Selected papers of the 2014 International Conference on Topology and its Applications

2014 International Conference on Topology and its Applications,
July 3-7, 2014, Nafpaktos,
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Editors

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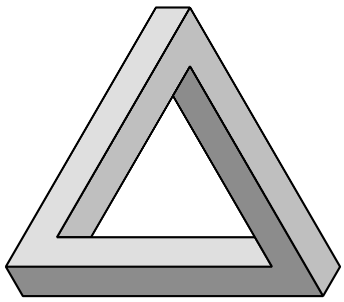

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**ISBN:** 978-618-80609-1-3

# Preface

The **2014 International Conference on Topology and its Applications** took place from July 3 to 7 in the **3$^{rd}$ High School of Nafpaktos, Greece**. It covered all areas of Topology and its Applications (especially General Topology, Set-Theoretic Topology, Geometric Topology, Topological Groups, Dimension Theory, Dynamical Systems and Continua Theory, Computational Topology, History of Topology). This conference was attended by 235 participants from 44 countries and the program consisted by 147 talks.

The Organizing Committee consisted of S.D. Iliadis (University of Patras), D.N. Georgiou (University of Patras), I.E. Kougias (Technological Educational Institute of Western Greece), A.C. Megaritis (Technological Educational Institute of Western Greece), and I. Boules (Mayor of the city of Nafpaktos).

**The Organizing Committee is very much indebted to the City of Nafpaktos for its hospitality and for its excellent support during the conference**.

The conference was sponsored by University of Patras, Technological Educational Institute of Western Greece, Municipality of Nafpaktos, New Media Soft – Internet Solutions, Loux Marlafekas A.B.E.E., TAXYTYPO – TAXYEKTYPOSEIS GRAVANIS EPE, Alpha Bank, and Wizard Solutions LTD.

This volume is a special volume under the title: "Selected papers of the 2014 International Conference on Topology and its Applications" which will be edited by the organizers (D.N. Georgiou, S.D. Iliadis, I.E. Kougias, and A.C. Megaritis) and published by the University of Patras. We thank the authors for their submissions.

*Editors*

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Selected papers of the 2014 International Conference on Topology and its Applications
(2015) Pages 49–59

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# On the class of semipre-$\theta$-open sets in topological spaces

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**Abstract**

In this paper we consider the class of $\beta\theta$-open sets in topological spaces and investigate some of their properties. We also present and study some weak separation axioms by involving the notion of $\beta\theta$-open sets.

*Key words:* $\beta\theta$-open set, $\beta\theta$-$D_1$ space, $\beta$-regular space, $\beta\theta$-$R$-continuous.
*1991 MSC:* 54C10.

## 1. Introduction and preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of semi-preopen sets (or $\beta$-open sets) introduced by Abd El-Monsef et al. [1] and Andrijević [2] introduced the notion of $\beta$-open set, which Andrijević called semi-preopen, completely independent of each other. In this paper, we adopt the word $\beta$-open for the sake of clarity. Noiri [7] used this notion and the semipre-closure [2] of a set to introduce the semipre-$\theta$-closure of a set. It is the object of this paper to further investigate the notion of $\beta\theta$-open sets by using the notion of the semipre-$\theta$-closure of a set. We also study some weak separation axioms defined by using the notion of $\beta\theta$-open sets.

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The $(X, \tau)$ and $(Y, \sigma)$ (or simply, $X$ and $Y$) denote topological spaces on which no separation axioms are assumed unless explicitly stated. A subset $A$ of a topological space $(X, \tau)$ is called $\beta$-open [1] if $A \subseteq Cl(Int(Cl(A)))$, where $Cl(A)$ and $Int(A)$ denote the closure and the interior of $A$, respectively. The complement of a $\beta$-open set is called $\beta$-closed [1]. The intersection of all $\beta$-closed sets containing $A$ is called the semipre-closure [1] of $A$ and is denoted by $spCl(A)$. A subset $A$ is called semipre-regular (or $\beta$-regular) if it is both $\beta$-open and $\beta$-closed. The family of all $\beta$-open (resp. $\beta$-closed, open, $\beta$-regular) subsets of $X$ is denoted by $\beta O(X, \tau)$ or $\beta O(X)$ (resp. $\beta C(X, \tau), O(X, \tau)$, $\beta R(X, \tau)$). We set

$$\beta O(X, x) = \{U : x \in U \in \beta O(X, \tau)\}$$

and

$$\beta C(X, x) = \{U : x \in U \in \beta C(X, \tau)\}.$$

Now we begin to recall some known notions which will be used in the sequel.

**Definition 1.1.** [7]. Let $A$ be a subset of $X$. The semipre-$\theta$-closure of $A$, denoted by $spCl_\theta(A)$, is the set of all $x \in X$ such that $\beta Cl(O) \cap A \neq \emptyset$ for every $O \in \beta O(X, x)$. A subset $A$ is called $\beta\theta$-closed if $A = spCl_\theta(A)$. The set $\{x \in X \mid \beta Cl(O) \subset A$ for some $O \in \beta O(X, x)\}$ is called the semipre-$\theta$-interior of $A$ and is denoted by $spInt_\theta(A)$. A subset $A$ is called $\beta\theta$-open if $A = spInt_\theta(A)$. The family of all $\beta\theta$-open (resp. $\beta\theta$-closed) subsets of $X$ is denoted by $\beta\theta O(X, \tau)$ or $\beta\theta O(X)$ (resp. $\beta\theta C(X, \tau)$). We set

$$\beta\theta O(X, x) = \{U : x \in U \in \beta\theta O(X, \tau)\}$$

and

$$\beta\theta C(X, x) = \{U : x \in U \in \beta\theta C(X, \tau)\}.$$

**Lemma 1.2.** [7]. For any subset $A$ of $X$, the following properties hold:
(1) $spCl_\theta(spCl_\theta(A)) = spCl_\theta(A)$.
(2) $spCl_\theta(A)$ is $\beta\theta$-closed.
(3) $spCl_\theta(A)$ is the intersection of all $\beta\theta$-closed sets containing $A$.
(4) $A \subset spCl(A) \subset spCl_\theta(A)$ and $spCl(A) = spCl_\theta(A)$ if $A \in \beta O(X)$.

Recall that a function $f : X \to Y$ is said to be:

(i) $\beta\theta$-continuous [7] if $f^{-1}(V)$ is $\beta\theta$-closed for every closed set $V$ in $Y$, equivalently if the inverse image of every open set $V$ in $Y$ is $\beta\theta$-open in $X$.

(ii) weakly $\beta$-irresolute ([7], Theorem 4.5) if $f^{-1}(V)$ is $\beta\theta$-open in $X$ for every $\beta\theta$-open set $V$ in $Y$, equivalently $spCl_\theta f^{-1}(V) \subset f^{-1}(spCl_\theta(V))$ for every subset $V$ of $Y$.

**2. Semipre-$\theta$-open sets**

We begin with the following result:

**Lemma 2.1.** Let $X$ be a topological space and $A \subset X$. The following statements hold:
(2) $X - spInt_\theta(A) = spCl_\theta(X - A)$.
(3) $spInt_\theta(spInt_\theta(A)) = spInt_\theta(A)$.
(4)$spInt_\theta(A)$ is $\beta\theta$-open.

**Lemma 2.2.** [7] For any subset $O$ of $X$, the following properties hold:
(1) $O$ is $\beta$-regular if and only if $O = spInt(spCl(O))$.
(2) $O$ is $\beta$-open if and only $spCl(O)$ is $\beta$-regular.

**Theorem 2.3.** If $O$ is $\beta$-open, then $spInt(spCl(O))$ is $\beta\theta$-open.

**Proof.** $spInt(spCl(O)) = (X - spCl(X - spCl(O)))$. Since $X - spCl(O)$ (=$A$, say) is $\beta$-open, $spCl(A) = pCl_\theta(A)$ (Lemma 1.2). Therefore there exists a subset $A = X - spCl(O)$ for which $X - spInt(spCl(O) = spCl_\theta(A)$. Hence $spInt(spCl(O)$ is $\beta\theta$-open. ■

**Corollary 2.4.** If $O$ is $\beta$-regular, then $O$ is $\beta\theta$-open.

**Proof.** It suffices to observe that, the subset $O$ is $\beta$-regular if and only if $O = spInt(spCl(O))$ (Lemma 2.2). ■

**Theorem 2.5.** For any subset $A$ of $X$, the following properties are equivalent:
(1) A subset $A$ is $\beta$-regular if and only if it is $\beta\theta$-open;
(2) $spCl_\theta(A)$ is $\beta$-regular for every subset $A$ of $X$.

**Proof.** (1) $\Rightarrow$ (2): Let $A$ be any subset of $X$. Then by Lemma 1.2, $X - spCl_\theta(A)$ is $\beta\theta$-open. This implies that $spCl_\theta(A)$ is $\beta$-regular.

(2) $\Rightarrow$ (1): Assume $spCl_\theta(O)$ is $\beta$-regular for every set $O$. Suppose $U$ is $\beta$-regular. Therefore $\beta$-regular is equivalent to $\beta\theta$-open. ■

**Theorem 2.6.** If $O$ is $\beta\theta$-open, then $O$ is the union of $\beta$-regular sets.

**Proof.** Since $O$ is $\beta\theta$-open, $O = spInt_\theta(O)$. For each $x \in O$, there exists $A_x \in \beta O(X)$ such that $x \in A_x \subset \beta Cl(A_x) \subset O$. Therefore, we obtain $O = \cup_{x \in O} \beta Cl(A_x)$ and $\beta Cl(A_x) \in \beta R(X)$. ■

**Corollary 2.7.** If $B$ is $\beta\theta$-closed, then $B$ is the intersection of $\beta$-regular sets.

**Remark 2.8.** (i) T. Noiri in ([7], Theorem 3.4) have proved that the intersection of arbitrary collection of $\beta\theta$-closed sets is $\beta\theta$-closed, hence by the complement, the union of arbitrary collection of $\beta\theta$-open sets is $\beta\theta$-open.
(ii) The intersection of $\beta\theta$-open sets may fail to be $\beta\theta$-open, as the following

example shows.

**Example 2.9.** Let $(X, \tau)$ be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then $O_1 = \{a, c\}$ and $O_2 = \{a, b\}$ are $\beta\theta$-open, but $O_1 \cap O_2 = \{a\}$ is not $\beta\theta$-open.

## 3. Semipre-$\theta$-$D_1$ Topological spaces

Now, we introduce new classes of topological spaces in terms of the concept of semipre-$\theta$-open sets.

**Definition 3.1.** A subset A of a topological space X is called a semipre-$\theta$ $D$-set if there are two sets $U, V \in \beta\theta O(X, \tau)$ such that $U \neq X$and $A = U - V$.

It is true that every $\beta\theta$-open set $U$ different from $X$ is a semipre-$\theta$ $D$-set if $A = U$ and $V = \emptyset$.

**Definition 3.2.** A topological space $(X, \tau)$ is called semipre-$\theta$-$D_0$ if for any pair of distinct points $x$ and $y$ of $X$ there exist a semipre-$\theta$ $D$-set of $X$ containing $x$ but not $y$ or a semipre-$\theta$ $D$-set of $X$ containing $y$ but not $x$.

**Definition 3.3.** A topological space $(X, \tau)$ is called semipre-$\theta$-$D_1$ if for any pair of distinct points $x$ and $y$ of $X$ there exist a semipre-$\theta$ $D$-set of $X$ containing $x$ but not $y$ and a semipre-$\theta$ $D$-set of $X$ containing $y$ but not $x$.

**Definition 3.4.** A topological space $(X, \tau)$ is called semipre- $\theta$-$D_2$ if for any pair of distinct points $x$ and $y$ of $X$ there exist disjoint semipre-$\theta$ $D$-sets $G$ and $E$ of $X$ containing $x$ and $y$, respectively.

**Definition 3.5.** A topological space $(X, \tau)$ is called $\beta\theta$-$T_0$ [5] if for any pair of distinct points in X, there exists a $\beta\theta$-open set containing one of the points but not the other.

**Definition 3.6.** A topological space $(X, \tau)$ is called $\beta\theta$-$T_1$ [5] if for any pair of distinct points $x$ and $y$ in $X$, there exist a $\beta\theta$-open $U$ in $X$ containing $x$ but not $y$ and a $\beta\theta$-open set $V$ in $X$ containing $y$ but not $x$.

**Definition 3.7.** A topological space $(X, \tau)$ is called $\beta\theta$-$T_2$ [5] if for any pair of distinct points $x$ and $y$ in $X$, there exist $\beta\theta$-open sets $U$ and $V$ in $X$ containing $x$ and $y$, respectively, such that $U \cap V = \emptyset$.

**Remark 3.8.** From Definition 3.1 to 3.7, we obtain the following diagram:

$$\begin{array}{ccccc} \beta\theta\text{-}T_2 & \rightarrow & \beta\theta\text{-}T_1 & \rightarrow & \beta\theta\text{-}T_0 \\ \downarrow & & \downarrow & & \downarrow \\ \text{semipre-}\theta\text{-}D_2 & \rightarrow & \text{semipre-}\theta\text{-}D_1 & \rightarrow & \text{semipre-}\theta\text{-}D_0 \end{array}$$

**Theorem 3.9.** [4] If a topological space $(X, \tau)$ is $\beta\theta$-$T_0$, then it is $\beta\theta$-$T_2$.

**Proof.** For any points $x \neq y$, let $V$ be a $\beta\theta$-open set such that $x \in V$ and $y \notin V$. Then, there exists $U \in \beta O(X, \tau)$ such that $x \in U \subset \beta Cl(U) \subset V$. By Lemma 2.2 $\beta Cl(U) \in \beta R(X, \tau)$. Then $\beta Cl(U)$ is $\beta\theta$-open and also $X - \beta Cl(U)$ is a $\beta\theta$-open set containing $y$. Therefore, $X$ is $\beta\theta$-$T_2$. ■

**Theorem 3.10.** For a topological space $(X, \tau)$, the six properties in the diagram are equivalent.

**Proof.** By Theorem 3.9, we have that $\beta\theta$-$T_0$ implies $\beta\theta$-$T_2$. Now we prove that semipre-$\theta$-$D_0$ implies $\beta\theta$-$T_0$. Let $(X, \tau)$ be semipre-$\theta$-$D_0$ so that for any pair of distinct points $x$ and $y$ of $X$, there exists a semipre-$\theta$ $D$-set $O$, says, such that $x \in O$ and $y \notin O$. Suppose $O = U - V$ for which $U \neq X$ and $U, V \in \beta\theta O(X, \tau)$. This implies that $x \in U$. For the case that $y \notin O$ we have $(i)$ $y \notin U$, $(ii)$ $y \in U$ and $y \in V$. For $(i)$, the space $X$ is $\beta\theta$-$T_0$ since $x \in U$ and $y \notin U$. For $(ii)$, the space $X$ is also $\beta\theta$-$T_0$ since $y \in V$ but $x \notin V$. ■

Let $x$ be a point of $X$ and $V$ a subset of $X$. $V$ is called a semipre-$\theta$-neighborhood of $x$ in $X$ if there exists a $\beta\theta$-open set $O$ of $X$ such that $x \in O \subset V$.

**Definition 3.11.** A point $x \in X$ which has only $X$ as the semipre-$\theta$-neighborhood is called a point common to all $\beta\theta$-closed sets (briefly semipre-$\theta$-*cc*).

**Theorem 3.12.** If a topological space $(X, \tau)$ is semipre-$\theta$-$D_1$, then $(X, \tau)$ has no semipre-$\theta$-*cc* point.

**Proof.** Since $(X, \tau)$ is semipre-$\theta$-$D_1$ , so each point $x$ of $X$ is contained in a semipre-$\theta$ $D$-set $O = U - V$ and thus in $U$. By definition $U \neq X$. This implies that $x$ is not a semipre-$\theta$-*cc* point. ■

**Definition 3.13.** A subset $A$ of a topological space $(X, \tau)$ is called a quasi semipre-$\theta$-closed set (briefly qspt-closed) if $spCl_\theta(A) \subset U$ whenever $A \subset U$ and $U$ is $\beta\theta$-open in $(X, \tau)$.

The complement of a quasi semipre-$\theta$-closed set is called quasi semipre-$\theta$-open (briefly qspt-open).

**Lemma 3.14.** Every $\beta\theta$-closed set is qspt-closed but not conversely.

**Example 3.15.** Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{c, d\}, X\}$. Set $A = \{a, b, d\}$. Then $spCl(A) = X$ and so $A$ is not $\beta$-closed. Hence $A$ is not $\beta\theta$-closed. Since $X$ is the only $\beta\theta$-open set containing $A$, $A$ is qspt-closed.

**Theorem 3.16.** For a topological space $(X, \tau)$, the following properties hold:
(1) For each points $x$ and $y$ in a topological space , $x \in spCl_\theta(\{y\})$ implies $y \in spCl_\theta(\{x\})$,
(2) For each $x \in X$, the singleton $\{x\}$ is qspt-closed in $(X, \tau)$.

**Proof.** (1) Let $y \notin spCl_\theta(\{x\})$. This implies that there exists $V \in \beta O(Y, y)$ such that $spCl(V) \cap \{x\} = \emptyset$ and $X - spCl(V) \in \beta R(X, x)$ which means that $x \notin spCl_\theta(\{y\})$.
(2) Suppose that $\{x\} \subset U \in \beta\theta O(X)$. This implies that there exists $V \in \beta O(X, \tau)$ such that $x \in V \subset spCl(V) \subset U$. Now we have $spCl_\theta(\{x\}) \subset spCl_\theta(V) = spCl(V) \subset U$. ■

Lemma 3.14 and Example 3.15 suggests the following natural definition.

**Definition 3.17.** A topological space $(X, \tau)$ is said to be $\beta\theta$-$T_{\frac{1}{2}}$ if every qspt-closed set is $\beta\theta$-closed.

**Theorem 3.18.** For a topological space $(X, \tau)$, the following are equivalent:
(1) $(X, \tau)$ is $\beta\theta$-$T_{\frac{1}{2}}$;
(2) $(X, \tau)$ is $\beta\theta$-$T_1$.

**Proof.** (1) $\to$ (2) : For distinct points $x, y$ of $X$ , $\{x\}$ is qspt-closed by Theorem 3.16. By hypothesis, $X - \{x\}$ is $\beta\theta$-open and $y \in X - \{x\}$. By the same token, $x \in X - \{y\}$ and $X - \{y\}$ is $\beta\theta$-open. Therefore $(X, \tau)$ is $\beta\theta$-$T_1$.

(2) $\to$ (1) : Suppose that $A$ is a qspt-closed set which is not $\beta\theta$-closed. There exists $x \in spCl_\theta(A) - A$. For each $a \in A$, there exists a $\beta\theta$-open set $V_a$ such that $a \in V_a$ and $x \notin V_a$. Since $A \subset \sqcup_{a \in V_a} V_a$ and $\sqcup_{a \in V_a} V_a$ is $\beta\theta$-open , we have $spCl_\theta(A) \subset \sqcup_{a \in V_a} V_a$ . Since $x \in spCl_\theta(A)$, there exists $a_0 \in A$ such that $x \in V_{a_0}$ . But this is a contradiction. ■

Recall that a topological space $(X, \tau)$ is called $\beta$-$T_2$ [6] if for any pair of distinct points $x$ and $y$ in $X$, there exist $\beta$-open sets $U$ and $V$ in $X$ containing $x$ and $y$, respectively, such that $U \cap V = \emptyset$.

**Theorem 3.19.** [5] For a topological space $(X, \tau)$, the following are equivalent:
(1) $(X, \tau)$ is $\beta\theta$-$T_2$,
(2) $(X, \tau)$ is $\beta$-$T_2$.

**Definition 3.20.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly $\beta$-irresolute [7] if for each $x \in X$ and each $V \in \beta O(Y, f(x))$, there is $U \in \beta O(X, x)$ such that $f(U) \subset spCl(V)$.

**Remark 3.21.** Noiri [7] proved that a function $f : (X, \tau) \to (Y, \sigma)$ is weakly $\beta$-irresolute if and only if $f^{-1}(V)$ is $\beta\theta$-closed (resp.$\beta\theta$-open) in $(X, \tau)$ for every $\beta\theta$-closed (resp. $\beta\theta$-open) set $V$ in $(Y, \sigma)$.

**Theorem 3.22.** If $f : (X, \tau) \to (Y, \sigma)$ is a weakly $\beta$-irresolute surjective function and $E$ is a semipre-$\theta$ $D$-set in $Y$, then the inverse image of $E$ is a semipre-$\theta$ $D$-set in $X$.

**Proof.** Let $E$ be a semipre-$\theta$ $D$-set in $Y$. Then there are $\beta\theta$-open sets $U$ and $V$ in $Y$ such that $E = U - V$ and $U \neq Y$. By weak $\beta$-irresoluteness of $f$, $f^{-1}(U)$ and $f^{-1}(V)$ are $\beta\theta$-open in $X$. Since $U \neq Y$, we have $f^{-1}(U) \neq X$. Hence $f^{-1}(E) = f^{-1}(U) - f^{-1}(V)$ is a semipre-$\theta$ $D$-set in $X$. ■

**Theorem 3.23.** If $(Y, \sigma)$ is semipre-$\theta$-$D_1$ and $f : (X, \tau) \to (Y, \sigma)$ is a weakly $\beta$-irresolute injection, then $(X, \tau)$ is semipre-$\theta$-$D_1$.

**Proof.** Suppose that $Y$ is a semipre-$\theta$-$D_1$ space. Let $x$ and $y$ be any pair of distinct points in $X$. Since $f$ is injective and $Y$ is semipre-$\theta$-$D_1$, there exist semipre-$\theta$ $D$-sets $U$ and $V$ of $Y$ containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin U$ and $f(x) \notin V$. By the above theorem, $f^{-1}(U)$ and $f^{-1}(V)$ are semipre-$\theta$ $D$-sets in $X$ containing $x$ and $y$, respectively. This implies that $X$ is a semipre-$\theta$-$D_1$ space. ■

**Theorem 3.24.** For a topological space $(X, \tau)$, the following statements are equivalent:
(1) $(X, \tau)$ is semipre-$\theta$-$D_1$;
(2) For each pair of distinct points $x, y \in X$, there exists a weakly $\beta$- irresolute surjective function $f : (X, \tau) \to (Y, \sigma)$, where $Y$ is a semipre-$\theta$-$D_1$ space such that $f(x)$ and $f(y)$ are distinct.

**Proof.** (1) $\to$ (2) : For every pair of distinct points of $X$, it suffices to take the identity function on $X$.

(2) $\to$ (1) : Let $x$ and $y$ be any pair of distinct points in $X$. By hypothesis, there exists a surjective weakly $\beta$-irresolute function $f$ of a space $X$ into a semipre-$\theta$-$D_1$ space $Y$ such that $f(x) \neq f(y)$. Therefore, there exist disjoint semipre-$\theta$ $D$-sets $U$ and $V$ of $Y$ containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin U$ and $f(x) \notin V$. Since $f$ is weakly $\beta$-irresolute and surjective, by Theorem 3.22, $f^{-1}(U)$ and $f^{-1}(V)$ are semipre-$\theta$ $D$-sets in $X$ containing $x$ and $y$, respectively, such that $y \notin f^{-1}(U)$ and $x \notin f^{-1}(V)$. Hence $X$ is a semipre-$\theta$ -$D_1$ space. ■

## 4. Additional Properties

Let $A$ be a subset of a topological space $(X, \tau)$. The semipre-$\theta$-kernel of $A \subset (X, \tau)$ [5], denoted by $spKer_\theta(A)$, is defined to be the set

$$\cap\{O \in \beta\theta O(X, \tau) \mid A \subset O\},$$

or equivalently to be the set $\{x \in X \mid spCl_\theta(\{x\}) \cap A \neq \emptyset\}$.

**Definition 4.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is $R$-continuous [8] (resp. $\beta\theta$-$R$-continuous, $\beta$-$R$-continuous) if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$ (resp. $V \in \beta O(Y, f(x))$), there exists an open subset $U$ of $X$ containing $x$ such that $Cl(f(U)) \subset V$(resp. $spCl_\theta(f(U)) \subset V$, $spCl(f(U)) \subset V$).

**Remark 4.2.** (i) Since $A \subset spCl(A) \subset spCl_\theta(A)$ for any set $A$, $\beta\theta$-$R$-continuity implies $\beta$-$R$-continuity.
(ii) Since the semipre-closure and semipre $\theta$-closure operators agree on $\beta$-open sets (Lemma 1.2), it follows that if $f : (X, \tau) \to (Y, \sigma)$ is $R$-$\beta$-continuous and $\beta$-open then $f$ is $\beta\theta$-$R$-continuous, where $f$ is called *$\beta$-open* if the image of each open set of X is $\beta$-open.

**Definition 4.3.** The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *spθ-c*-closed if for each point $(x, y) \in (X \times Y) - G(f)$, there exist subsets $U \in \beta O(X, x)$ and $V \in \beta\theta O(Y, y)$ such that $(spCl(U) \times V) \cap G(f) = \emptyset$.

**Lemma 4.4.** The graph $G(f)$ of $f : (X, \tau) \to (Y, \sigma)$ is *spθ-c*-closed in $X \times Y$ if and only if for each point $(x, y) \in (X \times Y) - G(f)$, there exist $U \in \beta O(X, x)$ and $V \in \beta\theta O(Y, y)$ such that $f(spCl(U)) \cap V = \emptyset$.

**Proof.** It follows immediately from Definition 4.3. ■

In [[8], Theorem 4.1], it is shown that the graph of an $R$-continuous function into a $T_1$-space is $\theta$-closed with respect to the domain. Here an analogous result is proved for $\beta\theta$-$R$-continuous functions.

**Theorem 4.5.** If $f : (X, \tau) \to (Y, \sigma)$ is $\beta\theta$-$R$-continuous weakly $\beta$-irresolute and $Y$ is $\beta$-$T_1$, then $G(f)$ is *spθ-c*-closed.

**Proof.** Assume $(x, y) \in (X \times Y) - G(f)$. Since $y \neq f(x)$ and $Y$ is $\beta$-$T_1$, there exists a $\beta$-open subset $V$ of $Y$ such that $f(x) \in V$ and $y \notin V$. The $\beta\theta$-$R$-continuity of $f$ implies the existence of an open subset $U$ of $X$ containing $x$ such that $spCl_\theta(f(U)) \subset V$. Therefore $(x, y) \in spCl(U) \times (Y - spCl_\theta(f(U))$ which is disjoint from $G(f)$ because if $a \in spCl(U)$, then since f is weakly $\beta$-irresolute, by [[7], Theorem 4.5] $f(a) \in f(spCl(U)) \subset spCl_\theta(f(U))$. Note that $Y - spCl_\theta(f(U))$ is $\beta\theta$-open. ■

It is proved in [[8], Theorem 3.1] that a function $f : (X, \tau) \to (Y, \sigma)$ is $R$-continuous if and only if for each $x \in X$ and each closed subset $F$ of $Y$ with $f(x) \notin F$, there exist open subsets $U \subset X$ and $V \subset Y$ such that $x \in U$, $F \subset V$ and $f(U) \cap V = \emptyset$. The following theorem is an analogous result for $\beta\theta$-$R$-continuous functions.

**Theorem 4.6.** Let $f : (X, \tau) \to (Y, \sigma)$ be a weakly $\beta$-irresolute function. Then $f$ is $\beta\theta$-$R$-continuous if and only if for each $x \in X$ and each $\beta$-closed subset $F$ of $Y$ with $f(x) \notin F$, there exist an open subset $U$ of $X$ containing $x$

and a $\beta\theta$-open subset $V$ of $Y$ with $F \subset V$ such that $f(spCl(U)) \cap V = \emptyset$.

**Proof.** Necessity. Let $x \in X$ and $F$ be a $\beta$-closed subset of $Y$ with $f(x) \in Y - F$. Since $f$ is $\beta\theta$-$R$-continuous, there exists an open subset $U$ of $X$ containing $x$ such that $spCl_\theta(f(U)) \subset Y - F$. Let $V = Y - spCl_\theta(f(U))$. Then $V$ is $\beta\theta$-open and $F \subset V$. Since $f$ is weakly $\beta$-irresolute , $f(spCl(U)) \subset spCl_\theta(f(U))$. Therefore $f(sCl(U)) \cap V = \emptyset$.
Sufficiency. Let $x \in X$ and $V$ be a $\beta$-open subset of $Y$ with $f(x) \in V$. Let $F = Y - V$. Since $f(x) \notin F$, there exists an open subset $U$ of $X$ containing $x$ and a $\beta\theta$-open subset $W$ of $Y$ with $F \subset W$ such that $f(spCl(U)) \cap W = \emptyset$. Then $f(spCl(U)) \subset Y - W$, thus $spCl_\theta(f(U)) \subset spCl_\theta(Y - W) = Y - W \subset Y - F = V$. Therefore $f$ is $\beta\theta$-$R$-continuous. ■

**Corollary 4.7.** Let $X$ and $Y$ be topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a weakly $\beta$-irresolute function. Then $f$ is $\beta\theta$-$R$-continuous if and only if for each $x \in X$ and each $\beta$-open subset $V$ of $Y$ containing $spCl_\theta(f(spCl(U))) \subset V$.

**Proof.** Assume $f$ is $\beta\theta$-$R$-continuous. Let $x \in X$ and $V$ be a $\beta$-open subset of $Y$ with $f(x) \in V$. Then there exists an open subset $U$ of $X$ containing $x$ such that $spCl_\theta(f(U)) \subset V$. Since $f$ is weakly $\beta$-irresolute, we have

$$spCl_\theta(f(spCl(U))) \subset spCl_\theta(spCl_\theta(f(U))) = spCl_\theta(f(U)) \subset V.$$

Thus $spCl_\theta(f(spCl(U))) \subset V$. The converse implication is immediate. ■

**Definition 4.8.** A topological space $(X, \tau)$ is said to be a $\beta$-$R_1$ if for $x, y \in X$ with $spCl(\{x\}) \neq spCl(\{y\})$, there exist disjoint $\beta$-open sets $U$ and $V$ such that $spCl(\{x\}) \subset U$ and $spCl(\{y\}) \subset V$.

**Proposition 4.9.** A space $X$ is $\beta$-$R_1$ if and only if for each $\beta$-open set $O$ and each $x \in O$ , $spCl_\theta(\{x\}) \subset O$.

**Proof.** Necessity. Assume that $X$ is $\beta$-$R_1$. Suppose that $O$ is a $\beta$-open subset of $X$ and $x \in O$. Let $y$ be an arbitrary element of $X - O$. Since $X$ is $\beta$-$R_1$, $spCl_\theta(\{y\}) = spCl(\{y\}) \subset X - O$. Hence we have that $x \notin spCl_\theta(\{y\})$ and $y \notin spCl_\theta(\{x\})$. It follows that $spCl_\theta(\{x\}) \subset O$.
Sufficiency. Assume now that, $y \in spCl_\theta(\{x\}) - spCl(\{x\})$ for some $x \in X$. Then there exists a $\beta$-open set $O$ containing $y$ such that $spCl(O) \cap \{x\} \neq \emptyset$ but $O \cap \{x\} = \emptyset$. Then $spCl_\theta(\{y\}) \subset O$ and $spCl_\theta(\{y\}) \cap \{x\} = \emptyset$. Hence $x \notin spCl_\theta(\{y\})$. Thus $y \notin spCl_\theta(\{x\})$. By this contradiction, we obtain $spCl_\theta(\{x\}) = spCl(\{x\})$ for each $x \in X$. Thus by ([3], Theorem 4.15) $X$ is $\beta$-$R_1$. ■

Now, we show that the range of a $\beta\theta$-$R$-continuous function satisfies the stronger $\beta$-$R_1$ condition.

**Theorem 4.10.** If $f : (X, \tau) \to (Y, \sigma)$ is a $\beta\theta$-$R$-continuous surjection, then $(Y, \sigma)$ is a $\beta$-$R_1$ space.

**Proof.** Let $V$ be a $\beta$-open subset of $Y$ and $y \in V$. Let $x \in X$ such that $y = f(x)$. Since $f$ is $\beta\theta$-$R$-continuous, there exists an open subset $U$ of $X$ containing $x$ such that $spCl_\theta(f(U)) \subset V$. Then $spCl_\theta(\{y\}) \subset spCl_\theta(f(U)) \subset V$. Therefore by Proposition 4.9, $Y$ is $\beta$-$R_1$. ■

We close this paper with a sample of the basic properties of $\beta\theta$-$R$-continuous functions concerning composition and restriction.

**Theorem 4.11.** If $f : (X, \tau) \to (Y, \sigma)$ is continuous and $g : (Y, \sigma) \to (Z, \gamma)$ is $\beta\theta$-$R$-continuous, then $g \circ f : (X, \tau) \to (Z, \gamma)$ is $\beta\theta$-$R$-continuous.

**Proof.** Let $x \in X$ and $W$ be a $\beta$-open subset of $Z$ containing $g(f(x))$. Since $g$ is $\beta\theta$-$R$-continuous, there exists an open subset $V$ of $Y$ containing $f(x)$ such that $spCl_\theta(g(V)) \subset W$. Since $f$ is continuous, there exists an open subset $U$ of $X$ containing $x$ such that $f(U) \subset V$ ; hence $spCl_\theta(g(f(U)) \subset W$. Therefore $g \circ f$ is $\beta\theta$-$R$-continuous. ■

**Theorem 4.12.** Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$ be functions. If $g \circ f : (X, \tau) \to (Z, \gamma)$ is $\beta\theta$-$R$-continuous and $f$ is an open surjection, then $g$ is $\beta\theta$-$R$-continuous.

**Proof.** Let $y \in Y$ and $W$ be a $\beta$-open subset of $Z$ containing $g(y)$. Since $f$ is surjective, there exists $x \in X$ such that $y = f(x)$ and $f$ is an open surjection, then $g$ is $\beta\theta$-$R$-continuous. Since $g \circ f$ is $\beta\theta$-$R$-continuous, there exists an open subset $U$ of $X$ containing $x$ such that $spCl_\theta(g(f(U)) \subset W$. Note that $f(U)$ is an open set containing $y$. Therefore $g$ is $\beta\theta$-$R$-continuous. ■

**Theorem 4.13.** If $f : (X, \tau) \to (Y, \sigma)$ is $\beta\theta$-$R$-continuous, $A \subset X$ and $f(A) \subset B \in \beta SO(Y, \sigma)$, then $f/A : A \to B$ is $\beta\theta$-$R$-continuous.

**Proof.** Let $x \in A$ and $V$ be a $\beta$-open subset of $B$ containing $f(x)$ (note that $f(A) \subset B$). Hence by [1, Lemma 2.7] $V$ be a $\beta$-open subset of $Y$ containing $f(x)$. Since $f$ is $\beta\theta$-$R$-continuous, there exists an open subset $U$ of $X$ containing $x$ such that $spCl_\theta(f(U)) \subset V$. Let $O = U \cap A$. Then $O$ is an open subset of $A$ containing $x$ such that $spCl_\theta((f/A)(O)) = spCl_\theta(f(O)) \subset spCl_\theta(f(U)) \subset V$. Therefore $f/A : A \to B$ is $\beta\theta$-$R$-continuous. ■

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## July 3-7, 2014 Nafpaktos, Greece

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**ISBN:** 978-618-80609-1-3