WEAK SEPARATION AXIOMS VIA PRE-REGULAR p-OPEN SETS

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ABSTRACT. In this paper, we obtain new separation axioms by using the notion of (δ, p)-open sets introduced by Jafari [3] via the notion of pre-regular p-open sets [2].

1. INTRODUCTION AND PRELIMINARIES

In what follows (X, τ) and (Y, σ) (or X and Y) denote topological spaces. If A is a subset of a space X, we denote the interior, the closure and the complement of A by $Int(A), Cl(A)$ and A^c, respectively. A subset A of a topological space (X, τ) is called preopen [5] if $A \subset Int\,(Cl(A))$, and preclosed if its complement is preopen; the preinterior $pInt(A)$ (resp. preclosure $pCl(A)$) of A is the largest preopen (resp. smallest preclosed) set contained in (resp. containing) A. It is evident that A is preopen (resp. preclosed) if and only if $A = pInt(A)$ (resp. $pCl(A)$). It is well known that $pInt(A) = A \cap Int\,(Cl(A))$, and that any union of preopen sets is preopen. A subset A of a topological space (X, τ) is called *pre-regular p-open* [2] if $A = pInt(pCl(A))$, and *pre-regular p-closed* if $A = pCl(pInt(A))$. It can be easily seen that $pInt(pCl(A))$ (resp. $pCl(pInt(A))$) is pre-regular p-open (resp. pre-regular p-closed) for any subset A of a space (X, τ). The collection of all pre-regular p-open (resp. pre-regular p-closed) subsets of a space (X, τ) will be denoted by $PRO(X, \tau)$ (resp. $PRC(X, \tau)$). Now we define the following notions which will be used in the sequel: A point $x \in X$ is called a (δ, p)-*cluster point* of A if $A \cap U \neq \emptyset$ for every pre-regular

2000 *Mathematics Subject Classification.* Primary 54D10.

Key words and phrases. (δ, p)-T_0, (δ, p)-T_1, (δ, p)-T_2, (δ, p)-D_0, (δ, p)-D_1, (δ, p)-D_2.

p-open set U of X containing x. The set of all (δ, p)-cluster points of A is called the (δ, p)-*closure* of A, denoted by $\delta Cl_p(A)$. If $\delta Cl_p(A) = A$, then A is called (δ, p)-*closed*. The complement of a (δ, p)-closed set is called (δ, p)-*open* [3]. We say that a set U in a topological space (X, τ) is a (δ, p)-*neighborhood* of a point x if U contains a (δ, p)-open set to which x belongs. We denote the collection of all (δ, p)-open (resp. (δ, p)-closed) sets by $\delta PO(X, \tau)$ (resp. $\delta PC(X, \tau)$).
Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers. For the concepts not defined here, we refer the reader to [1].
The following four propositions can be easily verified.

Proposition 1.1. *For subsets A and $A_i, i \in I$ of a space (X, τ), the following hold:*
(1) $A \subset \delta Cl_p(A)$.
(2) If $A \subset B$, then $\delta Cl_p(A) \subset \delta Cl_p(B)$.
(3) $\delta Cl_p(\cap\{A_i : i \in I\}) \subset \cap\{\delta Cl_p(A_i) : i \in I\}$.
(4) $\cup\{\delta Cl_p(A_i) : i \in I\} \subset \delta Cl_p(\cup\{A_i : i \in I\})$.

Proposition 1.2. *Any intersection of (δ, p)-closed sets in (X, τ) is (δ, p)-closed.*

Proposition 1.3. *Let A be a subset of a topological space (X, τ). Then*

$$\begin{aligned} \delta Cl_p(A) &= \cap\{F \in \delta PC(X, \tau) : A \subset F\} \\ &= \cap\{F \in PRC(X, \tau) : A \subset F\} \end{aligned}$$

Proposition 1.4. *Let A be a subset of a topological space (X, τ) and $x \in X$. Then $x \in \delta Cl_p(A)$ if and only if $U \cap A \neq \emptyset$ for every (δ, p)-open (pre-regular p-open) set U in X containing x.*

Corollary 1.5. *(1) $\delta Cl_p(A)$ is (δ, p)-closed in (X, τ) for any subset A of (X, τ).*
(2) A subset A of (X, τ) is (δ, p)-closed (resp. (δ, p)-open) if and only if A is the intersection (resp. union) of pre-regular p-closed (resp. pre-regular p-open) sets.

Proof. Follows immediately from Propositions 1.2 and 1.3. □

Corollary 1.6. *Let A be a subset of a topological space (X,τ). Then $\delta Cl_p(A)$ is the smallest (δ,p)-closed set in (X,τ) containing A.*

Proof. Follows from Proposition 1.1 (1), (2) and Corollary 1.5 (1). □

Remark 1.7. *It follows from Corollary 1.5 (2) that a singleton is (δ,p)-open if and only if it is pre-regular p-open.*

Remark 1.8. *It is clear also from Corollary 1.5 (2) that every pre-regular p-open set is (δ,p)-open. However, the converse is not true as the following simple example tells.*

Example 1.9. *Let $X=\{a,b,c\}$, $\tau=\{X,\emptyset,\{b,c\}\}$. Then $\{a,b\},\{a,c\}$ are pre-regular p-closed in (X,τ) as they are both preopen and preclosed. Thus by Corollary 1.5 (2), $\{a\}=\{a,b\}\cap\{a,c\}$ is (δ,p)-closed. However, $\{a\}$ is not pre-regular p-closed as $pInt(\{a\})=\emptyset$.*

Remark 1.10. *The union of even two (δ,p)-closed sets need not be (δ,p)-closed as seen from Example 1.9. Observe that $\{b\},\{c\}$ are pre-regular p-closed in (X,τ) as they are both preopen and preclosed. Thus by Remark 1.8, $\{b\},\{c\}$ are (δ,p)-closed. However, $\{b,c\}$ is not (δ,p)-closed (observe that $\{a\}$ is not pre-regular p-open as $pCl(\{a\})=\{a\}$ and $pInt(\{a\})=\emptyset$, thus by Remark 1.7, $\{a\}$ is not (δ,p)-open).*

We now discuss the product of two (δ,p)-open sets, to proceed, we introduce the following (probably) known result.

Lemma 1.11. *(1) Let A be a subset of a space X, B be a subset of a space Y. Then, $pInt(A\times B)=pInt(A)\times pInt(B)$.*
(2) The product of two preopen sets is preopen.
(3) The product of two preclosed sets is preclosed.
(4) Let A be a subset of a space X, B be a subset of a space Y. Then, $pCl(A\times B)\subset pCl(A)\times pCl(B)$.

Proof. (1) Follows from the fact that $pInt(A)=A\cap Int(Cl(A))$.

(2) Follows from (1).
(3) Let A be a preclosed subset of a space X, B be a preclosed subset of a space Y. Then by (2), $X \times (Y \backslash B)$, $(X \backslash A) \times Y$ are preopen subsets of $X \times Y$. Since any union of preopen sets is preopen, it follows that $(X \times Y) \backslash (A \times B) = (X \times (Y \backslash B)) \cup ((X \backslash A) \times Y)$ is preopen, that is, $A \times B$ is preclosed.
(4) By (3), $pCl\,(A) \times pCl\,(B)$ is preclosed, but $A \times B \subset pCl\,(A) \times pCl\,(B)$, so $pCl\,(A \times B) \subset pCl\,(A) \times pCl\,(B)$. □

Corollary 1.12. *Let A be a pre-regular p-open subset of a space X, B be a pre-regular p-open subset of a space Y. Then $A \times B$ is pre-regular p-open in $X \times Y$.*

Proof. It follows from Lemma 1.11 (1), (4) that

$$\begin{aligned} pInt\,(pCl\,(A \times B)) &\subset pInt\,(pCl\,(A) \times pCl\,(B)) \\ &= pInt\,(pCl\,(A)) \times pInt\,(pCl\,(B)) \\ &= A \times B \end{aligned}$$

Now

$$A \times B \subset pCl\,(A \times B)$$

but A, B are preopen, so it follows from Lemma 1.11 (2) that $A \times B$ is preopen, and thus

$$A \times B \subset pInt\,(pCl\,(A \times B))$$

Hence, $pInt\,(pCl\,(A \times B)) = A \times B$, that is, $A \times B$ is pre-regular p-open. □

Corollary 1.13. *(1) Let A be a (δ, p)-open subset of a space X, B be a (δ, p)-open subset of a space Y. Then, $A \times B$ is (δ, p)-open in $X \times Y$.*
(2) Let A be a (δ, p)-closed subset of a space X, B be a (δ, p)-closed subset of a space Y. Then, $A \times B$ is (δ, p)-closed in $X \times Y$.
(3) Let A be a subset of a space X, B be a subset of a space Y. Then, $\delta Cl_p\,(A \times B) \subset \delta Cl_p\,(A) \times \delta Cl_p\,(B)$.

Proof. (1) Follows from Corollaries 1.5 (2) and 1.12.
(2) Follows from (1).
(3) Follows from (2) and Corollary 1.6. □

2. $D(\delta, p)$-sets and associated separation axioms

Definition 2.1. *A subset A of a topological space X is called a $D(\delta, p)$-set if there are two $U, V \in \delta PO(X, \tau)$ such that $U \neq X$ and $A = U \backslash V$.*

Remark 2.2. *Letting $A = U$ and $V = \emptyset$ in the above definition, it is easy to see that every proper (δ, p)-open set U is a $D(\delta, p)$-set.*

Definition 2.3. *A topological space (X, τ) is called (δ, p)-D_0 if for any pair of distinct points x and y of X there exists a $D(\delta, p)$-set of X containing x but not y or a $D(\delta, p)$-set of X containing y but not x.*

Definition 2.4. *A topological space (X, τ) is called (δ, p)-D_1 if for any pair of distinct points x and y of X there exist a $D(\delta, p)$-set of X containing x but not y and a $D(\delta, p)$-set of X containing y but not x.*

Definition 2.5. *A topological space (X, τ) is called (δ, p)-D_2 if for any pair of distinct points x and y of X there exist disjoint $D(\delta, p)$-sets G and E of X containing x and y, respectively.*

Definition 2.6. *A topological space (X, τ) is called (δ, p)-T_0 (resp. pre-T_0 ([4], [6])) if for any pair of distinct points of X, there is a (δ, p)-open (resp. preopen) set containing one of the points but not the other.*

It is well known that every singleton of a space X is preopen or preclosed, thus it is clear that every space is pre-T_0.

Definition 2.7. *A topological space (X, τ) is called (δ, p)-T_1 (resp. pre-T_1 ([4], [6])) if for any pair of distinct points x and y of X, there are a (δ, p)-open (resp. preopen) set U in X containing x but not y and a (δ, p)-open set V in X containing y but not x.*

Definition 2.8. *A topological space (X, τ) is called (δ, p)-T_2 (resp. pre-T_2 ([4], [6])) if for any pair of distinct points x and y of X, there exist*

(δ, p)-open (resp. preopen) sets U and V in X containing x and y, respectively, such that $U \cap V = \emptyset$.

The following remark follows immediately from the definitions and Remark 2.2.

Remark 2.9. *(1) If (X, τ) is (δ, p)-T_i , then it is (δ, p)-T_{i-1}, $i = 1, 2$.*
(2) If (X, τ) is (δ, p)-T_i , then (X, τ) is (δ, p)-D_i , $i = 0, 1, 2$.
(3) If (X, τ) is (δ, p)-D_i , then it is (δ, p)-D_{i-1} , $i = 1, 2$.

Remark 2.10. *It is easy to see from Corollary 1.5 (2) that:*
(1) A topological space (X, τ) is (δ, p)-T_0 if and only if for any pair of distinct points of X, there is a pre-regular p-open set containing one of the points but not the other.
(2) A topological space (X, τ) is (δ, p)-T_1 if and only if for any pair of distinct points x and y of X, there are a pre-regular p-open U in X containing x but not y and a pre-regular p-open set V in X containing y but not x.
(3) A topological space (X, τ) is (δ, p)-T_2 if and only if for any pair of distinct points x and y of X, there exist pre-regular p-open sets U and V in X containing x and y, respectively, such that $U \cap V = \emptyset$.

Example 2.11. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then every preopen subset of (X, τ) is open, and thus the pre-regular p-open sets are $X, \emptyset, \{a\}, \{b\}$. Hence, it is clear from Remark 2.10 that (X, τ) is a (δ, p)-T_0 space that is not (δ, p)-T_1.*

Remark 2.12. *It is also easy to see from Remark 2.10 and the fact that every pre-regular p-open set is preopen, that if (X, τ) is (δ, p)-T_i, then it is pre-T_i, $i = 0, 1, 2$.*

Theorem 2.13. *A space (X, τ) is (δ, p)-T_2 if and only if it is pre-T_2.*

Proof. **Necessity.** Follows from Remark 2.12.
Sufficiency. Let $x, y \in X$ and $x \neq y$. Then by assumption, there exist disjoint preopen sets U, V containing x, y respectively. Since $U \cap V = \emptyset$

and V is preopen, $pCl\,(U) \cap V = \emptyset$ and thus, $pInt\,(pCl\,(U)) \cap V = \emptyset$. Similarly, since $pInt\,(pCl\,(U))$ is preopen, $pInt\,(pCl\,(U)) \cap pCl\,(V) = \emptyset$ and thus, $pInt\,(pCl\,(U)) \cap pInt\,(pCl\,(V)) = \emptyset$. Now $U \subset pInt\,(pCl\,(U))$ and $V \subset pInt\,(pCl\,(V))$ as U and V are preopen. Thus, $pInt\,(pCl\,(U))$ and $pInt\,(pCl\,(V))$ are disjoint pre-regular p-open sets containing x, y respectively. Hence by Remark 2.10 (3), (X, τ) is (δ, p)-T_2. □

Theorem 2.14. *For a topological space (X, τ), the following statements hold:*
(1) (X, τ) is (δ, p)-D_0 if and only if it is (δ, p)-T_0.
(2)(X, τ) is (δ, p)-D_1 if and only if it is (δ, p)-D_2.

Proof. (1) **Necessity.** Let (X, τ) be (δ, p)-D_0. Then for each distinct points $x, y \in X$, at least one of x, y, say x, belongs to a $D(\delta, p)$-set G but $y \notin G$. Suppose $G = U_1 \backslash U_2$ where $U_1 \neq X$ and U_1,$U_2 \in \delta PO(X, \tau)$. Then $x \in U_1$, and for $y \notin G$ we have two cases: (a) $y \notin U_1$; (b) $y \in U_1$ and $y \in U_2$. In case (a), U_1 contains x but does not contain y ; In case (b), U_2 contains y but does not contain x. Hence, X is (δ, p)-T_0.
Sufficiency. Follows from Remark 2.9 (2).
(2) **Necessity.** Let X be (δ, p)-D_1. Then for each distinct points $x, y \in X$, we have $D(\delta, p)$-sets G_1, G_2 such that $x \in G_1, y \notin G_1$; $y \in G_2, x \notin G_2$. Let $G_1 = U_1 \backslash U_2, G_2 = U_3 \backslash U_4$. From $x \notin G_2$, we have either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.
(1) $x \notin U_3$. From $y \notin G_1$, we obtain the following two subcases:
(a) $y \notin U_1$. From $x \in U_1 \backslash U_2$ we have $x \in U_1\backslash\ (U_2 \cup\ U_3)$ and from $y \in U_3 \backslash U_4$ we have $y \in U_3 \backslash (U_1 \cup U_4)$. It is easy to see that $(U_1 \backslash (U_2 \cup U_3)) \cap (U_3 \backslash (U_1 \cup U_4) = \emptyset$.
(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \backslash U_2, y \in U_2$ and $(U_1 \backslash U_2) \cap U_2 = \emptyset$.
(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \backslash U_4$, $x \in U_4$ and $(U_3 \backslash U_4) \cap U_4 = \emptyset$.
Hence, X is (δ, p)-D_2.

Sufficiency. Follows from Remark 2.9 (3). □

Corollary 2.15. *If* (X,τ) *is* (δ,p)-D_1*, then it is* (δ,p)-T_0*.*

Proof. Follows from Remark 2.9 (3) and Theorem 2.14 (1). □

The following diagram summarizes the implications among the introduced concepts and other related concepts.

$$\begin{array}{ccccccc} T_2 & \rightarrow & \text{pre-}T_2 & \leftrightarrow & (\delta,p)\text{-}T_2 & \rightarrow & (\delta,p)\text{-}D_2 \\ \downarrow & & \downarrow & & \downarrow & & \updownarrow \\ T_1 & \rightarrow & \text{pre-}T_1 & \leftarrow & (\delta,p)\text{-}T_1 & \rightarrow & (\delta,p)\text{-}D_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T_0 & \rightarrow & \text{pre-}T_0 & \leftarrow & (\delta,p)\text{-}T_0 & \leftrightarrow & (\delta,p)\text{-}D_0 \end{array}$$

Theorem 2.16. *Let* X *and* Y *be* (δ,p)-T_i*. Then* $X\times Y$ *is* (δ,p)-$T_i, i=0,1,2$*.*

Proof. Follows from Corollary 1.13 (3). □

Theorem 2.17. *A topological space* (X,τ) *is* (δ,p)-T_0 *if and only if for each pair of distinct points* x,y *of* X*,* $\delta Cl_p(\{x\})\neq\delta Cl_p(\{y\})$*.*

Proof. **Necessity.** Let (X,τ) be a (δ,p)-T_0 space and x, y be any two distinct points of X. Then there exists a (δ,p)-open set G containing x, say but not y, and therefore G^c is a (δ,p)-closed set which contains y but not x. Since $\delta Cl_p(\{y\})$ is the smallest (δ,p)-closed set containing y (Corollary 1.6), $\delta Cl_p(\{y\})\subset G^c$, and so $x\notin\delta Cl_p(\{y\})$. Thus by Proposition 1.1 (1), $\delta Cl_p(\{x\})\neq\delta Cl_p(\{y\})$.

Sufficiency. Suppose that $x,y\in X, x\neq y$. Then by assumption, $\delta Cl_p(\{x\})\neq\delta Cl_p(\{y\})$. Let z be a point of X such that $z\in\delta Cl_p(\{x\})$ and $z\notin\delta Cl_p(\{y\})$, say. We claim that $x\notin\delta Cl_p(\{y\})$. For, if $x\in\delta Cl_p(\{y\})$, then by Proposition 1.1 (2) and Corollary 1.5 (1), $\delta Cl_p(\{x\})\subset\delta Cl_p(\{y\})$, a contradiction with $z\notin\delta Cl_p(\{y\})$. Thus, $x\in(\delta Cl_p(\{y\}))^c$, but by Proposition 1.1 (1) and Corollary 1.5 (1), $(\delta Cl_p(\{y\}))^c$ is a (δ,p)-open set that does not contain y. Hence, (X,τ) is (δ,p)-T_0. □

Theorem 2.18. *A topological space (X, τ) is (δ, p)-T_1 if and only if the singletons of X are (δ, p)-closed.*

Proof. **Necessity.** Suppose (X, τ) is (δ, p)-T_1 and x is any point of X. Let $y \in \{x\}^c$. Then $x \neq y$ and so there exists a (δ, p)-open set U_y such that $y \in U_y$ but $x \notin U_y$. Consequently $y \in U_y \subset \{x\}^c$ i.e. $\{x\}^c = \bigcup\{U_y : y \in \{x\}^c\}$ which is (δ, p)-open by Proposition 1.2.
Sufficiency. Suppose $\{p\}$ is (δ, p)-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Then by assumption, $\{x\}^c$ is a (δ, p)-open set containing y but not x. Similarly $\{y\}^c$ is a (δ, p)-open set containing x but not y. Hence, X is (δ, p)-T_1. □

Definition 2.19. *A point $x \in X$ which has X as the only (δ, p)-neighborhood is called a $D(\delta, p)$-neat point.*

Remark 2.20. *It is clear that if a (δ, p)-T_0 topological space (X, τ) has a $D(\delta, p)$-neat point, then it is unique, because if x and y are both $D(\delta, p)$-neat point in X, then at least one of them say x has a (δ, p)-neighborhood U containing x but not y. But this is a contradiction since $U \neq X$.*

Theorem 2.21. *For a (δ, p)-T_0 topological space (X, τ), the following are equivalent:*
(1) (X, τ) is (δ, p)-D_1;
(2) (X, τ) has no $D(\delta, p)$-neat point.

Proof. (1)→(2): Since (X, τ) is (δ, p)-D_1, so each point x of X is contained in a $D(\delta, p)$-set $O = U \backslash V$ and thus in U. By definition $U \neq X$. Hence, x is not a $D(\delta, p)$-neat point.
(2)→(1): If X is (δ, p)-T_0, then for each distinct pair of points $x, y \in X$, there exists a (δ, p)-set U containing x, say but not y. Thus by Remark 2.2, U is a $D(\delta, p)$-set. If X has no $D(\delta, p)$-neat point, then y is not a $D(\delta, p)$-neat point. Thus, there exists a (δ, p)-open set V containing y such that $V \neq X$, and therefore, $y \in V \backslash U, x \notin V \backslash U$ and $V \backslash U$ is a $D(\delta, p)$-set. Hence, X is (δ, p)-D_1. □

Example 2.22. *Consider the space (X, τ) of Example 2.11. Then (X, τ) is (δ, p)-D_0 as it is (δ, p)-T_0. Since the pre-regular p-open sets are $X, \emptyset, \{a\}, \{b\}$, it follows from Corollary 1.5 (2) that the (δ, p)-open sets are $X, \emptyset, \{a\}, \{b\}, \{a, b\}$. Thus c is a $D(\delta, p)$-neat point of X. Hence, it follows from Theorem 2.21 that (X, τ) is not (δ, p)-D_1.*

Definition 2.23. *A function $f : (X, \tau) \to (Y, \sigma)$ is called (δ, p)-continuous if the inverse image of each (δ, p)-open set is (δ, p)-open.*

Theorem 2.24. *If $f : (X, \tau) \to (Y, \sigma)$ is a (δ, p)-continuous surjective function and E is a $D(\delta, p)$-set in Y, then the inverse image of E is a $D(\delta, p)$-set in X.*

Proof. Let E be a $D(\delta, p)$-set in Y. Then there are (δ, p)-open sets U_1 and U_2 in Y such that $E = U_1 \backslash U_2$ and $U_1 \neq Y$. By the (δ, p)- continuity of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are (δ, p)-open in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence, $f^{-1}(E) = f^{-1}(U_1) \backslash f^{-1}(U_2)$ is a $D(\delta, p)$-set. □

Theorem 2.25. *If (Y, σ) is (δ, p)-D_1 and $f : (X, \tau) \to (Y, \sigma)$ is (δ, p)-continuous and bijective, then (X, τ) is (δ, p)-D_1.*

Proof. Suppose that Y is a (δ, p)-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is (δ, p)-D_1, there exist $D(\delta, p)$-sets G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. By Theorem 2.24, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are $D(\delta, p)$-sets in X containing x and y respectively, and $y \notin f^{-1}(G_x), x \notin f^{-1}(G_y)$. Hence, X is (δ, p)-D_1. □

Theorem 2.26. *A topological space (X, τ) is (δ, p)-D_1 if and only if for each pair of distinct points $x, y \in X$, there exists a (δ, p)-continuous surjective function $f : (X, \tau) \to (Y, \sigma)$, where Y is a (δ, p)-D_1 space such that $f(x)$ and $f(y)$ are distinct.*

Proof. **Necessity.** For every pair of distinct points of X, it suffices to take the identity function on X.

Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a (δ, p)-continuous, surjective function f from X onto a (δ, p)-D_1 space Y such that $f(x) \neq f(y)$. Thus by Theorem 2.14 (2), there exist disjoint $D(\delta, p)$-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is (δ, p)-continuous and surjective, by Theorem 2.24, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $D(\delta, p)$-sets in X containing x and y, respectively. Hence again by Theorem 2.14 (2), X is (δ, p)-D_1. □

3. (δ, p)-R_0 SPACES AND (δ, p)-R_1 SPACES

Definition 3.1. *A topological space (X, τ) is said to be a (δ, p)-R_0 space if every (δ, p)-open set contains the (δ, p)-closure of each of its singletons.*

Definition 3.2. *A topological space (X, τ) is said to be (δ, p)-R_1 if for x, y in X with $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$, there exist disjoint (δ, p)-open sets U and V such that $\delta Cl_p(\{x\})$ is a subset of U and $\delta Cl_p(\{y\})$ is a subset of V.*

Theorem 3.3. *If (X, τ) is (δ, p)-R_1, then (X, τ) is (δ, p)-R_0.*

Proof. Let U be (δ, p)-open and $x \in U$. If $y \notin U$, then by Proposition 1.4, $x \notin \delta Cl_p(\{y\})$, and thus by Proposition 1.1 (1), $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$. Since (X, τ) is (δ, p)-R_1, there exists a (δ, p)-open V_y such that $\delta Cl_p(\{y\}) \subset V_y$ and $x \notin V_y$. Thus again by Propositions 1.1 (1) and 1.4, $y \notin \delta Cl_p(\{x\})$. Therefore, $\delta Cl_p(\{x\}) \subset U$, and hence, (X, τ) is (δ, p)-R_0. □

Definition 3.4. *A topological space (X, τ) is said to be (δ, p)-symmetric if for each $x, y \in X, x \in \delta Cl_p(\{y\})$ implies $y \in \delta Cl_p(\{x\})$.*

Theorem 3.5. *For a topological space (X, τ), the following properties are equivalent:*
(1) (X, τ) is a (δ, p)-R_0 space;
(2) (X, τ) is (δ, p)-symmetric.

Proof. (1)→(2): Assume X is (δ,p)-R_0. Let $x \in \delta Cl_p(\{y\})$ and U be any (δ,p)-open set such that $y \in U$. Now by hypothesis, $x \in U$. Therefore, every (δ,p)-open set which contain y containing x. Hence by Proposition 1.4, $y \in \delta Cl_p(\{x\})$.
(2)→(1): Let U be a (δ,p)-open set and $x \in U$. If $y \notin U$, then by Proposition 1.4, $x \notin \delta Cl_p(\{y\})$, and hence by assumption, $y \notin \delta Cl_p(\{x\})$. This implies that $\delta Cl_p(\{x\}) \subset U$. Hence, (X,τ) is (δ,p)-R_0. □

Theorem 3.6. *For a space (X,τ), the following are equivalent:*
(1) (X,τ) is (δ,p)-T_1;
(2) (X,τ) is (δ,p)-T_0 and (δ,p)-R_0.

Proof. (1)→(2): Follows from Remark 2.9 (1) and Theorem 2.18.
(2)→(1): Let $x, y \in X$ and $x \neq y$. Since X is (δ,p)-T_0, we may assume without loss of generality that $x \in G_1 \subset \{y\}^c$ for some $G_1 \in \delta PO(X,\tau)$. Thus by Proposition 1.4, $x \notin \delta Cl_p(\{y\})$, and hence by Theorem 3.5, $y \notin \delta Cl_p(\{x\})$. Thus again by Proposition 1.4, there exists $G_2 \in \delta PO(X,\tau)$ such that $y \in G_2 \subset \{x\}^c$. Hence, (X,τ) is (δ,p)-T_1. □

Corollary 3.7. *For a (δ,p)-R_0 topological space (X,τ), the following are equivalent:*
(1) (X,τ) is (δ,p)-T_0;
(2) (X,τ) is (δ,p)-D_1;
(3) (X,τ) is (δ,p)-T_1.

Proof. (1)→(3): Follows from Theorem 3.6.
(3)→(2): Follows from Remark 2.9 (2).
(2)→(1): Follows from Corollary 2.15. □

Theorem 3.8. *For a space (X,τ), the following are equivalent:*
(1) (X,τ) is (δ,p)-T_2;
(2) (X,τ) is (δ,p)-T_1 and (δ,p)-R_1.

Proof. Follows from Remark 2.9 (1) and Theorem 2.18. □

Remark 3.9. *It is clear from Theorems 3.6 and 3.8 that any space that is* (δ,p)-T_1 *but not* (δ,p)-T_2 *is* (δ,p)-R_0 *but not* (δ,p)-R_1.

Definition 3.10. *Let* A *be a subset of a space* X*. The* (δ,p)*-kernel of* A*, denoted by* $\delta Ker_p(A)$*, is defined to be the set* $\cap\{U \in \delta PO(X,\tau) : A \subset U\}$.

Lemma 3.11. *Let* (X,τ) *be a topological space and* $A \subset X$*. Then* $\delta Ker_p(A) = \{x \in X : \delta Cl_p(\{x\}) \cap A \neq \emptyset\}$.

Proof. Let $x \in \delta Ker_p(A)$ and $\delta Cl_p(\{x\}) \cap A = \emptyset$. Hence, $x \notin (\delta Cl_p(\{x\}))^c$ which is a (δ,p)-open set containing A (Corollary 1.5 (1)). This is absurd, since $x \in \delta Ker_p(A)$. Consequently, $\delta Cl_p(\{x\}) \cap A \neq \emptyset$. Next, let x such that $\delta Cl_p(\{x\}) \cap A \neq \emptyset$ and suppose that $x \notin \delta Ker_p(A)$. Then there exists a (δ,p)-open set U containing A and $x \notin U$. Let $y \in \delta Cl_p(\{x\}) \cap A$. Then by Proposition 1.4, $x \in U$, a contradiction. □

Lemma 3.12. *Let* (X,τ) *be a topological space and* $x \in X$*. Then* $y \in \delta Ker_p(\{x\})$ *if and only if* $x \in \delta Cl_p(\{y\})$.

Proof. Suppose that $y \notin \delta Ker_p(\{x\})$. Then there exists a (δ,p)-open set V containing x such that $y \notin V$. Therefore by Proposition 1.4, $x \notin \delta Cl_p(\{y\})$. The converse is similarly shown. □

Lemma 3.13. *The following statements are equivalent for any points* x *and* y *in a topological space* (X,τ)*:*
(1) $\delta Ker_p(\{x\}) \neq \delta Ker_p(\{y\})$;
(2) $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$.

Proof. (1)→(2): Suppose that $\delta Ker_p(\{x\}) \neq \delta Ker_p(\{y\})$, then there exists a point z in X such that $z \in \delta Ker_p(\{x\})$ and $z \notin \delta Ker_p(\{y\})$. From $z \in \delta Ker_p(\{x\})$ it follows that $\{x\} \cap \delta Cl_p(\{z\}) \neq \emptyset$ which implies $x \in \delta Cl_p(\{z\})$. By $z \notin \delta Ker_p(\{y\})$, we have $\{y\} \cap \delta Cl_p(\{z\}) = \emptyset$. Since $x \in \delta Cl_p(\{z\})$, it follows from Proposition 1.1 (2) and Corollary 1.5 (1) that $\delta Cl_p(\{x\}) \subset \delta Cl_p(\{z\})$, and thus $\{y\} \cap \delta Cl_p(\{x\}) = \emptyset$. Hence by Proposition 1.1 (1), $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$.

(2)→(1): Suppose that $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$. Then there exists a point z in X such that $z \in \delta Cl_p(\{x\})$ and $z \notin \delta Cl_p(\{y\})$. Thus it follows from Proposition 1.4 that there exists a (δ, p)-open set containing z and therefore x but not y, so $y \notin \delta Ker_p(\{x\})$. Hence, $\delta Ker_p(\{x\}) \neq \delta Ker_p(\{y\})$. □

Corollary 3.14. *A topological space (X, τ) is (δ, p)-R_1 if and only if for $x, y \in X, \delta Ker_p(\{x\}) \neq \delta Ker_p(\{y\})$, there exist disjoint (δ, p)-open sets U and V such that $\delta Cl_p(\{x\}) \subset U$ and $\delta Cl_p(\{y\}) \subset V$.*

Proof. Follows from Lemma 3.13. □

Theorem 3.15. *A topological space (X, τ) is a (δ, p)-R_0 space if and only for any x and y in X, $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$ implies $\delta Cl_p(\{x\}) \cap \delta Cl_p(\{y\}) = \emptyset$.*

Proof. **Necessity.** Assume (X, τ) is (δ, p)-R_0 and $x, y \in X$ such that $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$. Then, there exist $z \in \delta Cl_p(\{x\})$ such that $z \notin \delta Cl_p(\{y\})$ (or $z \in \delta Cl_p(\{y\})$ such that $z \notin \delta Cl_p(\{x\})$). Thus by Proposition 1.4, there exists $V \in \delta PO(X, \tau)$ such that $y \notin V$ and $z \in V$; hence again by Proposition 1.4, $x \in V$ and $x \notin \delta Cl_p(\{y\})$. Thus by Corollary 1.5 (1), $x \in (\delta Cl_p(\{y\}))^c \in \delta PO(X, \tau)$, but (X, τ) is (δ, p)-R_0, so $\delta Cl_p(\{x\}) \subset (\delta Cl_p(\{y\}))^c$. The proof for otherwise is similar.
Sufficiency. Let $V \in \delta PO(X, \tau)$ and let $x \in V$. We will show that $\delta Cl_p(\{x\}) \subset V$. Suppose $y \notin V$. Then by Propositions 1.1 (1) and 1.4, $x \notin \delta Cl_p(\{y\})$. Thus by Proposition 1.1 (1), $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$. By assumption, $\delta Cl_p(\{x\}) \cap \delta Cl_p(\{y\}) = \emptyset$, and thus again by Proposition 1.1 (1), $y \notin \delta Cl_p(\{x\})$. Hence, $\delta Cl_p(\{x\}) \subset V$, and therefore, (X, τ) is (δ, p)-R_0. □

Theorem 3.16. *A topological space (X, τ) is a (δ, p)-R_0 space if and only if for any points x and y in X, $\delta Ker_p(\{x\}) \neq \delta Ker_p(\{y\})$ implies $\delta Ker_p(\{x\}) \cap \delta Ker_p(\{y\}) = \emptyset$.*

Proof. **Necessity.** Suppose that (X,τ) is a (δ,p)-R_0 space. Thus by Lemma 3.13, for any points x and y in X, if $\delta Ker_p(\{x\}) \neq \delta Ker_p(\{y\})$, then $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$. Now we prove that $\delta Ker_p(\{x\}) \cap \delta Ker_p(\{y\}) = \emptyset$. Assume that $z \in \delta Ker_p(\{x\}) \cap \delta Ker_p(\{y\})$. By $z \in \delta Ker_p(\{x\})$ and Lemma 3.12, it follows that $x \in \delta Cl_p(\{z\})$. Thus by Theorem 3.15, $\delta Cl_p(\{x\}) = \delta Cl_p(\{z\})$. Similarly, we have $\delta Cl_p(\{y\}) = \delta Cl_p(\{z\}) = \delta Cl_p(\{x\})$, a contradiction. Hence, $\delta Ker_p(\{x\}) \cap \delta Ker_p(\{y\}) = \emptyset$.
Sufficiency. Let (X,τ) be a topological space such that for any points x and y in X, $\delta Ker_p(\{x\}) \neq \delta Ker_p(\{y\})$ implies $\delta Ker_p(\{x\}) \cap \delta Ker_p(\{y\}) = \emptyset$. Assume that $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$. Then by Lemma 3.13, $\delta Ker_p(\{x\}) \neq \delta Ker_p(\{y\})$, and therefore by assumption, $\delta Ker_p(\{x\}) \cap \delta Ker_p(\{y\}) = \emptyset$. Now if $z \in \delta Cl_p(\{x\})$, then by Lemma 3.12, $x \in \delta Ker_p(\{z\})$, and therefore, $\delta Ker_p(\{x\}) \cap \delta Ker_p(\{z\}) \neq \emptyset$. By hypothesis, $\delta Ker_p(\{x\}) = \delta Ker_p(\{z\})$. Thus $z \in \delta Cl_p(\{x\}) \cap \delta Cl_p(\{y\})$ implies that $\delta Ker_p(\{x\}) = \delta Ker_p(\{z\}) = \delta Ker_p(\{y\})$, a contradiction. Therefore, $\delta Cl_p(\{x\}) \neq \delta Cl_p(\{y\})$ implies that $\delta Cl_p(\{x\}) \cap \delta Cl_p(\{y\}) = \emptyset$, and thus by Theorem 3.15, (X,τ) is (δ,p)-R_0. □

Theorem 3.17. *For a topological space (X,τ), the following properties are equivalent :*
(1) (X,τ) is a (δ,p)-R_0 space;
(2) For any nonempty set A and $G \in \delta PO(X,\tau)$ such that $A \cap G \neq \emptyset$, there exists $F \in \delta PC(X,\tau)$ such that $A \cap F \neq \emptyset$ and $F \subset G$;
(3) For any $G \in \delta PO(X,\tau), G = \cup\{F \in \delta PC(X,\tau) : F \subset G\}$;
(4) For any $F \in \delta PC(X,\tau), F = \delta Ker_p(F)$;
(5) For any $x \in X, \delta Cl_p(\{x\}) \subset \delta Ker_p(\{x\})$.

Proof. (1)→(2): Let A be a nonempty set of X and $G \in \delta PO(X,\tau)$ such that $A \cap G \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in G \in \delta PO(X,\tau), \delta Cl_p(\{x\}) \subset G$. Set $F = \delta Cl_p(\{x\})$, then $F \in \delta PC(X,\tau)$ by Corollary 1.5 (1), $F \subset G$ and $A \cap F \neq \emptyset$.
(2)→(3): Let $G \in \delta PO(X,\tau)$, then $G \supset \cup\{F \in \delta PC(X,\tau) : F \subset G\}$. Let x be any point of G. There exists $F \in \delta PC(X,\tau)$ such that $x \in F$

and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in \delta PC(X,\tau) : F \subset G\}$ and hence $G = \cup\{F \in \delta PC(X,\tau) : F \subset G\}$.
(3)→(4): Clear.
(4)→(5): Let x be any point of X and $y \notin \delta Ker_p(\{x\})$. There exists $V \in \delta PO(X,\tau)$ such that $x \in V$ and $y \notin V$; hence $\delta Cl_p(\{y\}) \cap V = \emptyset$. By (4), $(\delta Ker_p(\delta Cl_p(\{y\}))) \cap V = \emptyset$ and there exists $G \in \delta PO(X,\tau)$ such that $x \notin G$ and $\delta Cl_p(\{y\}) \subset G$. Therefore, $\delta Cl_p(\{x\}) \cap G = \emptyset$ and thus by Proposition 1.4 and Corollary 1.5 (1), $y \notin \delta Cl_p \delta Cl_p(\{x\}) = \delta Cl_p(\{x\})$. Consequently, $\delta Cl_p(\{x\}) \subset \delta Ker_p(\{x\})$.
(5)→(1): Clear. □

Theorem 3.18. *For a topological space (X,τ), the following properties are equivalent:*
(1) (X,τ) is a (δ,p)-R_0 space;
(2) If F is (δ,p)-closed and $x \in F$, then $\delta Ker_p(\{x\}) \subset F$;
(3) If $x \in X$, then $\delta Ker_p(\{x\}) \subset \delta Cl_p(\{x\})$.

Proof. (1)→(2): Let F be (δ,p)-closed and $x \in F$. Then $\delta Ker_p(\{x\}) \subset \delta Ker_p(F)$. By (1), it follows from Theorem 3.17 that $\delta Ker_p(F) = F$. Thus, $\delta Ker_p(\{x\}) \subset F$.
(2)→(3): Since $x \in \delta Cl_p(\{x\})$ (Proposition 1.1 (1)) and $\delta Cl_p(\{x\})$ is (δ,p)-closed (Corollary 1.5 (1)), by (2), $\delta Ker_p(\{x\}) \subset \delta Cl_p(\{x\})$.
(3)→(1): Let $x \in \delta Cl_p(\{y\})$. Then by Lemma 3.12, $y \in \delta Ker_p(\{x\})$. By (3), $y \in \delta Cl_p(\{x\})$. Therefore, $x \in \delta Cl_p(\{y\})$ implies that $y \in \delta Cl_p(\{x\})$. Hence by Theorem 3.5, (X,τ) is (δ,p)-R_0. □

Corollary 3.19. *For a topological space (X,τ), the following properties are equivalent :*
(1) (X,τ) is a (δ,p)-R_0 space;
(2) $\delta Cl_p(\{x\}) = \delta Ker_p(\{x\})$ for all $x \in X$.

Proof. Follows from Theorems 3.17 and 3.18. □

Definition 3.20. *A filterbase F in a space X is called (δ,p)-convergent to a point x in X, if for any (δ,p)-open set U of X containing x, there exists B in F such that B is a subset of U.*

Definition 3.21. *A net $\{x_\alpha\}_{\alpha\in\Lambda}$ in a space X is called (δ,p)-convergent to a point x in X, if for any (δ,p)-open set U of X containing x, there exists $\alpha_0\in\Lambda$ such that $x_\alpha\in U$ for each $\alpha\geq\alpha_0$.*

Lemma 3.22. *Let (X,τ) be a topological space and let x and y be any two points in X such that every net in X (δ,p)-converging to y (δ,p)-converges to x. Then $x\in\delta Cl_p(\{y\})$.*

Proof. Suppose that $x_n=y$ for each $n\in\mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a net in X that (δ,p)-converges to y. Thus by assumption, $\{x_n\}_{n\in\mathbb{N}}$ (δ,p)-converges to x. Hence by Proposition 1.4, $x\in\delta Cl_p(\{y\})$. □

Theorem 3.23. *For a topological space (X,τ), the following statements are equivalent :*
(1) (X,τ) is a (δ,p)-R_0 space;
(2) If $x,y\in X$, then $y\in\delta Cl_p(\{x\})$ if and only if every net in X (δ,p)-converging to y (δ,p)-converges to x.

Proof. (1)→(2): Let $x,y\in X$ such that $y\in\delta Cl_p(\{x\})$. Let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net in X such that $\{x_\alpha\}_{\alpha\in\Lambda}$ (δ,p)-converges to y. Since $y\in\delta Cl_p(\{x\})$, by Theorem 3.5, $x\in\delta Cl_p(\{y\})$. Since $\{x_\alpha\}_{\alpha\in\Lambda}$ (δ,p)-converges to y and $x\in\delta Cl_p(\{y\})$, it follows from Proposition 1.4 that $\{x_\alpha\}_{\alpha\in\Lambda}$ (δ,p)-converges to x. Conversely, let $x,y\in X$ such that every net in X (δ,p)-converging to y (δ,p)-converges to x. Then $x\in\delta Cl_p(\{y\})$ by Lemma 3.22. By Theorem 3.5, $y\in\delta Cl_p(\{x\})$.
(2)→(1): Assume that x and y are any two points of X such that $y\in\delta Cl_p(\{x\})$. Suppose that $x_n=y$ for each $n\in\mathbb{N}$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a net in X that (δ,p)-converges to y. Since $y\in\delta Cl_p(\{x\})$ and $\{x_n\}_{n\in\mathbb{N}}$ (δ,p)-converges to y, it follows from (2) that $\{x_n\}_{n\in\mathbb{N}}$ (δ,p)-converges to x. Thus by Proposition 1.4, $x\in\delta Cl_p(\{y\})$. Hence by Theorem 3.5, (X,τ) is (δ,p)-R_0. □

4. Sober (δ,p)-R_0 spaces

Definition 4.1. *A topological space (X,τ) is said to be sober (δ,p)-R_0 if $\cap_{x\in X}\delta Cl_p(\{x\})=\emptyset$.*

Theorem 4.2. *A topological space* (X, τ) *is sober* (δ, p)-R_0 *if and only if* $\delta Ker_p(\{x\}) \neq X$ *for every* $x \in X$.

Proof. **Necessity.** Suppose that the space (X, τ) is sober (δ, p)-R_0. Assume that there is a point y in X such that $\delta Ker_p(\{y\}) = X$. Thus by Lemma 3.11, $y \in \cap_{x \in X} \delta Cl_p(\{x\})$, a contradiction.
Sufficiency. Assume that $\delta Ker_p(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap_{x \in X} \delta Cl_p(\{x\})$, then every (δ, p)-open set containing y must contain every point of X. This implies that the space X is the unique (δ, p)-open set containing y. Hence, $\delta Ker_p(\{y\}) = X$, a contradiction. □

Definition 4.3. *A function* $f : X \to Y$ *is called* (δ, p)*-closed if the image of every* (δ, p)*-closed subset of* X *is* (δ, p)*-closed in* Y.

Theorem 4.4. *If* $f : X \to Y$ *is an injective* (δ, p)*-closed function and* X *is sober* (δ, p)-R_0*, then* Y *is sober* (δ, p)-R_0.

Proof. Straightforward. □

Theorem 4.5. *If* X *is a sober* (δ, p)-R_0 *topological space and* Y *is any topological space, then the product space* $X \times Y$ *is sober* (δ, p)-R_0.

Proof. By showing that $\cap_{(x,y) \in X \times Y} \delta Cl_p(\{(x, y)\}) = \emptyset$, we are done. By Corollary 1.13 (3), we have:

$$\begin{aligned} \cap_{(x,y) \in X \times Y} \delta Cl_p(\{(x, y)\}) &\subset \cap_{(x,y) \in X \times Y} (\delta Cl_p(\{x\}) \times \delta Cl_p(\{y\})) \\ &= \cap_{x \in X} \delta Cl_p(\{x\}) \times \cap_{y \in Y} \delta Cl_p(\{y\}) \\ &\subset \emptyset \times Y \\ &= \emptyset \end{aligned}$$

□

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