Twin Primes Conjecture

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Abstract:

Twin prime conjecture, also known as **Polignac's conjecture**, in number theory, assertion that there are infinitely many twin primes, or pairs of primes that differ by 2. The first statement of the twin prime conjecture was given in 1846 by French mathematician Alphonse de Polignac, who wrote that any even number can be expressed in infinite ways as the difference between two consecutive primes.

Introduction.

The number of Twin primes: There are infinitely many twin primes. Two primes (*p, q)* are called twin primes if their difference is 2. Let $\pi_2(x)$ be the number of primes p such that $p<= x$ and $p + 2$ is also a prime.Then it is known :

$$\pi_2(x) \leq C_1 C_2 \frac{x}{(\log x)^2}\left(1 + O(\frac{\log\log x}{\log x})\right)$$

where $C_2 = \prod_{P>2}(1-(p-1)^{-2}) = 0.66016...$ is the twin-prime constant. Another constant C_1 is conjectured to be 2, by Hardy and Littlewood, but the best result so far is $C1 = 7 + \varepsilon$ obtained by Bombieri, Friedlander, and Iwaniec (1986). In practice this seems to be a exceptionally good estimate (even for small N) {seeTable 1}.

TABLE 1. Twin primes less than N

N	actual number	predicted integral	ratio
10^3	35	46	28
10^4	205	214	155
10^5	1224	1249	996
10^6	8169	8248	6917
10^7	58980	58754	50822
10^8	440312	440368	389107
10^9	3424506	3425308	3074426
10^{10}	27412679	27411417	24902848
10^{11}	224376048	224368865	205808661
10^{12}	1870585220	1870559867	1729364449
10^{13}	15834664872	15834598305	14735413063
10^{14}	135780321665	135780264894	127055347335
10^{15}	1177209242304	1177208491861	1106793247903

Brun in 1919 proved an interesting and important result as follows :

$$B = \sum_{p,p+2\,:\,\text{twin primes}} \left(\frac{1}{p} + \frac{1}{p+2}\right) < \infty.$$

B is now called the Brun's constant. (B = 1.90216054...)

Prime pairs {n, n+2k},q in N .

What if we replace the polynomials {n, n + 2} with {n, n + 2k}. In this case w(p) = 1 if p/2k and w(p) = 2 otherwise σύμφωνα με την σχέση..

$$\prod_p \frac{1 - w(p)/p}{(1-1/p)^k}.$$

so the adjustment factor becomes

$$C_{2,k} = C_2 \prod_{p|k, p>2} \frac{p-1}{p-2}.$$

The expected number of prime pairs f{p, p + 2k} with p <= N is

$$\pi_k(x) = 2c_{2,k} \int_2^N \frac{dx}{(\ell og x)^2} \approx \frac{2c_{2,k} N}{(\ell og N)^2}$$

For example, when searching for primes {n, n + 210} we expect to find (asymptotically) $\frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5}$ = 3.2 times as many primes as we find twins.

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TABLE 2. Prime pairs $\{n, n+2k\}$ with $n \leq N$

	$k=6$		$k=30$		$k=210$	
N	actual	predicted	actual	predicted	actual	predicted
10^3	74	86	99	109	107	118
10^4	411	423	536	558	641	653
10^5	2447	2493	3329	3316	3928	3962
10^6	16386	16491	21990	21981	26178	26358
10^7	117207	117502	156517	156663	187731	187976
10^8	879980	880730	1173934	1174300	1409150	1409141
10^9	6849047	6850611	9136632	9134141	10958370	10960950

Table 2 shows that this is indeed the case.
Erd˝os was the first to prove that there are infinitely many n for which $p_{n+1} - p_n$ is appreciably greater than log p_n, and Rankin proved that there are infinitely many n for which

$$p_{n+1} - p_n > c(\log p_n)\frac{(\log_2 p_n)(\log_4 p_n)}{(\log_3 p_n)^2}$$

where $\log_2 x$ = log log x and so on, and c is a positive constant. In the opposite direction, Bombieri and Davenport(1966) proved that there are infinitely many n for which

$$p_{n+1} - p_n < (0.46\cdots)\log p_n$$

Of course, if the "prime twins" conjecture is true, there are infinitely many n for which $p_{n+1} - p_n = 2$.There is a somewhat paradoxical situation in connection with the limit points of the sequence

$$\frac{p_{n+1} - p_n}{\log p_n}$$

Erd˝os,and Ricci (independently) have shown that the set of limit points has positive Lebesgue measure, and yet no number is known for which it can be asserted that it belongs to the set.

Existence theorems twin primes

Theorem 1.

Every positive integer if written in the form p=3k+1 or p=3k-1 , with $k \in N$ can be prime if and only if κ=2μ, $\mu \in N$ and the prime to be derived in the form 6μ + 1 or 6μ-1 respectively.

Proof ..
We assume that the prime p is written in the form p=3k+1 or p=3k-1 then p>2. We have 2 choices ....α) $k = 2\mu + 1 \Rightarrow p = 3(2\mu+1)+1 = 2(3\mu+2)$ ή $p = 3(2\mu+1)-1 = 2(3\mu+1)$ ie 2 / p which is absurd, because p is prime and not composite with p> 2. Β) The second case is summarized as $k = 2\mu \Rightarrow p = 3(2\mu)+1 = 6\mu+1$ or $p = 6\mu - 1$.

Theorem 2.(Wilson's)

An integer $p > 1$ is prime if and only when applicable the modulus

$(p-1)! = -1(\text{mod}\, p)$.

Theorem 3.

A positive integer m, It **can be written in the form** $\text{mod}(m,3) = u \Leftrightarrow m = k \cdot 3 + u, k \in N$ **with** $0 \le u < 3$**. According to Theorem 1 and by definition that two primes (*p, p')* are called q twin primes if** $p - p' = 2q, q \in N$ **, we get that every prime p of q-twin primes , had to be written in the form** ...

$$\mathbf{Mod}[\boldsymbol{p}, \mathbf{3}] = \boldsymbol{u} \wedge \boldsymbol{p} - \boldsymbol{p}' = \mathbf{2}\boldsymbol{q},\ q \in N,\ 0 \le u < 3.$$

Proof...

Generally accept as valid for a prime p belonging to N that ..

$\text{Mod}[p,3] = u \Leftrightarrow \{p = 3k + u, p' = 3k + (u - 2q),\ q,k \in N,\ 0 \le u < 3\}$ (1)

We examine three cases for q = 1 (Twin Primes).

i)From (1) if u=2 then $\{p = 3k + 2, p' = 3k\}$ which true when k = 1, because if k> 1 the p,p' not both first. Readily accepted result only pair primes(p,p')=(5,3).

ii) If u=1 then $\{p = 3k + 1, p' = 3k - 1\}$ with κ=2μ, resulting pairs of primes according to form pairs

according to form pairs, ie δηλαδή $\{p = 6\mu + 1, p' = 6\mu - 1\}$ where m in N and for certain values of μ.

iii) The case u = 0 is not valid, because one the prime of twin pair, it shows composite as a multiple of the number 3 and the other even.

Generalization when q> 1 and u ≠ 0.

From the general equation (1) is obtained .

$\mathrm{Mod}[p,3] = u \Leftrightarrow \{p = 3k + u, p' = 3k + (u - 2q), q,k \in N, 0 \le u < 3\}$ and distinguish two cases equivalent, with respect to the choice of q as an even or odd ...

i)If q=2μ+1 or q=2μ with u=2 ∧ u=1 ∧ u≠ 0 then $p' = 3k + 2 \wedge p = 3k + 2 - 2q \wedge p \le p'$ and

$p' = 3k + 1 \wedge p = 3k + 1 - 2q \wedge p \le p', \{p, p'\},$ it must be Primes and also apply $0 \le k \le$ IntegerPart$[m/3]$.

ii) In the most powerful form of force $\mathrm{Mod}[p,3] \neq 0 \wedge p - p' = 2q \wedge p < m, \{p', p\}$, Primes , with q=2μ+1 or q=2μ.

Example twin primes until the integer 100, in a language mathematica ...

***1rd* Method..**

in(1):=

m:=100;q:=3;

Reduce[Mod[p,3] ≠0 And p-p'==2q ∧ p<m,{p',p},Primes]

Count[Reduce[Mod[p,3] ≠0 And p-p'==2q ∧ p<=m ,{p',p},Primes],Except[False]]→

Out[2]:= {p' == 5 && p == 11) || (p' == 7 && p == 13) || (p' == 11 && p == 17) || (p' == 13 && p == 19) || (p' == 17 && p == 23) || (p' == 23 && p == 29) || (p'== 31 && p == 37) || (p'== 37 && p == 43) || (p'== 41 && p == 47) || (p' == 47 && p == 53) || (p' == 53 && p == 59) || (p' == 61 && p == 67) || (p' == 67 && p == 73) || (p' == 73 && p == 79) || (p' == 83 && p == 89}

Out[3]:=15

2rd Method..

in(1):=

m:=100;q:=2;

Cases[Table[Reduce[p'== 3k+2 And p== 3k+2-2q∧p<= p'∧p'<=m,{p,p'}, Primes], {k,0,IntegerPart[m/3]}],Except[False]]

Count[Table[Reduce[p'== 3k +2 And p== 3k+2-2q ∧p<= p '∧p'<= m,{p',p},Primes], {k,0,IntegerPart[m/3]}],Except[False]]

Out[2]: {p == 7 && p' == 11, p == 13 && p' == 17, p == 19 &&p' == 23, p == 37 && p' == 41, p == 43 &&p' == 47, p == 67 && p' == 71, p == 79 &&p' == 83}

Out[3]:=7

We are continuing the process of analysis with 1mod(3)…

in(1):=

m:=100;q:=2;

Cases[Table[Reduce[p'== 3k+1 And p== 3k+1-2q ∧p<= p '∧p'<= m,{p,p'},Primes], {k,0,IntegerPart[m/3]}],Except[False]]

Count[Table[Reduce[p'== 3k +1 And p== 3k+1-2q ∧p<= p '∧p'<= m,{p',p},Primes], {k,0,IntegerPart[m/3]}],Except[False]]

Out[2]: {p == 3 && p' == 7} ; **Out[3]:=**1**..**Therefore Number twin primes =7+1=8

Theorem 4.

For any integer n greater than 2, the pair {n (n + 2)} is a pair of twin primes if only if: $4[(n-1)!+1]+n=0 \bmod n(n+2)$ **. This characterization factorial and modular OF twin primes was discovered by P. A. Clement in 1949 [2].**

Proof. The sufficiency is obvious as divisions by n and n+2 separately reduce either Wilson's theorem or to a simple modification of it.

The necessity follows as easily,but we wish to indicate how (1) may be obtain directly.Thus,with n and n+2 both primes,we have

$$(n-1)!+1=0 \bmod n \quad (2)$$

$$(n+1)!+1=0 \bmod (n+2) \quad (3)$$

Reducing the factorial of (3) mod(n+2) and rewriting as an equation we obtain

$$2(n-1)!+1=k(n+2), k \in N \quad (4)$$

Using (2) we have $2k+1=0 \bmod n$ (5).Substitution of (5) in (4) determines the congruence of the theorem

Theorem 5.

The number of pairs of twins primes is infinite and this follows from Theorem 4, since the ratio g= $[4[(n-1)!+1]+n]/[\bmod n(\ n+2)\ ] \to \infty$ if $n \to \infty$.

Proof... Applicable to $(n-1)! > \frac{n^{n-1}}{e^{n-1}}$ (1) and the relation $[4[(n-1)!+1]+n]=g[n(n+2)]$ (2) where g belong in Z+. From (1) and (2) explicit that...

$$g > \frac{4 \cdot \frac{n^{n-1}}{e^{n-1}} + 4 + n}{n(n+2)} = \frac{4 \cdot n^{n-1}}{e^{n-1} \cdot (n+2) \cdot n} + \frac{4+n}{(n+2) \cdot n} \to \infty + 0 \to \infty$$ ie we see if $n \to \infty \Rightarrow g \to \infty$.

Therefore because $g > 1$ there are always twins primes and infinite as increases n, ie .If $n \to \infty \Rightarrow g \to \infty$ .If $g <= 1$ we would have limited couples primes and possibly to decrease in number..

Theorem 6.

Equivalence finding process primes couples to conjecture Goldbach to find pairs of twins primes method.

Proof..

According to Theorem 1, each positive integer m can write as $\bmod(m,3)=u$, $\Leftrightarrow m=k \cdot 3+u$, $k \in N$ με $0 \le u < 3$ and by definition that two primes (*p, p')* are called q twin primes άν $p - p' = 2q, q \in N$, we have that the pair q twin, had to be written in the form .. $\text{Mod}[p, 3] = u \wedge p - p' = 2q,\ q \in N$, $0 \le u < 3$. But if $p + (-p') = 2q, q \in N$ which means 2 choices ..

i)If q in N with u=2 ∨ u=1 with u≠ 0 then $p = 3k + 2 \wedge -p' = 3k + 2 - 2q \wedge p \le p'$ and $p = 3k + 1 \wedge -p' = 3k + 1 - 2q \wedge p \le p', \{p, p'\}$, it must be Primes και also into force $0 \le k \le$ IntegerPart$[m/3]$+1.

ii) If q in N with u=2 ∨ u=1 and u≠ 0 then $Mod(p', 3) \neq 0 \wedge p + p' = 2q \wedge p \le p'$ and $p \le 2q, \{p, p'\}, it\ must\ be$ Primes.

By applying program format language mathematica for both cases, for example if we take 2q = 200 => q = 100 and therefore ...

1^rd Method..

in(1):=

```
q:=100;
Cases[Table[Reduce[ p'== 3k+1 [And] p== -3k-1+2q  ∧p<= p',{p,p'},Primes],
{k,0,IntegerPart[2q/3]+1}],Except[False]]
Count[Table[Reduce[ p'== 3k +1 [And] p== -3k-1+2q ∧p<= p ',{p',p},Primes],
{k,0,IntegerPart[2q/3]+1}],Except[False]]
Cases[Table[Reduce[ p'== 3k+2 [And] p== -3k-2+2q  ∧p<= p ',{p,p'},Primes],
{k,0,IntegerPart[2q/3]+1}],Except[False]]
Count[Table[Reduce[ p'== 3k +2 [And] p== -3k-2+2q ∧p<= p ',{p',p},Primes],
{k,0,IntegerPart[2q/3]+1}],Except[False]]
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Out[2]:

{p == 97 && p’ == 103, p == 73 && p’ == 127, p == 61 && p’ == 139, p == 43 && p’ == 157, p == 37 && p’ == 163, p == 19 && p’ == 181, p == 7 && p’ == 193}

Out[3]:=7

Out[4]:={p == 3 && p’ == 197} ;**Out[5]:=**1

Therefore Number twin primes =7+1=8

2^rd Method..

in(1):=

```
q:=100;
Reduce[ Mod[p',3] ≠0 [And] p+p'==2q∧ p<= 2q ∧p'>p,{p',p},Primes]
Count[Reduce[ Mod[p',3]≠0 [And] p+p'==2q∧p<=2q ∧p'>p,{p',p},Primes],Except[False]]
```

Out[2]:=

(p’ == 103 && p == 97) || (p’ == 127 && p == 73) || (p’ == 139 && p == 61) || (p’ == 157 && p == 43) || (p’ == 163 && p == 37) || (p’ == 181 && p == 19) || (p’ == 193 && p == 7) || (p’ == 197 && p == 3)

Out[3]:=8

The number of Goldbach problem.. Every sufficiently large even number 2n is the sum of two primes. The Asymptotic formula for the number of representations is

$$r_2(2n) \sim C_2 \frac{4n}{(\log 2n)^2} \prod_{\substack{p>2 \\ p|n}} \frac{p-1}{p-2}$$

with

$$r_2(2n) = \#\{(p,q) :\ p,q \text{ primes},\ p+q=2n\}$$

where

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.66016\ldots$$

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