On Improper Integrals

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Abstract

The writing intends to point out aspects of conflict regarding some standard improper integrals.

Introduction

Two standard integrals frequently used in physics have been considered and the results have been analyzed to bring out some conflicting aspects. An interesting mathematical issue has been identified.

Section I

We consider the standard integral[1]

$$I = \int_{-\infty}^{+\infty} \frac{dk_0}{k^2 + s - i\varepsilon} \quad (1)$$

$$I = \int_{-\infty}^{+\infty} \frac{dk_0}{{k_0}^2 - \left|\vec{k}\right|^2 + s - i\varepsilon} = \frac{i\pi}{\sqrt{\left|\vec{k}\right|^2 + s}}$$

$$I = \int_{-\infty}^{+\infty} \frac{dk_0}{{k_0}^2 - A^2 - i\varepsilon}\ ; A^2 > 0\ (2)$$

While integration with respect to k_0 the variable $\left|\vec{k}\right|^2$ is held constant[asides s]

When $A^2 = \left|\vec{k}\right|^3 - s > 0$

$$I = \int_{-\infty}^{+\infty} \frac{dk_0}{{k_0}^2 + A^2 - i\varepsilon}; A^2 > 0 \quad (3)$$

We evaluate (2) and (3) ignoring the complex part

Evaluation of (2’), ignoring the imaginary part:

We evaluate the following improper integral by using limit concepts :

$$I = \int_{-\infty}^{+\infty} \frac{1}{x^{2} - a^2} dx \;\; (2')$$

Indefinite integral

$$\int \frac{1}{x^{2} - a^2} dx = ln \frac{x-a}{x+a}$$

The integral represented by (2') may be interpreted as

$$I = \frac{1}{2a} Lim_{q \to s, M \to \infty} \left[\left[ln \frac{x-a}{x+a} \right]_{-M}^{-q} + \left[ln \frac{x-a}{x+a} \right]_{-q}^{+q} + \left[ln \frac{x-a}{x+a} \right]_{q}^{M} \right]$$

$$= \frac{1}{2a} Lim_{q \to s, M \to \infty} \left[ln \left| \frac{-q-a}{-q+a} \right| - ln \left| \frac{-M-a}{-M+a} \right| + ln \left| \frac{q-a}{q+a} \right| - ln \left| \frac{-q-a}{-q+a} \right| + ln \left| \frac{M-a}{M+a} \right| - ln \left| \frac{q-a}{q+a} \right| \right]$$

$$= \frac{1}{2a} Lim_{q \to s, M \to \infty} \left[ln \frac{|q+a|}{|q-a|} - ln \frac{|M+a|}{|M-a|} + ln \frac{|q-a|}{|q+a|} - ln \frac{|q+a|}{|q-a|} + ln \frac{|M-a|}{|M+a|} - ln \frac{|q-a|}{|q+a|} \right]$$

$$= \frac{1}{2a} Lim_{q \to s, M \to \infty} \left[ln \frac{|q+a|}{|q-a|} - ln \frac{|q+a|}{|q-a|} + ln \frac{|q-a|}{|q+a|} - ln \frac{|q-a|}{|q+a|} + ln \frac{|M-a|}{|M+a|} - ln \frac{|M+a|}{|M-a|} \right]$$

$$= \frac{1}{2a} Lim_{M \to \infty} \, ln \frac{|M-a|}{|M+a|} - \frac{1}{2a} Lim_{M \to \infty} \frac{|M+a|}{|M-a|}$$

$$= \frac{1}{2a} [ln1 - ln1] = 0$$

$$\int_{-\infty}^{+\infty} \frac{1}{x^{2} - a^2} dx \;\; ; a^2 > 0 \; (4)$$

Next we pass on to the evaluation of

$$\int_{-\infty}^{+\infty} \frac{1}{x^{2} + a^2} dx \, ; a^2 > 0 \;\; (5)$$

The indefinite integral

$$\int \frac{1}{x^{2} + a^2} dx = \frac{1}{a} tan^{-1} \frac{x}{a} + C$$

Since the integrand an even function and positive everywhere on the x-axis

$$\int_{-\infty}^{+\infty} \frac{1}{x^{2} + a^2} dx = 2 \int_{0}^{+\infty} \frac{1}{x^{2} + a^2} dx \to \infty \;\; (5)$$

[The indefinite integral, in fact, is not required to come to this conclusion since we know that the integrand is positive everywhere on the x axis]

Section II

Standard result[2]

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+ns-i\varepsilon)^3} = \frac{i}{32\pi^2 ns} \quad (4)$$

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+ns-i\varepsilon)^3}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns+i\varepsilon)^3}{(k^2+ns-i\varepsilon)^3(k^2+ns+i\varepsilon)^3}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{[(k^2+ns)+i\varepsilon]^3}{[(k^2+ns)^2+\varepsilon^2]^3}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns)^3 - i\varepsilon^3 + 3i\varepsilon(k^2+ns)(k^2+ns+i\varepsilon)}{[(k^2+ns)^2+\varepsilon^2]^3}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns)^3 - i\varepsilon^3 + 3i\varepsilon(k^2+ns)^2 - 3\epsilon^2(k^2+ns)}{[(k^2+ns)^2+\varepsilon^2]^3}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns)^3 - 3\epsilon^2(k^2+ns)}{[(k^2+ns)^2+\varepsilon^2]^3} - i\int \frac{d^4k}{(2\pi)^4} \frac{\varepsilon^3 - 3\varepsilon(k^2+ns)^2}{[(k^2+ns)^2+\varepsilon^2]^3} = \frac{i}{32\pi^2 ns}$$

$$I_1 = \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns)^3 - 3\epsilon^2(k^2+ns)}{[(k^2+ns)^2+\varepsilon^2]^3} = 0; I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{\varepsilon^3 - 3\varepsilon(k^2+ns)^2}{[(k^2+ns)^2+\varepsilon^2]^3} = \frac{i}{32\pi^2 ns}$$

$$I_1 = \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns)^3 - 3\varepsilon^2(k^2+ns)}{[(k^2+ns)^2+\varepsilon^2]^3}$$
$$= \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns)^3}{[(k^2+ns)^2+\varepsilon^2]^3} - 3\varepsilon^2 \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns)}{[(k^2+ns)^2+\varepsilon^2]^3}$$

$$I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{\varepsilon^3 - 3\varepsilon(k^2+ns)^2}{[(k^2+ns)^2+\varepsilon^2]^3} = \varepsilon \int \frac{d^4k}{(2\pi)^4} \frac{\varepsilon^2 - 3(k^2+ns)^2}{[(k^2+ns)^2+\varepsilon^2]^3}$$

Calculations based on I_1

$$I_1 = \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns)^3}{[(k^2+ns)^2+\varepsilon^2]^3} - 3\varepsilon^2 \int \frac{d^4k}{(2\pi)^4} \frac{(k^2+ns)}{[(k^2+ns)^2+\varepsilon^2]^3} = 0$$

For → 0 ,

$$I_1 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+ns)^3}$$

Since

$I_1 = 0$ we have for $\epsilon \to 0$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+ns)^3} = 0 \ \ (A)$$

Differentiating (A) with respect to s we have

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+ns)^4} = 0 \ \ (B)$$

Calculations based on I_2

$$I_2 = \varepsilon \int \frac{d^4k}{(2\pi)^4} \frac{\varepsilon^2 - 3(k^2+ns)^2}{[(k^2+ns)^2+\varepsilon^2]^3}$$

Asides the fact that $\varepsilon \to 0$ we have theadditional strength of (B)

For $\varepsilon \to 0$[and recalling (B)]

$$I_2 = \varepsilon \int \frac{d^4k}{(2\pi)^4} \frac{\varepsilon^3 - 3\varepsilon(k^2+ns)^2}{[(k^2+ns)^2+\varepsilon^2]^3} = -3\varepsilon \times \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+ns)^4} = 0 \neq \frac{i}{32\pi^2 ns}$$

Asides the fact that $\varepsilon \to 0$ we have the additional fact that (B) does not tend to infinity in which case there would have been a possibility of the integral becoming convergent. On the contrary it evaluates to zero with $\varepsilon \to 0$.

I_2 does not work out to its standard value as given by (4)

Tracing the Error

Consider a region which is analytic everywhere except at a finite number of poles. We encircle apole by a closed curve Cwhich does not enclose any other pole.

$$f(z)dz = udx - vdy + i(udy + vdx)$$

CR equations:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

CR equations represent exact differential criteria in this context. Therefore

$$\Rightarrow dA = udx - vdy;$$

$$\Rightarrow dB = udy + vdx$$

$$f(z)dz = dA + idB$$

$$\oint f(z)dz = \oint dA + i\oint dB = 0$$

But

$\oint f(z)dz \neq 0$ since this curve encompasses pole.

Conclusion

As claimed , we have arrived at some conflicts with the two the standard integrals. The source of error has also been traced

References

1. Sakurai J. J., Advanced Quantum Mechanics, Pearson Education, India, Appendix E,p327
2. Sakurai J. J., Advanced Quantum Mechanics, Pearson Education, India, Appendix E,p327 https://www.linkedin.com/posts/anamitra-palit-1b628b5_on-improper-integrals-activity-6625891701691125760-p9Z