- closed. □

3. γg^*-$T_{\mathcal{I}}$-SPACE

Definition 3.1. A space $(X, \tau, \mathcal{I}, \gamma)$ is called an γg^*-$T_{\mathcal{I}}$-space if every $(\mathcal{I}, \gamma)$-g^*-closed subset of X is an τ^*-closed. We use the simple notation $g^*T_{\mathcal{I}}$-space, in case γ is the identity operator.

Theorem 3.1. *For a space $(X, \tau, \mathcal{I})$, the following conditions are equivalent.*

(a) *X is a g^*T_I-space.*
(b) *Each singleton of X is either closed or τ^*-open.*

Proof. **(a) ⇒ (b)** Let $x \in X$. If $\{x\}$ is not closed, then $A = X \backslash \{x\} \notin \tau$ and then A is trivially $\mathcal{I}$-g^*-closed. By (a), A is τ^*-closed. Hence $\{x\}$ is τ^*-open.
(b) ⇒ (a) Let A be a $\mathcal{I}$-g^*-closed and let $x \in \text{cl}^*(A)$. We have the following two cases.
case(i). $\{x\}$ is closed. By Theorem 2.10, $A^* - A$ does not contain any non-empty closed subset. This shows that $x \in A$.
case(ii). $\{x\}$ is τ^*-open. Then $\{x\} \cap A \neq \phi$. Hence $x \in A$.

Thus in both cases $\{x\}$ is in A and so $A = \text{cl}^* A$, that is A is τ^*-closed, which shows that X is a $g^*T_{\mathcal{I}}$-space. □

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[1] College of Vestsjaelland Syd, Herrestraede 11, 4200 Slagelse, Denmark.
[2] Department of Mathematics, Info Institute of Engineering, Coimbatore, Tamilnadu, India
[3] Department of Mathematics, Bannari Amman Institute of Technology, Sathyamangalam, Erode, Tamilnadu, India
E-mail address: [1] jafari@stofanet.dk, [2] selvam_mphil@yahoo.com, [3] rishwathpari@gmail.com