Topological Algebra and its Applications

Expansivity Theory

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EXPANSIVITY THEORY

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Abstract. In this paper we introduce and develop the concept of expansivity of a tuple whose entries are elements from the polynomial ring $\mathbb{R}[x]$. As an inverse problem, we examine how to recover a tuple from the expanded tuple at any given phase of expansion. We convert the celebrated Sendov conjecture concerning the distribution of zeros of polynomials and their critical points into this language and prove some weak variants of this conjecture. We also apply this to the existence of solutions to differential equations. In particular, we show that a certain system of differential equation has no non-trivial solution.

1. Introduction and motivation

The sendov conjecture is the assertion that any complex coefficient polynomial $P_n(x)$ of degree $n \geq 2$ with sufficiently small zeros must lie in the same unit disk with some zero of $P_n'(x)$. More formally if $|a_i| < 1$ such that $P_n(a_i) = 0$, then there exist some b_k with $P_n'(b_k) = 0$ such that

$$|a_i - b_k| < 1.$$

There has and is a flurry of research devoted to this problem and manifestly the current literature contains dozens of papers just for the problem. There has really been substantive progress ever since it was posed. For instance, It has been shown in [5] that the conjecture holds for zeros near the unit circle. In [1], the conjecture has been verified for degree at most six. This was improved further to polynomials of degree at most seven in [2] and polynomials of degree at most eight in [4]. The best result thus far concerning sendov conjecture is found in [3], where it was verified to hold for sufficiently large degree polynomials.
In this paper, we develop the theory of expansivity and convert the sendov conjecture into this new language as

Conjecture 1 (Sendov). Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial of degree $n \geq 2$. Let $\{b_i\}_{i=1}^{n}$ be the set of zeros of $P(x)$ such that $|b_i| \leq 1$ and let $\mathcal{S} = (a_nx^n, a_{n-1}x^{n-1}, \ldots, a_1x, a_0)$ be a tuple representation of $P(x)$. For each b_i, there exist some $\mathcal{S}_a \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S})]$ such that

$$|\mathrm{Id}_{n+1}(b_i\mathcal{S}_e - \mathcal{S}_a)| \leq 1.$$

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2. Notations

Though every notation in this paper has been thoroughly explained where it is used, we find it here appropriate to set the stage by highlighting them. Through out this paper a tuple will always be represented by $\mathcal{S}$ or $\mathcal{S}_j$ where j is contained in the natural indexing set $\mathbb{N}$. Ocassionally, we will use the tuple $\mathcal{S}_{\mathbb{R}}$ to denote a tuple of the base field $\mathbb{R}$ and $\mathcal{S}_{\mathbb{R}[x]}$ for a tuple of $\mathbb{R}[x]$. We set $\mathcal{S}_0 := (0, 0, \ldots, 0)$ and we call it the null tuple. Similarly we denote the tuple $\mathcal{S}_e := (1, 1, \ldots, 1)$ and we call it the unit tuple. We denote the rank of an expansion on $\mathcal{S}$ by $\mathcal{R}(\mathcal{S})$, the limit of expansion on $\mathcal{S}$ by $\lim(\mathcal{S}^m)$, the local number of expansion on $\mathcal{S}$ by $\mathcal{L}(\mathcal{S})$, the degree of an expansion on $\mathcal{S}$ by $deg(\mathcal{S})$, the dimension of an expansion on $\mathcal{S}$ by $dim(\mathcal{S})$, and the measure of an expansion on $\mathcal{S}$ by $\mathcal{N}(\mathcal{S})$. Also, we set $\mathcal{S}(a) := (f_1(a), f_2(a), \ldots, f_n(a))$, where $\mathcal{S} = (f_1, f_2, \ldots, f_n)$. We denote the nth phase expanded tuple of $\mathcal{S}$ by $\mathcal{S}^n$. Also for any function f and g with $f \asymp g$, we mean there exists some contant α_1 and α_2 such that

$$\alpha_1 f(n) \leq g(n) \leq \alpha_2 f(n)$$

for sufficiently large values of n.

3. Calculus on tuples of $\mathbb{R}[x]$

In this section we extend the concept of differentiation an integration on tuples whose entries are coming from the polynomial ring $\mathbb{R}[x]$.

3.1. Differentiation on tuples of $\mathbb{R}[x]$.

Definition 3.1. Let $\mathcal{S} = (f_1, f_2, \ldots, f_n)$ such that $f_i \in \mathbb{R}[x]$. By the derivative of $\mathcal{S}$, denoted $\nabla(\mathcal{S})$, we mean $\nabla(\mathcal{S}) = (\frac{df_1}{dx}, \frac{df_2}{dx}, \ldots, \frac{df_n}{dx})$. The value of the derivative at a, denoted by $\nabla_a(\mathcal{S})$ is given by $\nabla_a(\mathcal{S}) = (\frac{df_1(a)}{dx}, \frac{df_2(a)}{dx}, \ldots, \frac{df_n(a)}{dx})$.

Remark 3.2. We now examine some basic properties of derivative on tuples of $\mathbb{R}[x]$. These properties follow naturally from the properties of diffferentiation of functions.

3.2. Properties of differentiation on tuples of $\mathbb{R}[x]$.

Theorem 3.3. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be tuples whose entries are coming from $\mathbb{R}[x]$ and $c \in \mathbb{R}$. Then the following properties remain valid.*

(i) $\nabla(\mathcal{S}_1 \pm \mathcal{S}_2) = \nabla(\mathcal{S}_1) \pm \nabla(\mathcal{S}_2)$.

(ii) $\nabla(c\mathcal{S}_1) = c\nabla(\mathcal{S}_1)$.

Proof. (i) Assume the tuples $\mathcal{S}_1 := (f_1, f_2, \ldots, f_n)$ and $\mathcal{S}_2 := (g_1, g_2, \ldots, g_n)$, such that $f_i, g_i \in \mathbb{R}[x]$ for $i = 1, \ldots, n$. Then it follows from the additive property of tuples, that $\mathcal{S}_1 \pm \mathcal{S}_2 = (f_1, f_2, \ldots, f_n) \pm (g_1, g_2, \ldots, g_n) = (f_1 \pm$

$g_1, f_2 \pm g_2, \ldots, f_n \pm g_n)$. Applying definition 3.1 and the algebras on tuples, we have that

$$\begin{aligned}
\nabla(\mathcal{S}_1 \pm \mathcal{S}_2) &= \left(\frac{d(f_1 \pm g_1)}{dx}, \frac{d(f_2 \pm g_2)}{dx}, \ldots, \frac{d(f_n \pm g_n)}{dx}\right) \\
&= \left(\frac{df_1}{dx} \pm \frac{dg_1}{dx}, \frac{df_2}{dx} \pm \frac{dg_2}{dx}, \ldots, \frac{df_n}{dx} \pm \frac{dg_n}{dx}\right) \\
&= \left(\frac{df_1}{dx}, \frac{df_2}{dx}, \ldots, \frac{df_n}{dx}\right) \pm \left(\frac{dg_1}{dx}, \frac{dg_2}{dx}, \ldots, \frac{dg_n}{dx}\right) \\
&= \nabla(\mathcal{S}_1) \pm \nabla(\mathcal{S}_2).
\end{aligned}$$

(ii) Fix $c \in \mathbb{R}$ and suppose $\mathcal{S}_1 = (f_1, f_2, \ldots, f_n)$, such that $f_i \in \mathbb{R}[x]$. Then it follows that $c\mathcal{S}_1 := (cf_1, cf_2, \ldots, cf_n)$. We find by applying definition 3.1 and the fundamental algebras on tuples, that

$$\begin{aligned}
\nabla(c\mathcal{S}_1) &= \left(\frac{d(cf_1)}{dx}, \frac{d(cf_2)}{dx}, \ldots, \frac{d(cf_n)}{dx}\right) \\
&= \left(c\frac{df_1}{dx}, c\frac{df_2}{dx}, \ldots, c\frac{df_n}{dx}\right) \\
&= c\left(\frac{df_1}{dx}, \frac{df_2}{dx}, \ldots, \frac{df_n}{dx}\right) \\
&= c\nabla(\mathcal{S}_1).
\end{aligned}$$

□

Remark 3.4. The property (ii) in Theorem 3.3 tells us in partiular that, a derivative of any constant multiple of a tuple can be controlled by the derivatives of the tuple with entries the dilates of the original tuple.

3.3. **Integration on tuples of $\mathbb{R}[x]$.** In this section we carry out the complete opposite of the work done in the previous section, integration on tuples. The definition is natural and it comes in the following sequel.

Definition 3.5. Let $\mathcal{S} := (f_1, f_2, \ldots, f_n)$ such that $f_i \in \mathbb{R}[x]$ for $i = 1, \ldots, n$. Then the integral on the tuple $\mathcal{S}$, denoted $\Delta(\mathcal{S})$, is given by $\Delta(\mathcal{S}) = (\int f_1 dx, \int f_2 dx, \ldots, \int f_n dx)$.

Now a natural quest is to examine the properties of the concept of integration on tuples of $\mathbb{R}[x]$.

Theorem 3.6. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be tuples of $\mathbb{R}[x]$ and $c \in \mathbb{R}$, then the following properties hold:*

(i) $\Delta(\mathcal{S}_1 \pm \mathcal{S}_2) = \Delta(\mathcal{S}_1) \pm \Delta(\mathcal{S}_2)$.

(ii) $\Delta(c\mathcal{S}_1) = c\Delta(\mathcal{S}_1)$.

Proof. (i) Assume the tuple of $\mathbb{R}[x]$ namely $\mathcal{S}_1 = (f_1, f_2, \ldots, f_n)$ and $\mathcal{S}_2 = (g_1, g_2, \ldots, g_n)$. Then it follows that

$$\begin{aligned}\Delta(\mathcal{S}_1 \pm \mathcal{S}_2) &= \bigg(\int (f_1 \pm g_1)dx, \int (f_2 \pm g_2)dx, \ldots, \int (f_n \pm g_n)dx\bigg)\\ &= \bigg(\int f_1 dx \pm \int g_1 dx, \int f_2 dx \pm \int g_2 dx, \ldots, \int f_n dx \pm \int g_n dx\bigg)\\ &= \bigg(\int f_1 dx, \int f_2 dx, \ldots, \int f_n dx\bigg) \pm \bigg(\int g_1 dx, \int g_2 dx, \ldots, \int g_n dx\bigg)\\ &= \Delta(\mathcal{S}_1) \pm \Delta(\mathcal{S}_2).\end{aligned}$$

(ii) Fix $c \in \mathbb{R}$ and assume the tuple $\mathcal{S}_1 = (f_1, f_2, \ldots, f_n)$ of $\mathbb{R}[x]$. Then $c\mathcal{S}_1 = (cf_1, cf_2, \ldots, cf_n)$ and we find that

$$\begin{aligned}\Delta(c\mathcal{S}_1) &= \bigg(\int cf_1 dx, \int cf_2 dx, \ldots, \int cf_n dx\bigg)\\ &= \bigg(c\int f_1 dx, c\int f_2 dx, \ldots, c\int f_n dx\bigg)\\ &= c\bigg(\int f_1 dx, \int f_2 dx, \ldots, \int f_n dx\bigg)\\ &= c\Delta(\mathcal{S}_1).\end{aligned}$$

□

Having this extensions of integration and differentiation on tuples of $\mathbb{R}[x]$, we are now ready to launch the concept of expansivity of tuples of $\mathbb{R}[x]$. The concept of differentiation has an immediate effect, where as the concept of integration will be usefull for the inverse problem.

4. Expansion on a tuple of $\mathbb{R}[x]$

In this section we launch the concept of expansion of a tuple of $\mathbb{R}[x]$.

Definition 4.1. Let $\mathcal{S} = (f_1, f_2, \ldots, f_n)$ be a tuple of $\mathbb{R}[x]$. Then $\mathcal{S}$ is said to be expanded if

$$(f_1, f_2, \ldots, f_n) \longrightarrow \bigg(\sum_{i \neq 1} \frac{df_i}{dx}, \sum_{i \neq 2} \frac{df_i}{dx}, \ldots, \sum_{i \neq n} \frac{df_i}{dx}\bigg).$$

If $\mathcal{S}$ is the tuple then we denote by $\mathcal{S}^1$ the expanded tuple, and the value of the expanded tuple at $a \in \mathbb{R}$, denoted by $\mathcal{S}^1_a$, is given by

$$\mathcal{S}^1(a) = \bigg(\sum_{i \neq 1} \frac{df_i(a)}{dx}, \sum_{i \neq 2} \frac{df_i(a)}{dx}, \ldots, \sum_{i \neq n} \frac{df_i(a)}{dx}\bigg).$$

Remark 4.2. Through out the paper, in situations where it is not mentioned, a tuple of $\mathbb{R}[x]$ will always be understood to have at least two entries with distinct degrees. This will ensure the free flow of the expansion process.

Proposition 4.1. *Let $\mathcal{S}^1$ be the expanded tuple of the tuple $\mathcal{S} = (f_1, f_2, \ldots, f_n)$ of $\mathbb{R}[x]$ and $\{\mathcal{S}_i\}_{i=1}^{\infty}$ be the collection of all tuples of $\mathbb{R}[x]$. Then an expansion is the composite map*

$$\gamma^{-1} \circ \beta \circ \gamma \circ \nabla : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty},$$

where $\nabla(\mathcal{S}) = (f_1', f_2', \ldots, f_n')$ and

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \textit{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & 1 \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Proof. Pick an arbitrary tuple $\mathcal{S} := (f_1, f_2, \ldots, f_n) \in \{\mathcal{S}_i\}_{i=1}^{\infty}$, the collection of all tuples of $\mathbb{R}[x]$. By definition 4.1, we find that the expanded tuple $\mathcal{S}_1 = (f_2' + f_3' + \cdots + f_n', f_1' + f_3' + \cdots + f_n', \ldots, f_1' + f_2' + \cdots + f_{n-1}')$. Since γ is invertible we find that we can write

$$(f_2' + f_3' + \cdots + f_n', f_1' + f_3' + \cdots + f_n', \ldots, f_1' + f_2' + \cdots + f_{n-1}') = \gamma^{-1} \circ \beta \circ \gamma \circ \nabla(\mathcal{S}),$$

and the result follows immediately. ☐

Proposition 4.2. *Let $\{\mathcal{S}_i\}_{i=1}^{\infty}$ be the collection of all tuples of $\mathbb{R}[x]$, satisfying certain initial condition at each phase of expansion. Then an expansion $\gamma^{-1} \circ \beta \circ \gamma \circ \nabla : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty}$ is bijective.*

Proof. Since the composite of a bijective map is still bijective, it suffices to show that each of the map that contributes to an expansion is bijective. By Proposition 4.1, we find that an expansion is the composite map

$$\gamma^{-1} \circ \beta \circ \gamma \circ \nabla : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty},$$

where $\nabla(\mathcal{S}) = (f_1', f_2', \ldots, f_n')$ and

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & 1 \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Now suppose $\nabla(\mathcal{S}_1) = \nabla(\mathcal{S}_2)$ for any two tuples $\mathcal{S}_1$ and $\mathcal{S}_2$ having the same initial condition. That is, $\nabla(\mathcal{S}_1) = \nabla(\mathcal{S}_2)$ and $\mathcal{S}_1(a) = \mathcal{S}_2(a)$ for any $a \in \mathbb{R}$. It follows by the linearity of ∇ that $\nabla(\mathcal{S}_1 - \mathcal{S}_2) = \mathcal{S}_0$. It must be that $\mathcal{S}_1 - \mathcal{S}_2 = \mathcal{S}_b$, where $\mathcal{S}_b$ is a tuple of $\mathbb{R}$. Since both $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfies the same initial condition, it must be that $\mathcal{S}_b = \mathcal{S}_0$. This establishes injectivity. For any $\mathcal{S}_1 \in \{\mathcal{S}_i\}_{i=1}^{\infty}$, there is a unique tuple $\mathcal{S} \in \{\mathcal{S}_i\}_{i=1}^{\infty}$ satisfying certain initial condition and that $\nabla(\mathcal{S}) = \mathcal{S}_1$. Thus ∇ is indeed bijective. Now we proceed by showing that

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

is also bijective. Suppose that $\gamma(\mathcal{S}_1) = \gamma(\mathcal{S}_2)$, where $\mathcal{S}_1 = (f_1, f_2, \ldots, f_n)$ and $\mathcal{S}_2 = (g_1, g_2, \ldots, g_n)$. Then it follows that

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}$$

and it must be that $f_i = g_i$ for all $1 \leq i \leq n$. Thus $\mathcal{S}_1 = \mathcal{S}_2$. Surjectivity is very obvious, and γ is bijective. Finally, we remark that β is bijective, since the matrix

$$\begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & 1 \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix}$$

is invertible. Thus each of the maps is invertible and the result follows immediately. $\square$

Remark 4.3. The requirement that each tuple of $\mathbb{R}[x]$ satisfies certain initial condition at each phase of an expansion is very crucial here, and the whole theory hinges on this particular requirement. Without this an expansion would not be an invertible map, and so some of the theorem will break down.

Proposition 4.3. *An expansion $\gamma^{-1} \circ \beta \circ \gamma \circ \nabla : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty}$ is linear.*

Proof. It suffices to show that each of the operators $\nabla : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty}$, $\gamma : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty}$ and $\beta \circ \gamma : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty}$ is linear, since the map $\gamma : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty}$ is bijective. Let $\mathcal{S}_a = (f_1, f_2, \ldots, f_n), \mathcal{S}_b = (g_1, g_2, \ldots, g_n) \in \mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ and let $\lambda, \mu \in \mathbb{R}$, then it follows that

$$\begin{aligned}
\nabla(\lambda\mathcal{S}_a + \mu\mathcal{S}_b) &= \nabla(\lambda(f_1, f_2, \ldots, f_n) + \mu(g_1, g_2, \ldots, g_n)) \\
&= \nabla((\lambda f_1, \lambda f_2, \ldots, \lambda f_n) + (\mu g_1, \mu g_2, \ldots, \mu g_n)) \\
&= \nabla((\lambda f_1 + \mu g_1, \lambda f_2 + \mu g_2, \ldots \lambda f_n + \mu g_n)) \\
&= ((\lambda f_1 + \mu g_1)', (\lambda f_2 + \mu g_2)', \ldots, (\lambda f_n + \mu g_n)') \\
&= (\lambda f_1' + \mu g_1', \lambda f_2' + \mu g_2', \ldots, \lambda f_n' + \mu g_n') \\
&= (\lambda f_1', \lambda f_2', \ldots, \lambda f_n') + (\mu g_1', \mu g_2', \ldots, \mu g_n') \\
&= \lambda(f_1', f_2', \ldots, f_n') + \mu(g_1', g_2', \ldots, g_n') \\
&= \lambda\nabla(\mathcal{S}_a) + \mu\nabla(\mathcal{S}_b).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\gamma(\lambda\mathcal{S}_a + \mu\mathcal{S}_b) &= \begin{pmatrix} \lambda f_1 + \mu g_1 \\ \lambda f_2 + \mu g_2 \\ \vdots \\ \lambda f_n + \mu g_n \end{pmatrix} \\
&= \begin{pmatrix} \lambda f_1 \\ \lambda f_2 \\ \vdots \\ \lambda f_n \end{pmatrix} + \begin{pmatrix} \mu g_1 \\ \mu g_2 \\ \vdots \\ \mu g_n \end{pmatrix} \\
&= \lambda\gamma(\mathcal{S}_a) + \mu\gamma(\mathcal{S}_b).
\end{aligned}$$

Similarly

$$\begin{aligned}
\beta \circ \gamma(\lambda\mathcal{S}_a + \mu\mathcal{S}_b) &= \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & 1 \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} \lambda f_1 + \mu g_1 \\ \lambda f_2 + \mu g_2 \\ \vdots \\ \lambda f_n + \mu g_n \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & 1 \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \left\{ \begin{pmatrix} \lambda f_1 \\ \lambda f_2 \\ \vdots \\ \lambda f_n \end{pmatrix} + \begin{pmatrix} \mu g_1 \\ \mu g_2 \\ \vdots \\ \mu g_n \end{pmatrix} \right\} \\
&= \lambda \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & 1 \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & 1 \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} \\
&= \lambda(\beta \circ \gamma)(\mathcal{S}_a) + \mu(\beta \circ \gamma)(\mathcal{S}_b).
\end{aligned}$$

This proves the linearity of expansion. □

Proposition 4.4. *A tuple of* $\mathbb{R}[x]$ *can only admit a finite number of expansions.*

Proof. Proposition 4.1 informs us that an expansion is the map $\gamma^{-1} \circ \beta \circ \gamma \circ \nabla$. Pick arbitrarily a tuple $\mathcal{S}$ of $\mathbb{R}[x]$. Since the degrees of each entry of $\nabla(\mathcal{S})$ is one less than the degree of $\mathcal{S}$, it follows by induction that an expansion can only be applied at a finite number of time. □

It is very obvious from our setup an expanded tuple will certainly be a tuple of $\mathbb{R}[x]$, so the landscape of the theory would not be altered if we carry out further expansions on the expanded tuple, thereby obtaining another expanded tuple. This process can be carried out so long as the entries of the tuple do not vanish. This idea leads us to introduce the concept of phase expansions, the limits of expansion and the rank of an expansion.

4.1. **Phase expansions.** Consider the tuple $\mathcal{S}_1=(f_1,f_2,\ldots,f_n)$. By expanding the tuple we find that the expanded tuple is given by

$$\mathcal{S}^1=\bigg(\sum_{i\neq 1}\frac{df_i}{dx},\sum_{i\neq 2}\frac{df_i}{dx},\ldots,\sum_{i\neq n}\frac{df_i}{dx}\bigg),$$

which we can rewrite as the tuple $\mathcal{S}^1=(g_1,g_2,\ldots,g_n)$. We call this expansion the first phase expansion. We can go on to expand the tuple $\mathcal{S}^1=(g_1,g_2,\ldots,g_n)$ and by so doing we find that

$$\mathcal{S}^2=\bigg(\sum_{i\neq 1}\frac{dg_i}{dx},\sum_{i\neq 2}\frac{dg_i}{dx},\ldots,\sum_{i\neq n}\frac{dg_i}{dx}\bigg),$$

which we can rewrite as $\mathcal{S}^2=(h_1,h_2,\ldots,h_n)$. We call this expansion the second phase expansion. This expansion process can be carried out on each previously expanded tuple at certain number of times provided the entries of the tuple do not vanish. In general we denote the nth expanded tuple by $\mathcal{S}^n$ for $n\geq 1$. To make this expansion process meaningful we introduce the concept of the order, the rank and the limit of expansion. Before then let us consider the following example.

Example 4.4. Let us consider the tuple $\mathcal{S}=(x^4+x^2,x^5-x^3,x^2+1)$ of $\mathbb{R}[x]$. The first phase expanded tuple is given by $\mathcal{S}^1=(5x^4-3x^2+2x,4x^3+4x,5x^4+4x^3-3x^2+2x)$. The second phase expanded tuple is given by $\mathcal{S}^2=(20x^3+24x^2-6x+6,40x^3+12x^2-12x+4,20x^3+12x^2-6x+6)$. The third phase expanded tuple is given by $\mathcal{S}^3=(180x^2+48x-18,120x^2+72x-12,180x^2+72x-18)$. The fourth phase expanded tuple is given by $\mathcal{S}^4=(600x+144,720x+120,600x+120)$. The fifth phase expanded tuple is given by $\mathcal{S}^5=(1320,1200,1320)$ and it is essentially the last expanded tuple.

Remark 4.5. It is very important to notice especially that the fifth expanded tuple in Example 4.4 consist entirely of entries that are constants, so it is essentially the expanded tuple of the last non vanishing phase of expansion. Thus we introduce once again the notion of a rank of expansion.

4.2. **The rank of an expansion.**

Definition 4.6. Let $\mathcal{F}=\{\mathcal{S}_m\}_{m=1}^{\infty}$ be a family of tuples of $\mathbb{R}[x]$, each having at least two entries with distinct degrees. Then the value of n such that the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S})\neq\mathcal{S}_0$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n+1}(\mathcal{S})=\mathcal{S}_0$ where $\mathcal{S}_0=(0,0,\ldots,0)$ is called the degree of expansion and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S})$ is the rank of expansion, denoted by $\mathcal{R}(\mathcal{S})$.

Example 4.7. The expanded tuple $(1320,1200,1320)$ is the rank of the expansion of $\mathcal{S}$, since it is the last non-vanishing expanded tuple of $\mathcal{S}$. That is, $\mathcal{R}(\mathcal{S})=(1320,1200,1320)$.

Remark 4.8. Through out this paper all tuples are understood to have of the same number of entry, since the operations of addition and subtraction cannot be performed otherwise.

Proposition 4.5. *Let $\mathcal{F} = \{\mathcal{S}_k\}_{k=1}^{\infty}$ be a family of tuples of $\mathbb{R}[x]$. Suppose $\mathcal{S}_i$, $\mathcal{S}_j \in \mathcal{F}$ are of the same degree m of expansion. Then the following properties remain valid:*

(i) $\mathcal{R}(\mathcal{S}_i + \mathcal{S}_j) = \mathcal{R}(\mathcal{S}_i) + \mathcal{R}(\mathcal{S}_j)$.

(ii) $\mathcal{R}(c\mathcal{S}_i) = c\mathcal{R}(\mathcal{S}_i)$.

Proof. (i) Pick $\mathcal{S}_i$, $\mathcal{S}_j \in \mathcal{F}$, with the same degree m of expansion. Then it follows that $\mathcal{R}(\mathcal{S}_i + \mathcal{S}_j) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_i + \mathcal{S}_j) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{m-1}) \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(\mathcal{S}_i + \mathcal{S}_j)$. By applying the linearity property of the map, we find that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{m-1} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(\mathcal{S}_i + \mathcal{S}_j) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{m-1})((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(\mathcal{S}_i) + (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)(\mathcal{S}_j))$. By induction, it follows that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_i + \mathcal{S}_j) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_i) + (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_j) = \mathcal{R}(\mathcal{S}_i) + \mathcal{R}(\mathcal{S}_j)$.

(ii) The result follows by applying the linearity property of each of the m copies of the map.

□

Conjecture 2. Let $\mathcal{R}(\mathcal{S})$ be the rank of an expansion on $\mathcal{S}$, where $\mathcal{S}$ consist of polynomials with integer coefficients each of the same parity. Then there exist some tuple $(b, b, \ldots, b)$ with $b \in \mathbb{Z}$ such that all the entries of $\mathcal{R}(\mathcal{S}) + (b, b, \ldots, b)$ and $\mathcal{R}(\mathcal{S}) - (b, b, \ldots, b)$ are all prime.

Remark 4.9. It needs to be said that, Conjecture 2 is reminiscent of the Hardy-littlewood prime tuple conjecture

Theorem 4.10. *Let $\mathcal{S}_i$, $\mathcal{S}_j \in \{\mathcal{S}_k\}_{k=1}^{\infty}$, the family of tuples of $\mathbb{R}[x]$ such that each tuple has at least two entries with distinct degrees. Let the degree of expansion deg($\mathcal{S}_i$)=deg($\mathcal{S}_j$)= n. Then $\mathcal{R}(\mathcal{S}_i) = \mathcal{R}(\mathcal{S}_j)$ if and only if $\mathcal{S}_i - \mathcal{S}_j = (a_1, a_2, \ldots, a_n)$ for each $a_i \in \mathbb{R}$.*

Proof. Pick $\mathcal{S}_i, \mathcal{S}_j \in \{\mathcal{S}_k\}_{k=1}^{\infty}$, the family of tuples of $\mathbb{R}[x]$, such that deg($\mathcal{S}_i$)=deg($\mathcal{S}_j$). Asumme $\mathcal{S}_i - \mathcal{S}_j = (a_1, a_2, \ldots, a_n)$ for each $a_i \in \mathbb{R}$. Then applying n copies of expansion on both sides of the relation, we have that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_i - \mathcal{S}_j) = \mathcal{S}_0$, where $\mathcal{S}_0$ is the null tuple. Since an expansion is linear, we find that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_i) - (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j) = \mathcal{S}_0$. That is $\mathcal{R}(\mathcal{S}_i) = \mathcal{R}(\mathcal{S}_j) + \mathcal{S}_0 = \mathcal{R}(\mathcal{S}_j)$. Conversely suppose $\mathcal{R}(\mathcal{S}_i) = \mathcal{R}(\mathcal{S}_j)$, then by the properties of the rank, we find that $\mathcal{R}(\mathcal{S}_i - \mathcal{S}_j) = \mathcal{S}_0$. To avoid a contradiction, we are left with the only choice that the entries of $\mathcal{S}_i$ and $\mathcal{S}_j$ must differ by elements in $\mathbb{R}$. □

It is very important to notice imposing the condition $\mathcal{R}(\mathcal{S}) = \mathcal{R}(\mathcal{S}_j)$ on the tuples in $\{\mathcal{S}_k\}_{k=1}^{\infty}$ induces an equivalence relation and consequently partitions $\{\mathcal{S}_k\}_{k=1}^{\infty}$ into infinite disjoint classes. By denoting $\mathcal{S}_i \sim \mathcal{S}_j$ if and only if $\mathcal{R}(\mathcal{S}_i) = \mathcal{R}(\mathcal{S}_j)$. Then it follows that $\mathcal{S} \sim \mathcal{S}$, since $\mathcal{R}(\mathcal{S}) = \mathcal{R}(\mathcal{S})$, hence a reflexive relation. It is also clear that the relation is symmetric. Suppose $\mathcal{S}_a \sim \mathcal{S}_b$ and $\mathcal{S}_b \sim \mathcal{S}_c$. Then it follows that $\mathcal{R}(\mathcal{S}_a) = \mathcal{R}(\mathcal{S}_b)$ and $\mathcal{R}(\mathcal{S}_b) = \mathcal{R}(\mathcal{S}_c)$. Thus it follows that $\mathcal{R}(\mathcal{S}_a) = \mathcal{R}(\mathcal{S}_c)$, and therefore transitive. The next result tells us that a tuple of $\mathbb{R}[x]$ can be reduced to another tuple of the same rank by an expansion.

Theorem 4.11. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be tuples of $\mathbb{R}[x]$, with deg$(\mathcal{S}_1) >$ deg$(\mathcal{S}_2)$, satisfying certain initial conditions at each phase of expansion. If $\mathcal{R}(\mathcal{S}_1) = \mathcal{R}(\mathcal{S}_2)$, then there exist some j satisfying $1 \leq j <$ deg$(\mathcal{S}_1)$ such that $\mathcal{S}_1^j = \mathcal{S}_2$.*

Proof. Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are tuples of $\mathbb{R}[x]$. Let $deg(\mathcal{S}_1) = k_1$ and $deg(\mathcal{S}_2) = k_2$. By definition 4.6, we can write $\mathcal{R}(\mathcal{S}_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1}(\mathcal{S}_1)$ and $\mathcal{R}(\mathcal{S}_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_2}(\mathcal{S}_2)$. Under the assumption that $\mathcal{R}(\mathcal{S}_1) = \mathcal{R}(\mathcal{S}_2)$, we must have that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_2}(\mathcal{S}_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1}(\mathcal{S}_1)$ if and only if $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1-k_2}(\mathcal{S}_1) = \mathcal{S}_2$. Since $1 \leq k_1 - k_2 < k_1 = deg(\mathcal{S}_1)$, the result follows immediately. ☐

4.3. **The limit of an expansion.**

Definition 4.12. Let $\{\mathcal{S}^m\}_{m=1}^{\infty}$ be a family of expanded tuples of $\mathcal{S}$, having at least two entries with distinct degrees. Then the limit of expansion of $\mathcal{S}$ is the first expanded tuple $\mathcal{S}^j = (g_1, g_2, \ldots, g_n)$ such that $\deg(g_1) = \deg(g_2) = \cdots = \deg(g_n)$ for $n \geq 3$ and $1 \leq j \leq m$. Notation-wise, we denote simply by

$$\lim(\mathcal{S}^n) = \mathcal{S}^j,$$

the limit of the expansion.

Remark 4.13. To get our hands on this language, let us consider an example.

Example 4.14. Let us consider the tuple $\mathcal{S} = (x^4 + x^2, x^5 - x^3, x^2 + 1)$. The first phase expanded tuple is given by $\mathcal{S}^1 = (5x^4 - 3x^2 + 2x, 4x^3 + 4x, 5x^4 + 4x^3 - 3x^2 + 2x)$. The second phase expanded tuple is given by $\mathcal{S}^2 = (20x^3 + 24x^2 - 6x + 6, 40x^3 + 12x^2 - 12x + 4, 20x^3 + 12x^2 - 6x + 6)$. The third phase expanded tuple is given by $\mathcal{S}^3 = (180x^2 + 48x - 18, 120x^2 + 72x - 12, 180x^2 + 72x - 18)$. The fourth phase expanded tuple is given by $\mathcal{S}^4 = (600x + 144, 720x + 120, 600x + 120)$. The fifth phase expanded tuple is given by $\mathcal{S}^5 = (1320, 1200, 1320)$. Applying definition 4.12, we find that the limit of expansion is the expanded tuple $\mathcal{S}^2$.

Now we prove a result about the existence of the limit of expansion of any tuple of $\mathbb{R}[x]$ having at least two entries with distinct degrees. The method of proof is basically an argument by infinite descent.

Theorem 4.15. *Let $\{\mathcal{S}^m\}_{m=1}^{\infty}$ be a family of expansions of the tuple $\mathcal{S}$ of $\mathbb{R}[x]$, such that at least two entries have distinct degree. Then the limit of expansions $\lim(\mathcal{S}^n)$ of $\mathcal{S}$ exists.*

Proof. Let $\{\mathcal{S}^m\}_{m=1}^{\infty}$ be a family of expansions of the tuple $\mathcal{S}$ of $\mathbb{R}[x]$, having at least two entries with distinct degree. Suppose the limit of expansion does not exist, and let $\mathcal{S}^1 = (f_1, f_2, \ldots, f_n)$ be the first phase expansion of $\mathcal{S}$, then it follows that $\deg(f_i) \neq \deg(f_j)$ for some $1 \leq i, j \leq n$ with $i \neq j$. It follows in particular that $\mathcal{S}^1 \neq \mathcal{R}(\mathcal{S})$ and $\mathcal{S}^1 \neq \mathcal{S}_0$. Thus the second phase expansion exists and let

$\mathcal{S}^2 = (g_1, g_2, \ldots, g_n)$ be the second phase expanded tuple. Again, it follows from the hypothesis that $\deg(g_i) \neq \deg(g_j)$ for some $1 \leq i, j \leq n$, with $i \neq j$, and it follows in particular that $\mathcal{S}^2 \neq \mathcal{R}(\mathcal{S})$ and $\mathcal{S}^2 \neq \mathcal{S}_0$. Thus the third phase expansion exist. By induction it follows that the tuple $\mathcal{S}$ of $\mathbb{R}[x]$ admits infinite number of expansions, thereby contradicting Proposition 4.4. $\square$

Theorem 4.16. *Let $\{\mathcal{S}^n\}_{n=1}^{\infty}$ be a family of expanded tuples of the tuple $\mathcal{S}$ of $\mathbb{R}[x]$, such that at least two entries have distinct degrees and satisfying certain initial conditions at each phase of expansion. Then there exist some number k called the dimension of expansion (dim($\mathcal{S}$)), such that* $\lim(\mathcal{S}^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(\mathcal{S}))$ *for some $k <$ deg($\mathcal{S}$).*

Proof. Let $\mathcal{S}$ be any tuple of $\mathbb{R}[x]$ that can be expanded, with at least two entries having distinct degree. Then, the limit exists by Theorem 4.15 and since an expansion can only be applied at a finite number of time and the map $\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ is a recovery which exist, it is clear there will exist such number k, so that $\lim(\mathcal{S}^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(\mathcal{S}))$. We only need to show that k lies in the stated range. In anticipation of a contradiction, let us suppose $\lim(\mathcal{S}^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k(\mathcal{R}(\mathcal{S}))$ for any $k \geq deg(\mathcal{S})$. Since the map is a bijection, it follows that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k(\lim(\mathcal{S}^n)) = \mathcal{R}(\mathcal{S})$. It is easy to see that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k(\lim(\mathcal{S}^n) = \mathcal{S}_0$, in which case we have that $\mathcal{R}(\mathcal{S}) = \mathcal{S}_0$, and so the rank of an expansion is null, which is a contradiction by definition 4.6. $\square$

Remark 4.17. The above result is telling us that knowing the rank and the dimension is good enough to determining the limit of an expansion. We now leverage this result to prove an important property concerning the limits of an expansion, which tells us that the limit of an expansion on a tuple of $\mathbb{R}[x]$ is unique up to translation by a tuple of $\mathbb{R}$.

Theorem 4.18. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be tuples of the polynomial ring $\mathbb{R}[x]$, with their corresponding family of expanded tuples $\{\mathcal{S}_1^m\}_{m=1}^{\infty}$ and $\{\mathcal{S}_2^n\}_{n=1}^{\infty}$, satisfying certain initial conditions, and suppose the limit of each expansion exists. Then $\lim(\mathcal{S}_1^m) = \lim(\mathcal{S}_2^n)$ if and only if $\mathcal{S}_1 - \mathcal{S}_2 = (b_1, b_2, \ldots, b_n)$ where $b_i \in \mathbb{R}$ for $1 \leq i \leq n$.*

Proof. Invoking Theorem 4.16, we can write $\lim(\mathcal{S}_1^m) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^{k_1}(\mathcal{R}(\mathcal{S}_1))$ and $\lim(\mathcal{S}_2^n) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^{k_2}(\mathcal{R}(\mathcal{S}_2))$, for some $k_1, k_2 \in \mathbb{N}$. Suppose $\lim(\mathcal{S}_1^m) = \lim(\mathcal{S}_2^n)$, then we must have $(\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^{k_1}(\mathcal{R}(\mathcal{S}_1)) = (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^{k_2}(\mathcal{R}(\mathcal{S}_2))$ if and only if $(\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^{k_1 - k_2}(\mathcal{R}(\mathcal{S}_1)) = \mathcal{R}(\mathcal{S}_2)$. We claim that $k_1 = k_2$. Suppose $k_1 > k_2$, then it follows immediately that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_1 - k_2}(\mathcal{R}(\mathcal{S}_2)) = \mathcal{S}_0 = \mathcal{R}(\mathcal{S}_1)$, which is a contradiction. Again if $k_2 > k_1$, then we have $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_2 - k_1}(\mathcal{R}(\mathcal{S}_1)) = \mathcal{R}(\mathcal{S}_2) = \mathcal{S}_0$, which again is absurd. Therefore it must be that $k_1 = k_2$, and it follows that $\mathcal{R}(\mathcal{S}_1) = \mathcal{R}(\mathcal{S}_2)$. Now, thanks to Theorem 4.10, it must be that $\mathcal{S}_1 - \mathcal{S}_2 = (b_1, b_2, \ldots, b_n)$ where $b_i \in \mathbb{R}$ for $1 \leq i \leq n$. The converse, on the other hand, is straight-forward. $\square$

4.4. **The local number.**

Definition 4.19. Let $\mathcal{S}$ be a tuple of $\mathbb{R}[x]$ and $\{\mathcal{S}^m\}_{m=1}^{\infty}$ the family of expanded tuples of $\mathcal{S}$. Then by the local number of expansion, denoted $\mathcal{L}(\mathcal{S})$, we mean the value of n such that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S})=\lim(\mathcal{S}^m)$.

Invoking Theorem 4.16, It follows from the above definition that for any tuple of $\mathbb{R}[x]$ satisfying certain initial conditions at each phase of expansion,

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S})=(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^k(\mathcal{R}(\mathcal{S}))$$

if and only if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n+k}(\mathcal{S})=\mathcal{R}(\mathcal{S}).$$

By the definition of the rank of an expansion, it follows that

$$n+k=deg(\mathcal{S})$$

which we call the principal equation and where $\mathcal{L}(\mathcal{S})=n$, $dim(\mathcal{S})=k$ and $deg(\mathcal{S})$ are the local number, the dimension and the degree of expansion, respectively, on $\mathcal{S}$. It is interesting to recognize that the value of the local number $\mathcal{L}(\mathcal{S})$ in any case is bounded cannot be more than the dimesion of expansion. This assertion is confirmed in the following sequel.

Theorem 4.20. *Let $\mathcal{S}$ be a tuple of $\mathbb{R}[x]$, satisfying certain initial conditions at each phase with $deg(\mathcal{S})\geq 4$. If $dim(\mathcal{S})>2$, then the local number $\mathcal{L}(\mathcal{S})$ must satisfy the inequality*

$$0\leq\mathcal{L}(\mathcal{S})\leq 2.$$

Proof. Let us suppose on the contrary $\mathcal{L}(\mathcal{S})>2$. Then it follows from the principal equation that $dim(\mathcal{S})<deg(\mathcal{S})-2$, so that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{dim(\mathcal{S})+2}(\mathcal{S})\neq\mathcal{R}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{dim(\mathcal{S})+2}(\mathcal{S})\neq\mathcal{S}_0$. It follows that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{dim(\mathcal{S})+2}(\mathcal{S})=\mathcal{S}_1$. Theorem 4.11 gives $\mathcal{R}(\mathcal{S})=\mathcal{R}(\mathcal{S}_1)$, and we have that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{deg(\mathcal{S})}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{deg(\mathcal{S}_1)}(\mathcal{S}_1),$$

if and only if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{deg(\mathcal{S})-deg(\mathcal{S}_1)}(\mathcal{S})=\mathcal{S}_1.$$

It follows therefore that $deg(\mathcal{S})-deg(\mathcal{S}_1)=dim(\mathcal{S})+2$. Again, using the principal equation, we find that

$$\mathcal{L}(\mathcal{S})=deg(\mathcal{S}_1)+2.$$

It follows from the above equation that $deg(\mathcal{S}_1)+2=\mathcal{L}(\mathcal{S})=deg(\mathcal{S})-dim(\mathcal{S})<deg(\mathcal{S})-2$, so that $deg(\mathcal{S}_1)+4<deg(\mathcal{S})$. Since $deg(\mathcal{S})\geq 4$, it must be that $deg(\mathcal{S}_1)+4\leq 4$, and we have that $deg(\mathcal{S}_1)\leq 0$. This leaves us with the only choice that $deg(\mathcal{S}_1)=0$, contradicting the fact that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{dim(\mathcal{S})+2}(\mathcal{S})\neq\mathcal{R}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{dim(\mathcal{S})+2}(\mathcal{S})\neq\mathcal{S}_0$, and the proof is complete. □

Theorem 4.21. *Let $\mathcal{S}$ be a tuple of $\mathbb{R}[x]$ satisfying certain initial conditions at each phase of expansion, such that $\mathcal{S}$ is not a tuple of $\mathbb{R}$. Then the system has no non-trivial solution*

$$\lim(\mathcal{S}^n) = \mathcal{S}_0,$$

where $\mathcal{S}_0$ is the null tuple.

Proof. Let $\mathcal{S} \in \mathbb{R}[x]$ and suppose that the there exist some $a \in \mathbb{R}$ for $a \neq 0$ such that $\lim(\mathcal{S}^n)(a) = \mathcal{S}_0$. Then by Theorem 4.16 we can write

$$(\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^{dim(\mathcal{S})}(\mathcal{R}(\mathcal{S}))(a) = \mathcal{S}_0.$$

It follows from this relation

$$\begin{aligned}\mathcal{R}(\mathcal{S})(a) &= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{dim(\mathcal{S})}(\mathcal{S}_0)\\ &= \mathcal{S}_0.\end{aligned}$$

It follows that $\mathcal{R}(\mathcal{S})(a) = \mathcal{R}(\mathcal{S}) = \mathcal{S}_0$. This can only happen if $\mathcal{S}$ is a tuple of $\mathbb{R}$, which contradicts the requirement that $\mathcal{S}$ must not be a tuple of $\mathbb{R}$, and the proof is complete. $\square$

This result is extremely useful as it turns out, for it allows us to investigate the existence of a solution to certain systems of differential equations. The following sequel will illustrate this claim in great detail. It comes as an immediate consequence of the above result.

5. Application to solutions of systems of differential equations

Corollary 1. Let $f_1, f_2, \ldots f_n \in \mathbb{R}[x]$ such that $\deg(f_i) \neq \deg(f_j)$ for some $1 \leq i, j \leq n$ and satisfying $f_i(a) = b_{i1}, f_i'(a) = b_{i2}, \ldots f_i^n(a) = b_{in}$ for each $i = 1, 2, \ldots n$ for $a \in \mathbb{R}$. If

$$\deg(\sum_{i\neq 1}\frac{df_i}{dx}) = \deg(\sum_{i\neq 2}\frac{df_i}{dx}) = \ldots \deg(\sum_{i\neq n}\frac{df_i}{dx})$$

then the system

$$\begin{aligned} f_2' + f_3' + \cdots + f_n' &= 0\\ f_1' + f_3' + \cdots + f_n' &= 0\\ &\vdots\\ f_1' + f_2' + \cdots + f_{n-1}' &= 0\end{aligned}$$

has no non-trivial solution.

Proof. Suppose $f_1, f_2, \ldots, f_n \in \mathbb{R}[x]$ such that $\deg(f_i) \neq \deg(f_j)$ for some $1 \leq i, j \leq n$. Consider the tuple $\mathcal{S} := (f_1, f_2, \ldots, f_n)$ and it follows that $\mathcal{S}$ admits an expansion. Since

$$\deg(\sum_{i\neq 1}\frac{df_i}{dx}) = \deg(\sum_{i\neq 2}\frac{df_i}{dx}) = \ldots \deg(\sum_{i\neq n}\frac{df_i}{dx})$$

It follows by Theorem 4.20 that

$$\lim(\mathcal{S}^n)=\bigg(f_2'+f_3'+\cdots+f_n', f_1'+f_3'+\cdots+f_n',\ldots,f_1'+f_2'+\cdots+f_{n-1}'\bigg).$$

Since f_i for each $i=1,2,\ldots n$ with its higher order derivatives satisfies certain initial conditions, it follows that each phase of expansion of the tuple $\mathcal{S}$ satisfies certain initial condition. It follows that the hypothesis of Theorem 4.21 are satisfied and the system has no solution, thereby ending the proof. □

5.1. **The measure of an expansion.**

Definition 5.1. Let $\mathcal{S}$ be a tuple of $\mathbb{R}[x]$. Then by the measure of an expansion on $\mathcal{S}$, denoted $\mathcal{N}(\mathcal{S})$, we mean $\mathcal{N}(\mathcal{S})=||\mathcal{R}(\mathcal{S})||$, where $||\cdot||$ is the usual norm in $\mathbb{R}^n$.

It needs to be said that the measure of an expansion assigns values to an expansion of tuples of $\mathbb{R}[x]$. This interplay will enable us to undertake a very deep study on this particular concept in relation to expansion. We now show that the measure of an expansion is indeed a norm, in the following sequel.

Proposition 5.1. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be tuples of $\mathbb{R}[x]$, each having the same degree of expansion. Then the following properties of the measure of expansions remain valid.*

(i) $\mathcal{N}(\mathcal{S})\geq 0$. *(Positivity)*

(ii) $\mathcal{N}(\mu\mathcal{S})=\mu\mathcal{N}(\mathcal{S})$, *for* $\mu\in\mathbb{R}$. *(Homogeneity)*

(iii) $\mathcal{N}(\mathcal{S}_1+\mathcal{S}_2)\leq\mathcal{N}(\mathcal{S}_1)+\mathcal{N}(\mathcal{S}_2)$. *(Triangle inequality)*

Proof. (i) Clearly, $\mathcal{R}(\mathcal{S}_0)=\mathcal{S}_0$ and it follows that $\mathcal{N}(\mathcal{S}_0)=0$. Conversely suppose that $\mathcal{N}(\mathcal{S})=0$, then it follows that $||\mathcal{R}(\mathcal{S})||=0$. That means $\mathcal{R}(\mathcal{S})=\mathcal{S}_0$ and it follows by definition 4.6, that $\mathcal{S}=\mathcal{S}_0$. Thus the positivity property follows immediately.

(ii) Let $\mu\in\mathbb{R}$, then it follows that $\mathcal{N}(\mu\mathcal{S})=||\mathcal{R}(\mu\mathcal{S})||$. By the properties of the rank, it follows that $\mathcal{N}(\mu\mathcal{S})=||\mu\mathcal{R}(\mathcal{S})||=||\mu||||\mathcal{R}(\mathcal{S})||=\mu||\mathcal{R}(\mathcal{S})||=\mu\mathcal{N}(\mathcal{S})$. Thus the homogeneity property is also satisfied.

(iii) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be any n tuples of $\mathbb{R}[x]$, each having the same degree of expansion. Then $\mathcal{N}(\mathcal{S}_1+\mathcal{S}_2)=||\mathcal{R}(\mathcal{S}_1+\mathcal{S}_2)||$. Again the properties of the rank, it follows that $\mathcal{N}(\mathcal{S}_1+\mathcal{S}_2)=||\mathcal{R}(\mathcal{S}_1+\mathcal{S}_2)||=||\mathcal{R}(\mathcal{S}_1)+\mathcal{R}(\mathcal{S}_2)||\leq||\mathcal{R}(\mathcal{S}_1)||+||\mathcal{R}(\mathcal{S}_2)||=\mathcal{N}(\mathcal{S}_1)+\mathcal{N}(\mathcal{S}_2)$, and the triangle inequality is satisfied.

□

Remark 5.2. Proposition 5.1 does indicates that the measure of an expansion is a norm. It also assigns concrete values to expansions on the tuples of $\mathbb{R}[x]$. This measure becomes very large in magnitude if and only if the expansion process is very long. That is to say, if the degree of expansion is very large then we would expect the norm of expansion to be relatively large. This will become a criterion for determining the degree of expansion, which we shall discuss later. The next result tells us that the norm of the expansion of any tuple of $\mathbb{R}[x]$ is unique up to rearrangement of entries and translation by a tuple of $\mathbb{R}$. But before then, we prove a key lemma having to do with the fact that a permutation commutes with an expansion.

Lemma 5.3. *Let τ be any permutation on the set $\{1,2,\ldots,n\}$. Then $\tau\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)\circ\tau$.*

Proof. Let $\mathcal{S}=(f_1,f_2,\ldots,f_n)$, a tuple of $\mathbb{R}[x]$. Then it follows that $\tau(\mathcal{S})=\tau((f_1,f_2,\ldots,f_n))=(f_{\tau(1)},f_{\tau(2)},\ldots,f_{\tau(n)})$, and it follows by Proposition 4.1 that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)(\tau(\mathcal{S}))=(f'_{\tau(2)}+f'_{\tau(3)}+\cdots+f'_{\tau(n)},f'_{\tau(1)}+f'_{\tau(3)}+\cdots+f'_{\tau(n)},\ldots,f'_{\tau(1)}+f'_{\tau(2)}+\cdots+f'_{\tau(n-1)})$. On the other hand, by proposition 4.1, we observe that $\tau\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)(\mathcal{S})=\tau(((f'_2+f'_3+\cdots+f'_n,f'_1+f'_3+\cdots+f'_n,\ldots,f'_1+f'_2+\cdots+f'_{n-1})))=(f'_{\tau(2)}+f'_{\tau(3)}+\cdots+f'_{\tau(n)},f'_{\tau(1)}+f'_{\tau(3)}+\cdots+f'_{\tau(n)},\ldots,f'_{\tau(1)}+f'_{\tau(2)}+\cdots+f'_{\tau(n-1)})$. By comparing both sides, the result follows immediately. $\square$

Remark 5.4. We are now ready to prove the theorem.

Theorem 5.5. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be any two n tuples of $\mathbb{R}[x]$, each having the same degree of expansion. Then $\mathcal{N}(\mathcal{S}_1)=\mathcal{N}(\mathcal{S}_2)$ if and only if there exist a tuple $\mathcal{S}_a$ with $\deg(\mathcal{S}_a)<\deg(\mathcal{S}_1)$ and a permutation $\tau:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ such that $\mathcal{S}_2=\mathrm{Sgn}(\tau)\tau(\mathcal{S}_1)+\mathcal{S}_a$, where $\tau(\mathcal{S}_1)=\tau((f_1,f_2,\ldots,f_n)):=(f_{\tau(1)},f_{\tau(2)},\ldots,f_{\tau(n)})$.*

Proof. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ are any two n tuples of $\mathbb{R}[x]$, each having the same degree of expansion, and suppose there exist a tuple $\mathcal{S}_a$ such that $\deg(\mathcal{S}_a)<\deg(\mathcal{S}_2)$ and a permutation τ such that $\mathcal{S}_2=\mathrm{Sgn}(\tau)\tau(\mathcal{S}_1)+\mathcal{S}_a$. It follows from Theorem 4.10 that $\mathcal{R}(\mathcal{S}_2)=\mathcal{R}(\mathrm{Sgn}(\tau)\tau(\mathcal{S}_1)+\mathcal{S}_a)=\mathcal{R}(\mathrm{Sgn}(\tau)\tau(\mathcal{S}_1))=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{deg(\mathcal{S}_2)}(\mathrm{Sgn}(\tau)\tau(\mathcal{S}_1))=\mathrm{Sgn}(\tau)(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{deg(\mathcal{S}_2)}\circ\tau(\mathcal{S}_1)$. Applying Lemma 5.3 $deg(\mathcal{S}_1)=deg(\mathcal{S}_2)$ number of times, we find that

$$\begin{aligned}\mathcal{R}(\mathcal{S}_2)&=\mathrm{Sgn}(\tau)(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{deg(\mathcal{S}_2)}\circ\tau(\mathcal{S}_1)\\&=\mathrm{Sgn}(\tau)\tau\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{deg(\mathcal{S}_2)}(\mathcal{S}_1)\\&=\mathrm{Sgn}(\tau)\tau\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{deg(\mathcal{S}_1)}(\mathcal{S}_1)\\&=\mathrm{Sgn}(\tau)\tau(\mathcal{R}(\mathcal{S}_1)).\end{aligned}$$

It follows from the above derived relation that the rank of expansion of $\mathcal{S}_1$ is a permutation of the rank of expansion of $\mathcal{S}_2$ upto signs. Thus we must have that $\mathcal{N}(\mathcal{S}_1)=||\mathcal{R}(\mathcal{S}_1)||=||\mathcal{R}(\mathcal{S}_2)||=\mathcal{N}(\mathcal{S}_2)$. Conversely suppose $\mathcal{N}(\mathcal{S}_1)=\mathcal{N}(\mathcal{S}_2)$. Then it follows by definition 5.1 that $||\mathcal{R}(\mathcal{S}_1)||=||\mathcal{R}(\mathcal{S}_2)||$. It must be that $\mathcal{R}(\mathcal{S}_1)$ is a permutation of $\mathcal{R}(\mathcal{S}_2)$ upto signs. That is, there exists some permutation τ on $\mathcal{R}(\mathcal{S}_1)$ such that $\mathrm{Sgn}(\tau)\tau(\mathcal{R}(\mathcal{S}_1))=\mathcal{R}(\mathcal{S}_2)$. By Lemma 5.3, we can write $\mathcal{R}(\mathcal{S}_2)=\mathcal{R}(\mathrm{Sgn}(\tau)\tau(\mathcal{S}_1))$. Since $\deg(\mathcal{S}_1)=\deg(\mathcal{S}_2)$, Theorem 4.10 tells us that $\mathrm{Sgn}(\tau)\tau(\mathcal{S}_1)-\mathcal{S}_2=\mathcal{S}_b$, where $\mathcal{S}_b$ is a tuple of $\mathbb{R}$ and $\deg(\mathcal{S}_b)<\deg(\mathcal{S}_2)$, thereby ending the proof. $\square$

Conjecture 3. Let $\mathcal{S}$ be any tuple of $\mathbb{R}[x]$ with $deg(\mathcal{S}) > 1$ and satisfying certain initial conditions at each phase. Then the double inequality is valid

$$||\mathcal{S}(deg(\mathcal{S}))|| \asymp \mathcal{N}(\mathcal{S}).$$

Remark 5.6. Conjecture 3, in every sense of words, relates the degree of an expansion of any tuple of $\mathbb{R}[x]$ to their measure of expansion.

5.2. **The boundary of an expansion.** In this section we introduce the concept of the boundary of an expansion of tuple of the polynomial ring $\mathbb{R}[x]$.

Definition 5.7. Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of all tuples of $\mathbb{R}[x]$. By the boundary point of the nth phase expansion denoted $\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j)]$, we mean the set $\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j)] := \left\{(a_1, a_2, \ldots, a_m) : \mathrm{Id}_i[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n_{a_i}(\mathcal{S}_j)] = 0\right\}$.

But before then we prove the following proposition. It basically reinforces the fact that the boundary points decreases with expansion. That is, there will be fewer and fewer boundary points as expansion takes place for a very long time.

Proposition 5.2. *Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of all tuples of $\mathbb{R}[x]$ and let $\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_j)]$ be the boundary points of the nth phase expansion. Then*

$$\#\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_j)] > \#\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_j)]$$

for $1 \leq n_1 < n_2 < \deg(\mathcal{S}_j)$.

Proof. Recall that any polynomial of degree n has at most n roots. Since the degrees of the entries of polynomials in the ring $\mathbb{R}[x]$ decreases by 1 for successive phase of expansions, it follows that the boundary points must decrease with higher phases of expansions, thereby ending the proof. $\square$

Remark 5.8. Next we state and prove a proposition concerning the boundaries of any two tuples of the ring $\mathbb{R}[x]$. It basically says that once any two tuple share all their boundary at some phase of expansion then certainly they should be indistiguishable. We state in a more formal fashion:

Theorem 5.9. *Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of the polynomial ring $\mathbb{R}[x]$. For any $\mathcal{S}_a, \mathcal{S}_b \in \{\mathcal{S}_j\}_{j=1}^{\infty}$ with $\deg(\mathcal{S}_a) = \deg(\mathcal{S}_b)$, then $\mathcal{S}_a = \mathcal{S}_b + \mathcal{S}_{\mathbb{R}}$ if and only if*

$$\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_a)] = \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)]$$

for some $1 \leq n < \deg(\mathcal{S}_a) - 1 = \deg(\mathcal{S}_b) - 1$.

Proof. Suppose $\mathcal{S}_a = \mathcal{S}_b + \mathcal{S}_{\mathbb{R}}$, then by Theorem 4.10 it follows that $\mathcal{R}(\mathcal{S}_a) = \mathcal{R}(\mathcal{S}_b)$. There exist some $k \geq 1$ such that $(\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^{\deg(\mathcal{S}_a)-k}(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_a)$ and $(\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^{\deg(\mathcal{S}_b)-k}(\mathcal{S}_b) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$. It follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b).$$

Thus, $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_a)] = \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_b)]$. Conversely let $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_a)] = \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_b)]$ and suppose on the contrary that $\mathcal{S}_a = \mathcal{S}_b + \mathcal{S}_{\mathbb{R}[x]}$, then it follows that $\deg(\mathcal{S}_{\mathbb{R}[x]}) \leq \deg(\mathcal{S}_a) = \deg(\mathcal{S}_b)$. It follows from Proposition 5.2 that

$$\begin{aligned}\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_{\mathbb{R}[x]})] &\leq \#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_a)]\\ &= \#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_b)].\end{aligned}$$

On the other hand, we observe that

$$\begin{aligned}\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_a)] &\subset \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_a-\mathcal{S}_b)]\\ &\subset \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_{\mathbb{R}[x]})].\end{aligned}$$

Thus it follows that

$$\begin{aligned}\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_a)] &= \#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_b)]\\ &< \#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_{\mathbb{R}[x]})],\end{aligned}$$

a contradiction, thereby ending the proof. □

5.3. **The co-boundary of expansion.** In this section we introduce the concept of the co-boundary of an expansion. We launch formally the following terminology.

Definition 5.10. Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of the polynomial ring $\mathbb{R}[x]$, and let $\mathcal{S}_0$ be a boundary point of the nth phase expansion. Then by the free point generated by $\mathcal{S}_0$ we mean the tuples

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n_{a_i}(\mathcal{S}_j)$$

where $\mathcal{S}_0 = (a_1, a_2, \ldots, a_m)$. By co-boundary points generated by the boundary points, we mean points of the form $a_i\mathcal{S}_e$ for $1 \leq i \leq m$. The co-boundary points forms the co-boundary. Next we prove that the norms of free points of the nth

phase expansion for $n \geq 1$ cannot be small.

It is also reasonable to believe that the more distant are the boundary points for higher phase expansion as they become sparce. Thus we state in a more formall tone the following conjecture:

Conjecture 4. Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of all tuples of $\mathbb{R}[x]$ and let $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_j)]$ be the boundary of the nth phase expansion. Let $\mathcal{S}_k$, $\mathcal{S}_l \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_j)]$ be any two boundary points, then $\inf ||\mathcal{S}_k - \mathcal{S}_l|| \geq \epsilon$ for all $n \geq n_0$ for some $n_0 > 0$.

Remark 5.11. We are now ready to prove the theorem.

Theorem 5.12. *Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of the polynomial ring $\mathbb{R}[x]$, and let $\mathcal{S}_t \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_j)]$ be a boundary point of the nth phase expansion where $n < \deg\mathcal{S}_j$. Then*

$$||(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n_{a_i}(\mathcal{S}_j)|| > 0$$

where $\mathcal{S}_t = (a_1, a_2, \ldots, a_m)$ such that $a_i \neq a_j$ for some $1 \leq i, j \leq m$.

Proof. Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of the polynomial ring $\mathbb{R}[x]$, suppose $\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_j)] = k$ for some $k > 1$ and let $\mathcal{S}_t \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_j)]$ be a boundary point of the nth phase expansion where $n < \deg \mathcal{S}_j$ and suppose on the contrary that

$$||(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n_{a_i}(\mathcal{S}_j)|| = 0$$

where $\mathcal{S}_t = (a_1, a_2, \ldots, a_m)$. Then it follows that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n_{a_i}(\mathcal{S}_j) = \mathcal{S}_0$ for $1 \leq i \leq m$. Thus it follows that the co-boundary point $a_i\mathcal{S}_e$ is also a boundary point. Thus

$$a_i\mathcal{S}_e \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_j)],$$

for $1 \leq i \leq m$. Since $\mathcal{S}_t = (a_1, a_2, \ldots, a_m)$ is such that $a_i \neq a_j$ for some $1 \leq i, j \leq m$, It follows that $\mathcal{S}_t \neq a_i\mathcal{S}_e$ for some $1 \leq i \leq m$. Thus

$$\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_j)] > k$$

thereby contradicting the size of the boundary. This completes the proof. □

Remark 5.13. The above result places some sort of barrier between the boundary and co-boundary points. In esense, the boundary and the co-boundary points generated should not overlap.

Corollary 2. Let $f_1, f_2, \ldots f_n \in \mathbb{R}[x]$, then the system

$$\begin{aligned} f_2' + f_3' + \cdots + f_n' &= 0 \\ f_1' + f_3' + \cdots + f_n' &= 0 \\ &\vdots \\ f_1' + f_2' + \cdots + f_{n-1}' &= 0 \end{aligned}$$

has no non-trivial solution.

Next we introduce a classification scheme of all tuples of the polynomial ring $\mathbb{R}[x]$. This scheme is based pretty much on the boundary points of a given phase of expansion.

Definition 5.14. Let $\mathcal{S} \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_j)]$ for $n \geq 0$. Then by the phase identifier, we mean the value $|\mathrm{Id}_i(\mathcal{S})|$. We say the phase identifier is weak if $\inf|\mathrm{Id}_i(\mathcal{S})| \leq 1$; otherwise we say it is strong.

In the language of expansivity, we state the celebrated Sendov conjecture which states that any zero of a polynomial must certainly lie in the same unit disc with some zero of the critical point. We restate the conjecture in this language. It is also important to make an analogy with the original formulation Sendov conjecture and the sendov conjecture formulated in this language. The zeros of the polynomial $P_n(x)$ of degree n correspond to the co-boundary points and the zeros of $P_n'(x)$ corresponds to the boundary points in the language of expansivity. Thus we restate the sendov conjecture in this language as follows:

Conjecture 5 (Sendov). Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n \geq 2$. Let $\{b_i\}_{i=1}^{n}$ be the set of zeros of $P(x)$ such that $|b_i| \leq 1$ and let $\mathcal{S} = (a_n x^n, a_{n-1}x^{n-1}, \ldots, a_1 x, a_0)$ be a tuple representation of $P(x)$. For each b_i, there exist some $\mathcal{S}_a \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S})]$ such that

$$|\mathrm{Id}_{n+1}(b_i \mathcal{S}_e - \mathcal{S}_a)| \leq 1.$$

Remark 5.15. The sendov conjecture, in the language of expansivity, can be stated as saying that any co-boundary point of the trivial expansion with weak phase identifier must in some sense be close to some boundary point of the first phase expansion with a weak phase identifier.

5.4. **The speed of an expansion.**

Definition 5.16. Let $\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of the polynomial ring $\mathbb{R}[x]$. Let $\mathcal{S} \in \{\mathcal{S}_j\}_{j=1}^{\infty}$, then by the speed of expansion, denoted $\upsilon(\mathcal{S})$, we mean

$$\upsilon(\mathcal{S}) = \frac{\mathcal{N}(\mathcal{S})}{\deg(\mathcal{S})}.$$

Remark 5.17. Next we relate the concept of the speed $\upsilon(\mathcal{S})$ of expansion of a tuple of the ring $\mathbb{R}[x]$ to the concept of the measure of expansion. We show that the speed of expansion is unique upto measure.

Proposition 5.3. *Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of n tuples of the ring $\mathbb{R}[x]$ and let $\mathcal{S}_a$, $\mathcal{S}_b \in \mathcal{F}$. If $\mathcal{N}(\mathcal{S}_a) = \mathcal{N}(\mathcal{S}_b)$, then $\upsilon(\mathcal{S}_a) = \upsilon(\mathcal{S}_b)$.*

Proof. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of the tuples of the ring $\mathbb{R}[x]$ and suppose $\mathcal{N}(\mathcal{S}_a) = \mathcal{N}(\mathcal{S}_b)$ for $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$, then it follows by Theorem 5.5 that $\mathcal{S}_a = \mathrm{Sgn}(\tau)\tau(\mathcal{S}_b) + \mathcal{S}_k$, where $\deg(\mathcal{S}_k) < \deg(\mathcal{S}_b)$ and $\tau : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ is some permutation. It follows by Theorem 5.5 that $\deg(\mathcal{S}_a) = \deg(\mathcal{S}_b)$. By definition 5.16, the result follows immediately. $\square$

Theorem 5.1 reveals that the measure of an expansion of elements in the collection $\mathcal{F}$ of the tuples in the ring $\mathbb{R}[x]$ is a norm. The speed of an expansion $\upsilon(\mathcal{S})$ inherits this property given the profound relationship with the measure. The following proposition verifies that claim.

Proposition 5.4. *The speed of an expansion $\upsilon(\mathcal{S})$ is a norm.*

Proof. Let $\mathcal{F}$ be a collection of tuples of the polynomial ring $\mathbb{R}[x]$. Let $\mathcal{S} \in \mathcal{F}$, then it follows that $\upsilon(\mathcal{S}) > 0$ since $\mathcal{N}(\mathcal{S}) > 0$. In the case $\upsilon(\mathcal{S}) = 0$, then it follows by definition 5.16 that $\mathcal{N}(\mathcal{S}) = 0$. Using Theorem 5.1, it follows that $\mathcal{S} = \mathcal{S}_0$. Conversely, if $\mathcal{S} = \mathcal{S}_0$, then it follows that $\mathcal{N}(\mathcal{S}) = \mathcal{N}(\mathcal{S}_0) = 0$, and it follows that $\upsilon(\mathcal{S}) = 0$.

Now let $a \in \mathbb{R}$ for $a > 0$, then it follows that

$$\begin{aligned}\upsilon(a\mathcal{S}) &= \frac{\mathcal{N}(a\mathcal{S})}{\deg(a\mathcal{S})}\\ &= \frac{a\mathcal{N}(\mathcal{S})}{\deg(\mathcal{S})}\\ &= a\upsilon(\mathcal{S})\end{aligned}$$

since, by Theorem 5.5, the measure $\mathcal{N}(\mathcal{S})$ is a norm. Also we observe that

$$\begin{aligned}\upsilon(\mathcal{S}_1 + \mathcal{S}_2) &= \frac{\mathcal{N}(\mathcal{S}_1 + \mathcal{S}_2)}{\deg(\mathcal{S}_1 + \mathcal{S}_2)}\\ &\leq \frac{\mathcal{N}(\mathcal{S}_1) + \mathcal{N}(\mathcal{S}_2)}{\deg(\mathcal{S}_1 + \mathcal{S}_2)}\\ &\leq \frac{\mathcal{N}(\mathcal{S}_1)}{\deg(\mathcal{S}_1)} + \frac{\mathcal{N}(\mathcal{S}_1)}{\deg(\mathcal{S}_2)}\\ &= \upsilon(\mathcal{S}_1) + \upsilon(\mathcal{S}_2)\end{aligned}$$

thereby ending the proof. □

Next we relate the notion of the boundary of expansion to the speed of expansion. We prove that once the boundary of two tuples of the polynomial ring $\mathbb{R}[x]$ coincides at some phase of expansion, then certainly they should have the same speed of expansion.

Proposition 5.5. *Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of the tuples of the polynomial ring $\mathbb{R}[x]$. For any $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$ with $\deg(\mathcal{S}_a) = \deg(\mathcal{S}_b)$, if there exist some $n \geq 1$ such that*

$$\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_a)] = \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)],$$

then $\upsilon(\mathcal{S}_a) = \upsilon(\mathcal{S}_b)$.

Proof. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ and let $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$ with $\deg(\mathcal{S}_a) = \deg(\mathcal{S}_b)$. Suppose $\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_a)] = \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)]$ for some $n \geq 1$, then it follows from Theorem 5.9 that $\mathcal{S}_a = \mathcal{S}_b + \mathcal{S}_{\mathbb{R}}$. By Theorem 4.10, it follows that $\mathcal{N}(\mathcal{S}_a) = \mathcal{N}(\mathcal{S}_b)$. The result follows by applying definition 5.16. □

5.5. **Momentum of phase expansions.**

Definition 5.18. Let $\{\mathcal{S}_j\}_{j=1}^{\infty} = \mathcal{F}$ be any collection of the tuples of $\mathbb{R}[x]$. By the momentum of the nth phase expansion, denoted $\mathcal{M}(\mathcal{S}_j^n)$, we mean

$$\mathcal{M}(\mathcal{S}_j^n) := \upsilon(\mathcal{S}_j^{n_j})\mathcal{H}(\mathcal{S}_j^n)$$

where $\mathcal{B}^n$ denotes the set of boundary points of the nth phase expansion, and

$$\mathcal{H}(\mathcal{S}_j^n) = \sum_{\mathcal{S}_k \in \mathcal{B}^n} ||\mathcal{S}_k||$$

is the mass of the nth phase expansion with $||\mathcal{S}_k|| := \sqrt{\sum\limits_{i=1}^{n} |a_i|^2}$ for $\mathcal{S}_k = (a_1, a_2, \ldots, a_n)$.

6. **Inverse problems**

In this section we devote our attention to studying ways of recovering a tuple from expanded tuples of $\mathbb{R}[x]$ at any given phase of expansion. Using Proposition 4.1 and the concepts of integration, which can be viewed as an inverse of differentiation on tuples of $\mathbb{R}[x]$, we find that for any given expanded tuple, say $\mathcal{S}_1$ and given that the composite map $\gamma^{-1}\circ\beta\circ\gamma\circ\nabla : \{\mathcal{S}_i\}_{i=1}^{\infty} \longrightarrow \{\mathcal{S}_i\}_{i=1}^{\infty}$ is bijective, $\gamma^{-1}\circ\beta\circ\gamma\circ\nabla(\mathcal{S}) = \mathcal{S}_1$ if and only if $\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma(\mathcal{S}_1) = \mathcal{S}$. In most cases we would like to keep track of the original tuple, that satisfies certain initial conditions. Say at $a \in \mathbb{R}$, the sought-after tuple satisfies $\mathcal{S}_a = (a_1, a_2, \ldots, a_n)$, then in such case the one copy map $\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma(\mathcal{S}_1)$ will be evaluated at a, denoted by $\Delta_a \circ \gamma_a^{-1} \circ \beta_a^{-1} \circ \gamma_a$.

6.1. **Inverse problem for second phase expansions.** Given a tuple $\mathcal{S} := (f_1, f_2, \ldots, f_n)$ of $\mathbb{R}[x]$ expanded up to the second phase, with $\mathcal{S}^1$ and $\mathcal{S}^2$ denoting the first and the second phase expanded tuple, respectively. Then to recover the original tuple $\mathcal{S}$, we first need to recover the first phase expanded tuple from the second, and then finally the original from the first. That is, we can recover the first phase expanded tuple from the composite map

$$\Delta_a \circ \gamma_a^{-1} \circ \beta_a^{-1} \circ \gamma_a(\mathcal{S}^2) = \mathcal{S}^1,$$

and the original is obtained by

$$\Delta_b \circ \gamma_b^{-1} \circ \beta_b^{-1} \circ \gamma_b(\mathcal{S}^1) = \mathcal{S}.$$

Thus we find that $\Delta_b \circ \gamma_b^{-1} \circ \beta_b^{-1} \circ \gamma_b(\Delta_a \circ \gamma_a^{-1} \circ \beta_a^{-1} \circ \gamma_a(\mathcal{S}^2)) = \mathcal{S}$ and

$$(\Delta_a \circ \gamma_a^{-1} \circ \beta_a^{-1} \circ \gamma_a)^2(\mathcal{S}^2) = \mathcal{S}.$$

if and only if $a = b$ for $a, b \in \mathbb{R}$.

6.2. **Inverse problem for higher phase expansions.** It turns out from the set up, in order to recover a tuple from the nth phase expanded tuple, we only need n copies of the recovery map of each phase. The recovery process is fairly within reach by applying the n copies of the map $\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma$ to the nth phase expanded tuple, for values of n reasonably small. In practice, it will suffice to apply the n distinct copies of the map $(\Delta_{a_1} \circ \gamma_{a_1}^{-1} \circ \beta_{a_1}^{-1} \circ \gamma_{a_1}) \circ (\Delta_{a_2} \circ \gamma_{a_2}^{-1} \circ \beta_{a_2}^{-1} \circ \gamma_{a_2}) \circ \cdots \circ (\Delta_{a_n} \circ \gamma_{a_n}^{-1} \circ \beta_{a_n}^{-1} \circ \gamma_{a_n})$ to the nth phase expanded tuple. However this process becomes less efficient and very brutal if the phase expansion number n is sufficiently large. So we ask a fairly natural question, as follows:

Question 1. What is the most efficient way of recovering a tuple from the nth expanded tuple for sufficiently large values of n?

Theorem 6.1. *The set* $\mathcal{T} := \{\mathrm{Id}\} \cup \bigcup_{k=1}^{\infty} \left\{(\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^k\right\} \cup \bigcup_{k=1}^{\infty} \left\{(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k\right\}$ *forms a group.*

Proof. Let $\mathcal{S}$ be any tuple of $\mathbb{R}[x]$ satisfying certain initial conditions at each phase of expansion, then $\mathrm{Id}(\mathcal{S}) = \mathcal{S}$, where $\mathrm{Id}(\mathcal{S}) = \mathrm{Id}(f_1, f_2, \ldots, f_n) := (\mathrm{Id}(f_1), \mathrm{Id}(f_2), \ldots, \mathrm{Id}(f_n)) = (f_1, f_2, \ldots, f_n) = \mathcal{S}$. That is, Id leaves each tuple of $\mathbb{R}[x]$ invariant. Again pick arbitrarily a tuple $\mathcal{S}$ of $\mathbb{R}[x]$. Then $(\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^l \circ (\Delta \circ \gamma^{-1} \circ \beta^{-1} \circ \gamma)^m(\mathcal{S}) =$

$(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^m\circ(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^l(\mathcal{S})=(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^{m+l}(\mathcal{S})$. That is, applying m and l copies of a recovery map is the same as applying $m+l$ copies of the recovery map to the tuple $\mathcal{S}$. Thus $(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^l\circ(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^m\in\mathcal{T}$. A similar characterization applies to expansions. For the mixed maps with distinct copies, we find that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^l\circ(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^m=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l-m}$ is an expansion provided $l-m>0$ and is a recovery map in the case $l-m<0$. Each of the either situations is still contained in the set $\mathcal{T}$. Thus the set is closed. Again pick an arbitrary tuple whose degree of expansion is n, then $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^l\circ(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^l=(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^l\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^l=\mathrm{Id}\in\mathcal{T}$ for $l\leq n$. Thus each copy of an expansion has a recovery and vice-versa in the set $\mathcal{T}$. The associative property is easy to verify. Hence the collection is a group. □

7. **Embedding and extension of expansions**

In this section, we introduce the notion of an embedding and an extension of a phase of an expansion.

Definition 7.1. Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Let $\mathcal{S}_a,\mathcal{S}_b\in\mathcal{F}$, then we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_1}(\mathcal{S}_b)$ is an embedding of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_2}(\mathcal{S}_a)$ if

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_1}(\mathcal{S}_b)]\subset\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_2}(\mathcal{S}_a)]$$

for some $n_1,n_2\in\mathbb{N}$. Conversely, we say $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_2}(\mathcal{S}_a)$ is an extension of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_1}(\mathcal{S}_b)$

The notion of an embedding and an extension of expansion could be adapted in practice. Intuitively, as the cell wall of a living organism becomes turgid the membranes expands to make room for this behaviour. This notion reinforces that we can in practice pinch any two portion of the membrane and join these ends together, thereby obtaining a membrane similar to the previous membrane but now with fewer materials of the previous membrane. In relation to our work, a natural question to ask is whether there exists a tuple whose boundary of expansion represents this boundary, and if it does how does this tuple relates to the tuple of the actual expansion. The sequel will be devoted to investigate these things in far greater detail.

Proposition 7.1. *Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of n tuples of $\mathbb{R}[x]$ and $\mathcal{S}_a,\mathcal{S}_b\in\mathcal{F}$. If $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_2}(\mathcal{S}_a)$ is an embedding of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_1}(\mathcal{S}_b)$, then*

$$\mathcal{H}(\mathcal{S}_a^{n_2})<\mathcal{H}(\mathcal{S}_b^{n_1}).$$

Proof. Let $\mathcal{S}_a,\mathcal{S}_b\in\mathcal{F}$ and suppose $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_2}(\mathcal{S}_a)$ is an embedding of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_1}(\mathcal{S}_b)$. Then it follows by definition 7.1

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_2}(\mathcal{S}_a)]\subset\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_1}(\mathcal{S}_b)]$$

for some $n_1,n_2\in\mathbb{N}$. The result follows from this condition by leveraging definition 5.18. □

We now prove the statement we made in the previous remark, concerning the finite process of embedding of expansions.

Theorem 7.2. *Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{R}[x]$. Then there exist some $\mathcal{S}_a \in \mathcal{F}$ that do not admit an embedding.*

Proof. Suppose the collection $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ admits an embedding for all $\mathcal{S}_a \in \mathcal{F}$. Then for some $\mathcal{S}_1 \in \mathcal{F}$, it follows by definition 7.1 there exist some $\mathcal{S}_2 \in \mathcal{F}$ such that

$$\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_2)] \subset \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_1)].$$

It follows from Proposition 7.1 $\mathcal{H}(\mathcal{S}_2^{n_2}) < \mathcal{H}(\mathcal{S}_1^{n_1})$. Again, since $\mathcal{S}_2 \in \mathcal{F}$, it follows that there exist some $\mathcal{S}_3 \in \mathcal{F}$ such that

$$\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_3}(\mathcal{S}_3)] \subset \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_2)]$$

and it follows in light Proposition 7.1 that $\mathcal{H}(\mathcal{S}_3^{n_3}) < \mathcal{H}(\mathcal{S}_2^{n_2})$. Since the collection is infinite, it follows by induction

$$\mathcal{H}(\mathcal{S}_1^{n_1}) > \mathcal{H}(\mathcal{S}_2^{n_2}) > \cdots > \mathcal{H}(\mathcal{S}_n^{n_n}) > \cdots > \mathcal{H}(\mathcal{S}_{n+1}^{k_{n+1}}) > \cdots > \mathcal{H}(\mathcal{S}_r^{k_r}) = 0.$$

Thus we obtain sequence of masses eventually descending to zero. This cannot happen since the mass $\mathcal{H}(\mathcal{S}_j^{n_j})$ for $j \geq 1$ of elements in the collection $\mathcal{F}$ satisfies

$$\mathcal{H}(\mathcal{S}_j^{n_j}) > 0.$$

This completes the proof of the theorem. □

Remark 7.3. Next we connect results on the notion of an embedding of expasions to the momemtum of expansion and, hence, the speed of an expansion in the following sequel.

Proposition 7.2. *Let $\mathcal{F} = \{\mathcal{S}\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Let $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$ and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_a)$ is an embedding of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_b)$. If $\mathcal{M}(\mathcal{S}_a^{n_1}) = \mathcal{M}(\mathcal{S}_b^{n_2})$, then*

$$\nu(\mathcal{S}_b^{n_2}) < \nu(\mathcal{S}_a^{n_1}).$$

Proof. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ and let $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$ and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_2}(\mathcal{S}_a)$ is an embedding of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_1}(\mathcal{S}_b)$, then it follows from Proposition 7.1 $\mathcal{H}(\mathcal{S}_a^{n_2}) < \mathcal{H}(\mathcal{S}_b^{n_1})$. Using the equation

$$\mathcal{M}(\mathcal{S}_j^{n}) = \nu(\mathcal{S}_j^{n})\mathcal{H}(\mathcal{S}_j^{n})$$

with the condition $\mathcal{M}(\mathcal{S}_a^{n_1}) = \mathcal{M}(\mathcal{S}_b^{n_2})$, the result follows immediately. □

7.1. **The index of expansion.** In this section we introduce the concept of the index $\mathcal{I}(\mathcal{S}_j)$ of expansion of the tuple $\mathcal{S}_j$. We launch more formally the terminology:

Definition 7.4. Let $\mathcal{P} := \{\mathcal{S}_j\}_{j=1}^{n}$ be a finite collection of tuples of $\mathbb{R}[x]$. Then by the index of the m th phase expansion of the tuple $\mathcal{S}_k$ for $1 \leq k \leq n$, we mean the ratio

$$\mathcal{I}(\mathcal{S}_k^m) = \frac{\sum\limits_{j=1}^{n} \mathcal{M}(\mathcal{S}_j^m)}{\mathcal{M}(\mathcal{S}_k^m)}.$$

Remark 7.5. Next we establish an inequality that relates the index of expansion of a tuple to the largest size of the number of embeddings of expansion, in the following result.

Theorem 7.6. *Let $\mathcal{P} := \{\mathcal{S}_j\}_{j=1}^{n}$ and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_k}(\mathcal{S}_k)$ $(1 \leq k \leq n)$ admits an embedding $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_j}(\mathcal{S}_j)$ for all $1 \leq j \leq n$. If $\nu(\mathcal{S}_k^{n_k}) \geq \nu(\mathcal{S}_j^{n_j})$ for all $1 \leq j \leq n$, then*

$$\mathcal{I}(\mathcal{S}_k^{n_k}) < n.$$

Proof. Let $\mathcal{P} := \{\mathcal{S}_j\}_{j=1}^{n}$ and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_k}(\mathcal{S}_k)$ $(1 \leq k \leq n)$ admits an embedding $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_j}(\mathcal{S}_j)$ for all $1 \leq j \leq n$. Then it follows that

$$\begin{aligned}\sum_{j=1}^{n} \mathcal{M}(\mathcal{S}_j^{n_j}) &= \mathcal{M}(\mathcal{S}_1^{n_1}) + \mathcal{M}(\mathcal{S}_2^{n_2}) + \cdots + \mathcal{M}(\mathcal{S}_k^{s_k}) + \cdots + \mathcal{M}(\mathcal{S}_n^{n_n}) \\ &= \nu(\mathcal{S}_1^{n_1})\mathcal{H}(\mathcal{S}_1^{n_1}) + \nu(\mathcal{S}_2^{n_2})\mathcal{H}(\mathcal{S}_2^{n_2}) + \cdots + \nu(\mathcal{S}_k^{n_k})\mathcal{H}(\mathcal{S}_k^{n_k}) + \cdots + \nu(\mathcal{S}_n^{n_n})\mathcal{H}(\mathcal{S}_n^{n_n}) \\ &\leq \nu(\mathcal{S}_1^{n_1})\mathcal{H}(\mathcal{S}_k^{n_k}) + \nu(\mathcal{S}_2^{n_2})\mathcal{H}(\mathcal{S}_k^{n_k}) + \cdots + \nu(\mathcal{S}_k^{n_k})\mathcal{H}(\mathcal{S}_k^{n_k}) + \cdots + \nu(\mathcal{S}_n^{n_n})\mathcal{H}(\mathcal{S}_k^{n_k}) \\ &\leq n\nu(\mathcal{S}_k^{n_k})\mathcal{H}(\mathcal{S}_k^{n_k}) \\ &= n\mathcal{M}(\mathcal{S}_k^{n_k})\end{aligned}$$

and the inequality is established. □

Corollary 3. Let $\mathcal{P} := \{\mathcal{S}_j\}_{j=1}^{n}$ and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_k}(\mathcal{S}_k)$ $(1 \leq k \leq n)$ admits an embedding $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_j}(\mathcal{S}_j)$ for all $k < j \leq n$ for $k \geq 1$. If $\nu(\mathcal{S}_k^{n_k}) \geq \nu(\mathcal{S}_j^{n_j})$ for all $k < j \leq n$ $(k \geq 1)$, then

$$\sum_{r=1}^{n} \mathcal{I}(\mathcal{S}_r^{n_r}) < \frac{n(n+1)}{2}.$$

Proof. The result follows by applying Theorem 7.6. □

7.2. **Application of mass embedding to the sendov conjecture.** In this section we prove a weak variant of the sendov conjecture under the assumption that the first phase of an expansion of any tuple $\mathcal{S}_a \in \{\mathcal{S}_j\}_{j=1}^{\infty}$ is an embedding of the trivial expansion. We give a formall statement in the following result:

Theorem 7.7. *Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$ and let $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$. Suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{1}(\mathcal{S}_a)$ is an embedding of $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{0}(\mathcal{S}_b) = \mathcal{S}_b$, where $\mathcal{S}_b = (P(x), P(x), \ldots, P(x))$ with $P(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for $n \geq 3$ and $\mathcal{S}_a = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{1}(\mathcal{S}_c)$, where $\mathcal{S}_c = (a_n x^n, a_{n-1}x^{n-1}, \ldots, a_1 x, a_0)$, a* *tuple representation of $P(x)$. If the mass $\mathcal{H}(\mathcal{S}_b^0)$ of the trivial expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S}_b)$ satisfies*

$$\mathcal{H}(\mathcal{S}_b^0) < \delta$$

where $1 > \delta > 0$ is sufficiently small, then for each boundary point of the trivial expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S}_b)$ of the form $b_i\mathcal{S}_e$, there exist a boundary point $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S}_a)]$ such that

$$||b_i\mathcal{S}_e - \mathcal{S}_0|| < 1.$$

Proof. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$ and let $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$. Suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S}_a)$ is an embedding of $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S}_b) = \mathcal{S}_b$, where

$$\mathcal{S}_b = (P(x), P(x), \ldots, P(x))$$

with $P(x) := a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $\mathcal{S}_a = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S}_c)$, where $\mathcal{S}_c = (a_nx^n, a_{n-1}x^{n-1}, \ldots, a_1x, a_0)$, a tuple representation of $P(x)$. Futhermore, suppose that for any boundary point $b_i\mathcal{S}_e$ of the trivial expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S}_b) = \mathcal{S}_b$

$$||b_i\mathcal{S}_e - \mathcal{S}_0|| \geq 1$$

for all $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S}_a)]$. Since $\mathcal{H}(\mathcal{S}_b^0) < \delta$ and $1 > \delta > 0$ is sufficiently small, It follows that the mass $\mathcal{H}(\mathcal{S}_a^1) \geq 1$. Since $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S}_a)$ is an embedding of $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S}_b) = \mathcal{S}_b$, it follows from proposition 7.1

$$1 > \mathcal{H}(\mathcal{S}_b^0) > \mathcal{H}(\mathcal{S}_a^1) \geq 1.$$

This inequality is absurd. This completes the proof of the Theorem. □

Remark 7.8. Theorem 7.7 is close to proving the sendov conjecture. It tells us that for any polynomial $P(x)$ with sufficiently small zeros, we can find a zero of $P'(x)$ that is somewhat close under the assumption that the set of zeros of $P'(x)$ are subsets of the zeros of $P(x)$. This result is much weaker than the statement of the sendov conjecture.

It is very important to notice that we could in some way rigorize Theorem 7.7 by removing the mass-embedding the condition in place of the assumption that the mass of each phase of expansion diminishes. In principle the sendov conjecture would be proven in full if we could prove unconditionally the mass diminishes with higher phase expansions. At the moment we carry on this assumption to prove Sendov's conjecture. A sequel to this paper will be devoted to studying phase expansions in relation to their masses.

Theorem 7.9. *Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$ and let $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$. Suppose $\mathcal{H}(\mathcal{S}_b^0) > \mathcal{H}(\mathcal{S}_a^1)$, where $\mathcal{S}_b = (P(x), P(x), \ldots, P(x))$ with $P(x) := a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with $n \geq 3$ and $\mathcal{S}_a = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S}_c)$, where $\mathcal{S}_c = (a_nx^n, a_{n-1}x^{n-1}, \ldots, a_1x, a_0)$, a tuple representation of $P(x)$. If the mass $\mathcal{H}(\mathcal{S}_b^0)$ of the trivial expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S}_b)$ satisfies*

$$\mathcal{H}(\mathcal{S}_b^0) < \delta$$

where $1 > \delta > 0$ is sufficiently small, then for each boundary point of the trivial expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S}_b)$ of the form $b_i\mathcal{S}_e$, there exist a boundary point $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S}_a)]$ such that

$$||b_i\mathcal{S}_e - \mathcal{S}_0|| < 1.$$

Proof. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$ and let $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{F}$. Suppose $\mathcal{H}(\mathcal{S}_b^0) > \mathcal{H}(\mathcal{S}_a^1)$, where

$$\mathcal{S}_b = (P(x), P(x), \ldots, P(x))$$

with $P(x) := a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $\mathcal{S}_a = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S}_c)$, where $\mathcal{S}_c = (a_nx^n, a_{n-1}x^{n-1}, \ldots, a_1x, a_0)$, a tuple representation of $P(x)$. Futhermore, suppose that for any boundary point of the form $b_i\mathcal{S}_e$ of the trivial expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^0(\mathcal{S}_b) = \mathcal{S}_b$

$$||b_i\mathcal{S}_e - \mathcal{S}_0|| \geq 1$$

for all $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^1(\mathcal{S}_a)]$. Since $\mathcal{H}(\mathcal{S}_b^0) < \delta$ and $1 > \delta > 0$ is sufficiently small and $n \geq 3$, It follows that the mass $\mathcal{H}(\mathcal{S}_a^1) \geq 1$ and we obtain the inequality

$$1 > \mathcal{H}(\mathcal{S}_b^0) > \mathcal{H}(\mathcal{S}_a^1) \geq 1.$$

This inequality is absurd, thereby ending the proof of the Theorem. □

Remark 7.10. It is important to make a distiction between Theorem 7.7 and Theorem 7.9. On face value they seem almost the same. The mass-embedding condition as supposed in Theorem 7.7 certainly evokes a diminishing state of the mass of expansion. However, the converse as espoused in Theorem 7.9 does not neccesarily evoke a mass-embedding regime. Indeed the mass of an expansion at a given phase could be smaller than the mass of another expansion not because it is an embedding.

Theorem 7.11. *Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Then for each $\mathcal{S} \in \mathcal{F}$*

$$\sum_{k=0}^{deg(\mathcal{S})-1} \nu(\mathcal{S}^k) = \nu(\mathcal{S})deg(\mathcal{S})\log(deg(\mathcal{S})) + deg(\mathcal{S})\nu(\mathcal{S})\gamma + O(\nu(\mathcal{S})),$$

where $\gamma = 0.5772\cdots$ is the euler-macheroni constant.

Proof. Clearly

$$\begin{aligned}\sum_{k=0}^{deg(\mathcal{S})-1}\nu(\mathcal{S}^k)&=\nu(\mathcal{S})+\nu(\mathcal{S}^1)+\cdots+\nu(\mathcal{S}^{deg(\mathcal{S})-1})\\&=\frac{\mathcal{N}(\mathcal{S})}{deg(\mathcal{S})}+\frac{\mathcal{N}(\mathcal{S}^1)}{deg(\mathcal{S}^1)}+\cdots+\frac{\mathcal{N}(\mathcal{S}^{deg(\mathcal{S})-1})}{deg(\mathcal{S}^{deg(\mathcal{S})-1})}\\&=\mathcal{N}(\mathcal{S})\bigg(\frac{1}{deg(\mathcal{S})}+\frac{1}{deg(\mathcal{S}^1)}+\cdots+\frac{1}{deg(\mathcal{S}^{deg(\mathcal{S})-1})}\bigg)\\&=\mathcal{N}(\mathcal{S})\bigg(\frac{1}{deg(\mathcal{S})}+\frac{1}{deg(\mathcal{S})-1}+\cdots+\frac{1}{2}+1\bigg)\\&=\mathcal{N}(\mathcal{S})\sum_{m=1}^{deg(\mathcal{S})}\frac{1}{m}\\&=\nu(\mathcal{S})deg(\mathcal{S})\sum_{m=1}^{deg(\mathcal{S})}\frac{1}{m}\end{aligned}$$

thereby establishing the formula. □

Remark 7.12. As it will turn out in the sequel, this formula will become extremely useful in studying the diminishing state of the behaviour of the mass of expansions. For the time being, we use this formula to prove that the mass diminishes at some succesive phase of expansion.

Theorem 7.13. *Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Suppose $\mathcal{S}\in\mathcal{F}$, then*

$$\mathcal{H}(\mathcal{S}^n)>\mathcal{H}(\mathcal{S}^{n+1})$$

for some $0\leq n\leq deg(\mathcal{S})-2$.

Proof. Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$ and specify $\mathcal{S}\in\mathcal{F}$. Consider the finite collection $\mathcal{P}=\{\mathcal{S}^k\}_{k=0}^{deg(\mathcal{S})-1}$. Suppose on the contrary that

$$\mathcal{H}(\mathcal{S}^n)\leq\mathcal{H}(\mathcal{S}^{n+1})$$

for all $0\leq n\leq deg(\mathcal{S})-2$. Then it follows by an application of Theorem 7.11 that

$$\begin{aligned}\sum_{k=0}^{deg(\mathcal{S})-1}\mathcal{M}(\mathcal{S}^k)&=\sum_{k=0}^{deg(\mathcal{S})-1}\mathcal{H}(\mathcal{S}^k)\nu(\mathcal{S}^k)\\&\leq\mathcal{H}(\mathcal{S}^{deg(\mathcal{S})-1})\sum_{k=0}^{deg(\mathcal{S})-1}\nu(\mathcal{S}^k)\\&\ll\mathcal{H}(\mathcal{S}^{deg(\mathcal{S})-1})\nu(\mathcal{S})deg(\mathcal{S})\log(deg(\mathcal{S}))\\&\ll\mathcal{H}(\mathcal{S}^{deg(\mathcal{S})-1})\nu(\mathcal{S}^{deg(\mathcal{S})-1})deg(\mathcal{S})\log(deg(\mathcal{S}))\\&\ll\mathcal{M}(\mathcal{S}^{deg(\mathcal{S})-1})deg(\mathcal{S})\log(deg(\mathcal{S})),\end{aligned}$$

and it follows that the index of expansion $\mathcal{I}((\mathcal{S}^{deg(\mathcal{S})-2})^1)\ll deg(\mathcal{S})\log(deg(\mathcal{S}))$, thereby contradicting the upper bound in Theorem 7.6. □

Remark 7.14. Next we classify all phase of expansions with decreasing mass. We launch the following classification scheme in that regard.

Definition 7.15. Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Then for any $\mathcal{S}_k\in\mathcal{F}$, we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_k)$ is regular if $\mathcal{H}(\mathcal{S}_k^n)>\mathcal{H}(\mathcal{S}_k^{n+1})$ for some $0\leq n\leq deg(\mathcal{S}_k)-2$.

Theorem 7.16. *Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$ and suppose the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_k)$ with ($n\leq deg(\mathcal{S}_k)-3$) is regular, for $\mathcal{S}_k\in\mathcal{F}$. If*

$$\mathcal{H}(\mathcal{S}_k^n)<\delta$$

for $0<\delta<1$ sufficiently small, then for each $\mathcal{S}_1\in Z[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_k)]$, there exist some $\mathcal{S}_0\in\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n+1}(\mathcal{S}_k)]$ such that

$$||\mathcal{S}_1-\mathcal{S}_0||<1.$$

Proof. Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Pick arbitrarily $\mathcal{S}_k\in\mathcal{F}$ and suppose the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_k)$ is regular. Suppose on the contrary that for each $\mathcal{S}_1\in Z[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_k)]$, then

$$||\mathcal{S}_1-\mathcal{S}_0||\geq 1$$

for all $\mathcal{S}_0\in\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n+1}(\mathcal{S}_k)]$. Since $\mathcal{H}(\mathcal{S}_k^n)<\delta$ with $0<\delta<1$ sufficiently small and $n\leq deg(\mathcal{S}_k)-3$, it follows that $\mathcal{H}(\mathcal{S}_k^{n+1})\geq 1$. Under the regularity condition, it must be that

$$1>\delta>\mathcal{H}(\mathcal{S}_k^n)>\mathcal{H}(\mathcal{S}_k^{n+1})\geq 1$$

which is absurd. This completes the proof of the theorem. □

Remark 7.17. The combination of Theorem 7.13 and Theorem 7.16 roughly speaking affirms that the sendov conjecture is true at some phase of expansion.

Conjecture 6 (The mass law). Let $\mathcal{S}\in\mathcal{F}$ with $deg(\mathcal{S})>1$, where $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ is a collection of tuples of $\mathbb{R}[x]$. Then

$$\mathcal{H}(\mathcal{S}^1)\gg\frac{||\mathcal{S}(deg(\mathcal{S}))||deg(\mathcal{S})}{\mathcal{N}(\mathcal{S})\log(deg(\mathcal{S}))}.$$

We could demonstrate the validity of this conjecture under certain assumptions, namely that the mass decreases uniformly with each successive phase of expansion. Additionally, under the assumption of Conjecture 3, then we have

$$\begin{aligned}\sum_{k=0}^{deg(\mathcal{S})-1}\mathcal{M}(\mathcal{S}^k)&\geq\sum_{k=0}^{deg(\mathcal{S})-1}\mathcal{H}(\mathcal{S}^k)\nu(\mathcal{S}^{deg(\mathcal{S})-1})\\&\gg||\mathcal{S}(deg(\mathcal{S}))||\sum_{k=0}^{deg(\mathcal{S})-1}\mathcal{H}(\mathcal{S}^k)\\&\gg||\mathcal{S}(deg(\mathcal{S}))||deg(\mathcal{S})\mathcal{H}(\mathcal{S}^{deg(\mathcal{S})-1})\\&\gg||\mathcal{S}(deg(\mathcal{S}))||deg(\mathcal{S}).\end{aligned}$$

Again, we observe that, by using Theorem 7.11, then we have the upper bound

$$\begin{aligned}\sum_{k=0}^{deg(\mathcal{S})-1}\mathcal{M}(\mathcal{S}^k) &\ll \mathcal{H}(\mathcal{S}^1)\sum_{k=0}^{deg(\mathcal{S})-1}\nu(\mathcal{S}^k)\\ &= (1+o(1))\mathcal{H}(\mathcal{S}^1)\nu(\mathcal{S})deg(\mathcal{S})\log(deg(\mathcal{S}))\\ &= (1+o(1))\mathcal{H}(\mathcal{S}^1)\mathcal{N}(\mathcal{S})\log(deg(\mathcal{S})).\end{aligned}$$

By combining the upper bound and the lower bound establishes the lower bound for the mass of the first phase expansion. This derivation, it must be said, is not rigorous. We could in principle make this process rigorous without having to resort to unproven conjectures. At the moment this quest seems out of reach, since ascertaining the diminishing state of the mass of expansion in a sufficiently uniform way and establishing the measure inequality seems to be a hard enough problem.

8. **Isomorphic boundaries and expansions**

In this section we introduce the concept of isomorphic boundaries and expansions. We consider this notion in-depth and investigate it's connection to the already developed concepts.

Definition 8.1. Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Then we say the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)$ (resp. boundaries) are isomorphic if $\mathcal{H}(\mathcal{S}_l^{n_l})=\mathcal{H}(\mathcal{S}_k^{n_k})$, and we write

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$$

to denote the isomorphism.

It is important to point out the notion of isomorphism of expansions induces an equivalence relation and, thus, partitions expansions to equivalent classes. Indeed

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]$$

since $\mathcal{H}(\mathcal{S}_k^{n_k})=\mathcal{H}(\mathcal{S}_k^{n_k})$. The symmetric property also holds. For the transitivity property, we have

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$$

implies $\mathcal{H}(\mathcal{S}_k^{n_k})=\mathcal{H}(\mathcal{S}_l^{n_l})$ and

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]$$

implies $\mathcal{H}(\mathcal{S}_l^{n_l})=\mathcal{H}(\mathcal{S}_h^{n_h})$ and it follows that

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)].$$

Proposition 8.1. *Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Let $\mathcal{S}_l,\mathcal{S}_k\in\mathcal{F}$. Suppose $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)$ is regular. If*

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k+1}(\mathcal{S}_k)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l+1}(\mathcal{S}_l)]$$

then $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)$ is also regular.

Proof. Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Let $\mathcal{S}_l,\mathcal{S}_k\in\mathcal{F}$. Suppose $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$. Then it follows by definition 8.1

$$\mathcal{H}(\mathcal{S}_k^{n_k})=\mathcal{H}(\mathcal{S}_l^{n_l}).$$

Since $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)$ is regular, it follows that $\mathcal{H}(\mathcal{S}_k^{n_k})=\mathcal{H}(\mathcal{S}_l^{n_l})>\mathcal{H}(\mathcal{S}_k^{n_k+1})$. Since

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k+1}(\mathcal{S}_k)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l+1}(\mathcal{S}_l)]$$

It follows by definition 8.1 $\mathcal{H}(\mathcal{S}_l^{n_l})=\mathcal{H}(\mathcal{S}_k^{n_k})>\mathcal{H}(\mathcal{S}_k^{n_k+1})=\mathcal{H}(\mathcal{S}_l^{n_l+1})$, and the result follows immediately. □

9. **Boundary deformation of expansions**

In this section we introduce the notion of deformation of boundaries of expansions. We launch more formally the following terminology:

Definition 9.1. Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. We say the boundary of the expansion $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$ is a deformation of the boundary of expansion $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]$ if there exist a map

$$\pi:\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]\longrightarrow\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$$

such that $\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]>\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$ with $\mathcal{H}(\mathcal{S}_h^{n_h})=\mathcal{H}(\mathcal{S}_l^{n_l})$.

Remark 9.2. Next we relate results on deformation of the boundaries of expansions with isomorphism of expansions. We highlight this relationship in the following result.

Theorem 9.3. *Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$ and suppose $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$ is a deformation of $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]$. If $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_r}(\mathcal{S}_r)]$ is also a deformation of $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]$, then*

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_r}(\mathcal{S}_r)]\simeq\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$$

Theorem 9.4. *Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$ and suppose $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$ is a deformation of $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]$. If $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_r}(\mathcal{S}_r)]$ is a deformation of $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$, then $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_r}(\mathcal{S}_r)]$ is a deformation of $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]$*

Proof. Let $\mathcal{F}=\{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$ and suppose $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$ is a deformation of $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]$. Then by definition 9.1, It follows that there exist some map

$$\pi_1:\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]\longrightarrow\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$$

such that $\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)]>\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$ with $\mathcal{H}(\mathcal{S}_l^{n_l})=\mathcal{H}(\mathcal{S}_h^{n_h})$. Again, since $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_r}(\mathcal{S}_r)]$ is a deformation of $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$, It follows from definition 9.1 that there exist a mapping

$$\pi_2:\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]\longrightarrow\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_r}(\mathcal{S}_r)]$$

such that $\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)] > \#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_r}(\mathcal{S}_r)]$ with $\mathcal{H}(\mathcal{S}_l^{n_l}) = \mathcal{H}(\mathcal{S}_r^{n_r})$. By choosing $\beta = \pi_2\circ\pi_1$, It follows that

$$\beta : \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)] \longrightarrow \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_r}(\mathcal{S}_r)].$$

Since $\mathcal{H}(\mathcal{S}_h^{n_h}) = \mathcal{H}(\mathcal{S}_l^{n_l}) = \mathcal{H}(\mathcal{S}_r^{n_r})$ and

$$\begin{aligned}\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_h}(\mathcal{S}_h)] &> \#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]\\ &> \#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_r}(\mathcal{S}_r)],\end{aligned}$$

the result follows immediately. □

10. **Overlapping and non-overlapping expansions**

In this section we study the concept of overlapping of expansions. To begin with, we launch the folowing terminology:

Definition 10.1. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Then the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)$ are said to be overlapping if

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]\bigcap\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)] \neq \emptyset.$$

We denote this overlapping region by $\mathcal{O}(\mathcal{S}_l^{n_l}, \mathcal{S}_k^{n_k})$. We call

$$\mathcal{D}^l[\mathcal{O}(\mathcal{S}_l^{n_l}, \mathcal{S}_k^{n_k})] = \frac{\#\mathcal{O}(\mathcal{S}_l^{n_l}, \mathcal{S}_k^{n_k})}{\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]}$$

the density of the overlapping region relative to the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)$ and

$$\mathcal{D}^k[\mathcal{O}(\mathcal{S}_l^{n_l}, \mathcal{S}_k^{n_k})] = \frac{\#\mathcal{O}(\mathcal{S}_l^{n_l}, \mathcal{S}_k^{n_k})}{\#\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]}$$

the density of the overlapping region relative to the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)$.

11. **Associate expansions**

In this section we introduce the notion of associate of expansion. We study how this property interacts with the notion of isomorphism and their interplay. We first launch the following language.

Definition 11.1. Let $\mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ be a collection of tuples of $\mathbb{R}[x]$. Let $\mathcal{S}_k, \mathcal{S}_l \in \mathcal{F}$, then we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)$ is an associate of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)$ if for each

$$\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$$

there exist an $\mathcal{S}_1 \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]$ such that $\mathcal{S}_0 = m\mathcal{S}_1$ for some $m \in \mathbb{N}$.

Proposition 11.1. *Let $\mathcal{S}_k, \mathcal{S}_l \in \mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ and suppose $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)] \simeq \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)]$. Suppose $\mathcal{S}_a \neq r\mathcal{S}_b$ ($r \in \mathbb{N}$) for $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)]$. If $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)$ is an associate of $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)$, then*

$$\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_l}(\mathcal{S}_l)] = \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n_k}(\mathcal{S}_k)].$$

Proof. Let $\mathcal{S}_k, \mathcal{S}_l \in \mathcal{F} = \{\mathcal{S}_j\}_{j=1}^{\infty}$ and suppose $\mathcal{S}_a \neq r\mathcal{S}_b$ ($r \in \mathbb{N}$) for $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_k}(\mathcal{S}_k)]$. Suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_k}(\mathcal{S}_k)$ is an associate of $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_l}(\mathcal{S}_l)$, then by definition 11.1, for each $\mathcal{S}_0 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_k}(\mathcal{S}_k)]$, there exist a unique $\mathcal{S}_1 \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_l}(\mathcal{S}_l)]$ such that $\mathcal{S}_0 = m\mathcal{S}_1$ for some $m \in \mathbb{N}$. Since

$$\mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_k}(\mathcal{S}_k)] \simeq \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{n_l}(\mathcal{S}_l)]$$

It follows from definition 8.1

$$\mathcal{H}(\mathcal{S}_l^{n_l}) = \mathcal{H}(\mathcal{S}_k^{n_k}).$$

Then we must have

$$\begin{aligned}\mathcal{H}(\mathcal{S}_l^{n_l}) &= \sum_{\mathcal{S}_a \in \mathcal{B}^{n_l}} ||\mathcal{S}_a|| \\ &= \sum_{\substack{\mathcal{S}_b \in \mathcal{B}^{n_k} \\ \mathcal{S}_a = m_b\mathcal{S}_b}} ||m_b\mathcal{S}_b|| \\ &= \sum_{\substack{\mathcal{S}_b \in \mathcal{B}^{n_k} \\ \mathcal{S}_a = m_b\mathcal{S}_b}} m_b||\mathcal{S}_b|| \\ &= \sum_{\mathcal{S}_b \in \mathcal{B}^{n_l}} ||\mathcal{S}_b||.\end{aligned}$$

It follows that we can take $m_b = 1$, and the result follows immediately. $\square$

12. **Sub-expansions**

In this section we introduce the concept of sub-expansions of an expansion. We launch the following terminology in that respect.

Definition 12.1. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$ be any two expansions with $m < n$, then we say $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a)$ is a sub-expansion of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$, denoted

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$$

if there exist some $j \geq 1$ such that $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{m+j}(\mathcal{S}_b)$. We say the sub-expansion is proper if $m + j = n$. We denote this proper sub-expansion by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$$

Remark 12.2. Next we prove a result that indicates that the regularity condition on an expansion can be localized as well as extended through expansions. We formalize this statement in the following result.

Proposition 12.1. *Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$, a proper sub-expansion. Then $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_a)$ is regular if and only if $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^n(\mathcal{S}_b)$ is regular.*

Proof. Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_a) < (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_b)$ and suppose $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_a)$ is regular. Then it follows that $\mathcal{H}(\mathcal{S}_a^m) > \mathcal{H}(\mathcal{S}_a^{m+1})$ for some $1 \leq m \leq deg(\mathcal{S}_a) - 2$. By definition 12.1, it follows that there exist some $j \geq 1$ such that we can write $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_a) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m+j}(\mathcal{S}_b)$. Since the expansion is proper, it follows that $m + j = n$ and we have

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_a) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_b).$$

It follows that $\mathcal{H}(\mathcal{S}_a^m) = \mathcal{H}(\mathcal{S}_b^n)$. Since

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m+1}(\mathcal{S}_a) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{n+1}(\mathcal{S}_b)$$

the regularity of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^n(\mathcal{S}_b)$ also follows. The converse on the other hand follows a similar approach. □

13. **Distribution of the boundary points of expansion**

In this section we study the distribution of the boundary points of any phase of expansion. We first introduce the notion of an integration of polynomials along the boundaries of various phases of expansion, which we then use as our main tool. We launch the following definition in that direction:

Definition 13.1. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n, then we call the tuple

$$\begin{aligned}\mathcal{S}_f &:= (a_n x^n, a_{n-1}x^{n-1}, \ldots, a_1 x, a_0)\\ &= (g_1(x), g_2(x), \ldots, g_n(x))\end{aligned}$$

the tuple representation of f. By the integral of $f(x)$ along the boundary of the m th phase expansion of $\mathcal{S}_f$, we mean the formal integral defined by

$$\int\limits_{\substack{\mathcal{B}^m\\ m<n}} f(t)dt := \sum_{i=1}^{\#\mathcal{B}^m-1} \sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{B}^m\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}} \overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)} \cdot \overrightarrow{O\mathcal{S}_e},$$

where

$$\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f) = \left(\int\limits_{a_1}^{b_1} g_1(x)dx, \int\limits_{a_2}^{b_2} g_2(x)dx, \ldots, \int\limits_{a_n}^{b_n} g_n(x)dx\right),$$

where $\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}$ and $\overrightarrow{O\mathcal{S}_e}$ are the position vectors of $\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)$ and $\mathcal{S}_e$ respectively with $\mathcal{S}_i = (a_1, a_2, \ldots, a_n)$ and $\mathcal{S}_{i+1} = (b_1, b_2, \ldots, b_n)$.

Remark 13.2. It is in practice very difficult to ascertain the local distribution of the boundary points of an expansion. However, we can now show that if we shrink the space bounded by the boundary of an expansion, then the boundary points must be closely packed together in some sense. We use the integral proposed in definition 13.1 as a black box.

Theorem 13.3. *Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ be a polynomial of degree n. Then*

$$\bigg|\bigg|\int\limits_{\substack{\mathcal{B}^m\\ m<n}}f(t)dt\bigg|\bigg|>\epsilon$$

for some $\epsilon>0$ if and only if $||\mathcal{S}_i-\mathcal{S}_{i+1}||>\delta$ for $\delta:=\delta(n)>0$ for some

$$\mathcal{S}_i\in\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)]$$

with $1\leq i\leq \#\mathcal{B}^m-1$ and $||\mathcal{S}_i-\mathcal{S}_{i+1}||<||\mathcal{S}_i-\mathcal{S}_j||$ for $i+1\neq j$.

Proof. Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\in\mathbb{R}[x]$ be a polynomial of degree n and suppose

$$\bigg|\bigg|\int\limits_{\substack{\mathcal{B}^m\\ m<n}}f(t)dt\bigg|\bigg|>\epsilon$$

for some $\epsilon>0$. By a repeated application of the triangle inequality, we find that

$$\begin{aligned}\bigg|\bigg|\int\limits_{\substack{\mathcal{B}^m\\ m<n}}f(t)dt\bigg|\bigg| &\leq \sum_{i=1}^{\#\mathcal{B}^m-1}\sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{B}^m\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}}||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}||||\overrightarrow{O\mathcal{S}_e}||\\ &=\sqrt{n}\sum_{i=1}^{\#\mathcal{B}^m-1}\sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{B}^m\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}}||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}||\\ &\leq(\#\mathcal{B}^m-1)\sqrt{n}\max\left\{||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}||\right\}_{\substack{i=1\\ ||\mathcal{S}_i||<||\mathcal{S}_{i+1}||}}^{\#(\mathcal{B}^m-1)}.\end{aligned}$$

Since the inequality

$$\begin{aligned}||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}|| &=\sqrt{|\int\limits_{a_1}^{b_1}g_1dx|^2+\cdots+|\int\limits_{a_n}^{b_n}g_ndx|^2}\\ &\leq M\sqrt{|a_1-b_1|^2+\cdots+|a_n-b_n|^2}\end{aligned}$$

is valid, it follows that there exist some $\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)]$ with $||\mathcal{S}_i-\mathcal{S}_{i+1}||<||\mathcal{S}_i-\mathcal{S}_j||$ for all $i+1\neq j$. It follows that for some closest pair of boundary point, the inequality

$$\frac{\epsilon}{(\#\mathcal{B}^m-1)M\sqrt{n}}<\sqrt{|a_1-b_1|^2+\cdots+|a_n-b_n|^2}$$

is valid, and thus it must be that $||\mathcal{S}_i-\mathcal{S}_{i+1}||>\delta$ by choosing $\delta=\frac{\epsilon}{(\#\mathcal{B}^m-1)M\sqrt{n}}$. Conversely, suppose there exist some closest boundary points $\mathcal{S}_i,\mathcal{S}_{i+1}\in\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)]$ such that

$$||\mathcal{S}_i-\mathcal{S}_{i+1}||>\delta$$

for some $\delta := \delta(n) > 0$. Then it follows that $\sqrt{|a_1 - b_1|^2 + \cdots + |a_n - b_n|^2} > \delta$. By choosing $R = \min\{|g_i(x)| : x \in [a_i, b_i]\}_{i=1}^{n}$, we find that

$$\begin{aligned}||\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}|| &= \sqrt{|\int\limits_{a_1}^{b_1} g_1 dx|^2 + \cdots + |\int\limits_{a_n}^{b_n} g_n dx|^2} \\ &\geq R\sqrt{|a_1 - b_1|^2 + \cdots + |a_n - b_n|^2} \\ &= \delta R.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{i=1}^{\#\mathcal{B}^m - 1} \sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1} \in \mathcal{B}^m \\ ||\mathcal{S}_i|| < ||\mathcal{S}_{i+1}||}} \overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)} \cdot \overrightarrow{O\mathcal{S}_e} &> \sum_{i=1}^{\#\mathcal{B}^m - 1} \sum_{\substack{\mathcal{S}_i,\mathcal{S}_{i+1} \in \mathcal{B}^m \\ ||\mathcal{S}_i|| < ||\mathcal{S}_{i+1}||}} \delta R ||\overrightarrow{O\mathcal{S}_e}|| \cos\alpha \\ &= \delta(\#\mathcal{B}^m - 1)R\sqrt{n}\cos\alpha\end{aligned}$$

where α is the angle between the vectors $\overrightarrow{O\Delta_{\mathcal{S}_i,\mathcal{S}_{i+1}}(\mathcal{S}_f)}$ and $\overrightarrow{O\mathcal{S}_e}$. It follows that

$$\left|\left| \int\limits_{\substack{\mathcal{B}^m \\ m<n}} f(t)dt \right|\right| > \delta(\#\mathcal{B}^m - 1)R\sqrt{n}|\cos\alpha|.$$

The result follows by taking

$$\delta := \frac{\epsilon}{(\#\mathcal{B}^m - 1)R\sqrt{n}|\cos\alpha|}.$$

□

It is somewhat clear Theorem 13.3 partially solves Conjecture 4. Indeed the space bounded by boundaries increases with expansions. Thus Theorem 13.3 in the affirmative tells us that we can use the area as a yardstick to determine the distribution of the points of any given phase of expansion. We leverage this new tool to study the mass of the corresponding phases of expansions in the following sequel. The result below is a consequence of Theorem 13.3.

Corollary 4. Let $f(x) \in \mathbb{R}[x]$ be a polynomial of degree n. If

$$\left|\left| \int\limits_{\substack{\mathcal{B}^m \\ m<n}} f(t)dt \right|\right| < 1$$

with $||\mathcal{S}_i|| < 1$ for some $\mathcal{S}_i \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)]$, then $\mathcal{H}(\mathcal{S}_f^m) < \epsilon$ for some $\epsilon := \epsilon(n) > 0$.

Proof. Let $f(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial of degree n. Suppose

$$\left|\left| \int\limits_{\substack{\mathcal{B}^m \\ m<n}} f(t)dt \right|\right| < 1$$

then it follows by applying Theorem 13.3

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| < 1$$

for all $i \leq 1 \leq \#\mathcal{B}^m - 1$. Since $||\mathcal{S}_i|| < 1$ for some $i \leq 1 \leq \#\mathcal{B}^m$, it follows that $||\mathcal{S}_j|| < 1$ for all $1 \leq j \leq \#\mathcal{B}^m$ and the result follows immediately. □

Remark 13.4. With this new tool available, we can now establish a uniform version of the diminishing state of the mass of phases of an expansion for certain types of expansions whose phase boundaries are produced from the expansion of some part of the boundary. We state this result in a more formal manner but at the compromise of taking sufficiently small boundaries.

Theorem 13.5. *Let $f(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial of degree n. Suppose*

$$\sum_{m=1}^{n-1} \left|\left| \int\limits_{\substack{\mathcal{B}^m \\ m<n}} f(t)dt \right|\right| < 1.$$

If

$$\bigcap_{m=1}^{n-1} \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)] \neq \emptyset$$

then $\mathcal{H}(\mathcal{S}_f^m) > \mathcal{H}(\mathcal{S}_f^{m+1})$ for all $1 \leq m \leq n-1 = deg(\mathcal{S}_f) - 1$.

Proof. Let $f(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial of degree n and suppose

$$\sum_{m=1}^{n-1} \left|\left| \int\limits_{\substack{\mathcal{B}^m \\ m<n}} f(t)dt \right|\right| < 1.$$

Then it follows from Theorem 13.3

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| < 1$$

for $\mathcal{S}_i \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)]$ for all $1 \leq i \leq \#\mathcal{B}^m - 1$ for each $1 \leq m \leq n-1 = deg(\mathcal{S}_f) - 1$. It follows that $||\mathcal{S}_i|| \approx ||\mathcal{S}_{i+1}||$. Since

$$\bigcap_{m=1}^{n-1} \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)] \neq \emptyset$$

It follows that the boundary points of each phase of expansions are of size comparable to the size of the boundary points of other phases of expansions. This very fact completes the proof, since the boundary points decreases with successive phases of expansions. □

Remark 13.6. Next we demonstrate in the upcoming result that this special integral can also be used as criterion for determining the sub-expansions of an expansion, provided it is small enough. We state the result in a formal manner as follows.

Theorem 13.7. *Let $f(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial of degree n. If*

$$\left|\left|\int\limits_{\substack{\mathcal{B}^{m_1}\\ m_1<n}} f(t)dt\right|\right| < \left|\left|\int\limits_{\substack{\mathcal{B}^{m_2}\\ m_2<n}} f(t)dt\right|\right| < 1,$$

then $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m_1}(\mathcal{S}_f) < (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m_2}(\mathcal{S}_f)$.

Proof. Let $f(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial of degree n and let

$$\left|\left|\int\limits_{\substack{\mathcal{B}^{m_1}\\ m_1<n}} f(t)dt\right|\right| < \left|\left|\int\limits_{\substack{\mathcal{B}^{m_2}\\ m_2<n}} f(t)dt\right|\right| < 1,$$

and suppose on the contrary

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m_2}(\mathcal{S}_f) < (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m_1}(\mathcal{S}_f).$$

Then by appealing to Theorem 13.3, it must certainly be that $||\mathcal{S}_i - \mathcal{S}_{i+1}|| < ||\mathcal{S}_j - \mathcal{S}_{j+1}|| < 1$ with $\mathcal{S}_i, \mathcal{S}_{i+1} \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m_2}(\mathcal{S}_f)]$ and $\mathcal{S}_j, \mathcal{S}_{j+1} \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m_1}(\mathcal{S}_f)]$ such that

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| = \mathrm{Inf}\left\{||\mathcal{S}_i - \mathcal{S}_k|| : \mathcal{S}_k \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m_2}(\mathcal{S}_f)]\right\}$$

and

$$||\mathcal{S}_j - \mathcal{S}_{j+1}|| = \mathrm{Inf}\left\{||\mathcal{S}_j - \mathcal{S}_l|| : \mathcal{S}_l \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m_1}(\mathcal{S}_f)]\right\}.$$

It follows that the boundary points of the two distinct boundaries of expansions are of size comparable to each other, upto a very small error. Since points on the boundary becomes sparce for higher phase of expansions, It follows that

$$\left|\left|\int\limits_{\substack{\mathcal{B}^{m_2}\\ m_2<n}} f(t)dt\right|\right| \leq \left|\left|\int\limits_{\substack{\mathcal{B}^{m_1}\\ m_1<n}} f(t)dt\right|\right|$$

which contradicts the hypothesis, thereby ending the proof. $\square$

Remark 13.8. It is important to point out that this pass is somewhat easy; the pass from the area bounded by the boundary of expansion to information about the sub-expansions of an expansion, If we allow for only sufficiently small areas. The converse on the other hand may not neccessarily be true. Unfortunately we cannot affirmatively opine on that particular behaviour, but we do have a strong belief that can be done if we impose some extra conditions.

14. **Interior and exterior points of expansion**

We devote this section to study the interior and the exterior points of the boundary of expansions. We also digress into the concept of the neighbourhood of the boundary of an expansion and their interplay with some concept in Topology. We begin by launching the following terminology to ease our work.

Definition 14.1. Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})$ be an expansion for $1\leq m\leq deg(\mathcal{S})-1$ with boundary $\mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})]$, then by the interior of the expansion, denoted $\mathrm{Int}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})]$, we mean the set

$$\mathrm{Int}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})]:=\left\{\mathcal{S}_a\in\mathbb{R}^n:||\mathcal{S}_a||<||\mathcal{S}_j||\text{ or }||\mathcal{S}_a||>||\mathcal{S}_j||,\text{ for most }\mathcal{S}_j\in\mathcal{B}^m\right\}.$$

Points in the interior of expansion are called the interior points of expansion. The interior is said to be an upper interior, denoted by $\mathrm{Int}_u[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})]$, if each interior point belongs to the set

$$\mathrm{Int}_u[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})]:=\left\{\mathcal{S}_a\in\mathbb{R}^n:||\mathcal{S}_a||>||\mathcal{S}_j||,\text{ for most }\mathcal{S}_j\in\mathcal{B}^m\right\}.$$

Otherwise it is a lower interior, denoted by

$$\mathrm{Int}_l[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})]:=\left\{\mathcal{S}_a\in\mathbb{R}^n:||\mathcal{S}_a||<||\mathcal{S}_j||,\text{ for most }\mathcal{S}_j\in\mathcal{B}^m\right\}.$$

Similarly the exterior of an expansion, denoted $\mathrm{Ext}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})]$, is given by the set

$$\mathrm{Ext}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})]=\left\{\mathcal{S}_a\in\mathbb{R}^n:||\mathcal{S}_a||>||\mathcal{S}_j||\text{ or }||\mathcal{S}_a||<||\mathcal{S}_j||,\text{ for all }\mathcal{S}_j\in\mathcal{B}^m\right\}.$$

A similar characterization also holds for exterior of an expansion as does the interior of an expansion.

Remark 14.2. Next we show that we can actually use the interior of an expansion to determine the mass of an expansion. We use the integral proposed as our main tool. We state the result as follows:

Proposition 14.1. *Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ be a polynomial of degree n, and let $\mathcal{S}_f$ be the tuple representation of f. Suppose*

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m\\ m<n}}f(t)dt\right|\right|<1,$$

and

$$\begin{aligned}\mathrm{Int}_l[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)]&:=\left\{\mathcal{S}_a\in\mathbb{R}^n:||\mathcal{S}_a||<||\mathcal{S}_j||,\text{ for most }\mathcal{S}_j\in\mathcal{B}^m\right\}\\&=\mathrm{Int}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)].\end{aligned}$$

If

$$\sum_{\substack{\mathcal{S}_a\in\mathcal{R}\\ \mathcal{R}\subset\mathrm{Int}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)]\\ \#\mathcal{R}=\#\mathcal{B}^m}}||\mathcal{S}_a||>\epsilon$$

for some $\epsilon>0$ and $||\mathcal{S}_a-\mathcal{S}_j||\geq 1$, then $\mathcal{H}(\mathcal{S}_f^m)>\epsilon$.

Proof. Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ be a polynomial of degree n, and let $\mathcal{S}_f$ be the tuple representation of f and suppose

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m\\ m<n}}f(t)dt\right|\right|<1,$$

then it follows from Theorem 13.3 that $||\mathcal{S}_i|| \approx ||\mathcal{S}_j||$ for any pair of points $\mathcal{S}_i, \mathcal{S}_j \in \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)]$. Since

$$\begin{aligned}\mathrm{Int}_l[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)] &:= \{\mathcal{S}_a \in \mathbb{R}^n : ||\mathcal{S}_a|| < ||\mathcal{S}_j||, \text{ for most } \mathcal{S}_j \in \mathcal{B}^m\}\\ &= \mathrm{Int}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)].\end{aligned}$$

and the exceptional set of the interior is negligible, it follows that

$$\begin{aligned}\sum_{\substack{\mathcal{S}_a \in \mathcal{R}\\ \mathcal{R} \subset \mathrm{Int}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)]\\ \#\mathcal{R} = \#\mathcal{B}^m}} ||\mathcal{S}_a|| &< \sum_{\mathcal{S}_b \in \mathcal{Z}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S}_f)]} ||\mathcal{S}_b||\\ &= \mathcal{H}(\mathcal{S}_f^m)\end{aligned}$$

thereby ending the proof. ☐

Remark 14.3. Next we show that all points not on the boundary of an expansion occupying a small enough space must neccessarily be exterior points. We give a formall statement in the following proposition.

Proposition 14.2. *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n, and let $\mathcal{S}_f$ be the tuple representation of f. Suppose*

$$\left|\left| \int\limits_{\substack{\mathcal{B}^m\\ m<n}} f(t)dt \right|\right| < \delta,$$

where $0 < \delta < 1$, then $\mathrm{Ext}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)] \neq \emptyset$.

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n, and let $\mathcal{S}_f$ be the tuple representation of f. Suppose

$$\left|\left| \int\limits_{\substack{\mathcal{B}^m\\ m<n}} f(t)dt \right|\right| < \delta,$$

where $0 < \delta < 1$. Then it follows from Theorem 13.3

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| < \epsilon$$

for $\epsilon > 0$ sufficiently small for all $1 \leq i \leq \#\mathcal{B}^m - 1$. It follows that $||\mathcal{S}_k|| \approx ||\mathcal{S}_l||$ for all $\mathcal{S}_k, \mathcal{S}_l \in \mathcal{B}^m$. Now choose $\mathcal{S}_a$ such that $||\mathcal{S}_a - \mathcal{S}_i|| \geq 1$ for all $1 \leq i \leq \#\mathcal{B}^m$. Then it follows that $\mathcal{S}_a \notin \mathcal{B}^m$. Without loss of generality, let us assume that $||\mathcal{S}_a|| < ||\mathcal{S}_k||$, then it follows that $||\mathcal{S}_a|| < ||\mathcal{S}_l||$. The result follows by inducting this argument on other boundary points. ☐

Theorem 14.4. *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n, and let $\mathcal{S}_f$ be the tuple representation of f. Suppose*

$$\left|\left| \int\limits_{\substack{\mathcal{B}^m\\ m<n}} f(t)dt \right|\right| > \delta,$$

for $\delta > 0$, then $\mathrm{Int}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S})] \neq \emptyset$.

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n, and let $\mathcal{S}_f$ be the tuple representation of f and suppose

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m\\ m<n}} f(t)dt\right|\right| > \delta,$$

for $\delta > 0$ and suppose on the contrary that, $\mathrm{Int}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)] = \emptyset$. By appealing to Theorem 13.3, it follows that

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \epsilon$$

for $\epsilon > 0$ and $1 \leq i \leq \#\mathcal{B}^m - 1$, with

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| = \mathrm{Inf}\left\{||\mathcal{S}_i - \mathcal{S}_j|| : \mathcal{S}_j \in \mathcal{B}^m\right\}.$$

That is to say, points on the boundary of expansion are mostly spaced out. Under the assumption that $\mathrm{Int}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)] = \emptyset$, it follows that for any $\mathcal{S}_l \notin \mathcal{B}^m$, then it must be that

$$\mathcal{S}_l \in \mathrm{Ext}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)].$$

Now let us choose $\mathcal{S}_l$ such that

$$||\mathcal{S}_l|| = \frac{1}{\#\mathcal{B}^m} \sum_{\mathcal{S}_i \in \mathcal{B}^m} ||\mathcal{S}_i||,$$

then it follows that $\mathcal{S}_l \notin \mathcal{B}^m$. For suppose $\mathcal{S}_l \in \mathcal{B}^m$, then it follows that

$$\frac{1}{\#\mathcal{B}^m} \sum_{\mathcal{S}_i \in \mathcal{B}^m} ||\mathcal{S}_i|| = ||\mathcal{S}_j||,$$

for some $\mathcal{S}_j \in \mathcal{B}^m$. It follows that

$$\sum_{\mathcal{S}_i \in \mathcal{B}^m} ||\mathcal{S}_i|| = \#\mathcal{B}^m ||\mathcal{S}_j||.$$

This contradicts the assumption that $||\mathcal{S}_i - \mathcal{S}_{i+1}|| > \epsilon$ for $\epsilon > 0$. Since $\mathcal{S}_l \in \mathrm{Ext}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)]$, then without loss of generality we can assume that

$$\frac{1}{\#\mathcal{B}^m} \sum_{\mathcal{S}_i \in \mathcal{B}^m} ||\mathcal{S}_i|| < ||\mathcal{S}_k||$$

for all $\mathcal{S}_k \in \mathcal{B}^m$. By choosing $||\mathcal{S}_k|| = \min\left\{||\mathcal{S}_j|| : \mathcal{S}_j \in \mathcal{B}^m\right\}$, then it follows that

$$\sum_{\mathcal{S}_i \in \mathcal{B}^m} ||\mathcal{S}_i|| < \#\mathcal{B}^m ||\mathcal{S}_k||$$

which is absurd, thereby ending the proof. □

15. **The neighbourhood of expansion**

In this section we introduce the concept of the neighbourhood of an expansion. We use this as a carviat for the study in the following sequel. We launch more officially the following language.

Definition 15.1. Let $\mathcal{B}^m = \mathcal{Z}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)]$ be the boundary of an expansion. Then by the neighbourhood of $\mathcal{S}_j \in \mathcal{B}^m$ with radius ϵ, denoted $\mathcal{E}_\epsilon(\mathcal{S}_j)$, we mean the set

$$\mathcal{E}_\epsilon(\mathcal{S}_j) := \{\mathcal{S}_a : ||\mathcal{S}_a - \mathcal{S}_j|| < \epsilon \text{ for } \mathcal{S}_j \in \mathcal{B}^m\}.$$

Remark 15.2. Next we prove a result that relates the region bounded by the boundary of an expansion to the distribution of points in the points near the boundary.

Proposition 15.1. *Let* $f(x) := a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{R}[x]$ *be a polynomial of degree* n *and suppose*

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m \\ m<n}} f(t)dt\right|\right| < \delta$$

for $\delta > 0$ *sufficiently small, then*

$$\mathcal{E}_1(\mathcal{S}_j)\bigcap \mathcal{E}_{\frac{1}{2}}(\mathcal{S}_{j+1}) \neq \emptyset$$

for $\mathcal{S}_j, \mathcal{S}_{j+1} \in \mathcal{B}^m$.

Proof. Let $f(x) := a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{R}[x]$ be a polynomial of degree n and suppose

$$\left|\left|\int\limits_{\substack{\mathcal{B}^m \\ m<n}} f(t)dt\right|\right| < \delta$$

for $\delta > 0$ sufficiently small, then it follows from Theorem 13.3

$$||\mathcal{S}_i - \mathcal{S}_{i+1}|| < \epsilon$$

for $\epsilon > 0$ sufficiently small for $\mathcal{S}_i, \mathcal{S}_{i+1} \in \mathcal{B}^m$ with $1 \leq i \leq \#\mathcal{B}^m - 1$. It follows that any two boundary points are sufficiently close to each other. The result follows from this fact. For suppose $\mathcal{E}_1(\mathcal{S}_j)\bigcap \mathcal{E}_{\frac{1}{2}}(\mathcal{S}_{j+1}) = \emptyset$, then it follows that for all $\mathcal{S}_a \in \mathcal{E}_{\frac{1}{2}}(\mathcal{S}_{j+1})$, then $\mathcal{S}_a \notin \mathcal{E}_1(\mathcal{S}_j)$. It must be that $||\mathcal{S}_j - \mathcal{S}_a|| \geq 1$. Thus it follows that

$$\epsilon + ||\mathcal{S}_{j+1} - \mathcal{S}_a|| \geq 1$$

and it follows that $\frac{1}{2} > ||\mathcal{S}_{j+1} - \mathcal{S}_a|| \geq 1 - \epsilon$. This is absurd, since $\epsilon > 0$ is sufficiently small, thereby ending the proof. □

16. **Rotation of the boundary of expansion**

In this section we introduce the concept of rotation of the boundary of an expansion. This concept will form the classification scheme for various types of an expansion. We launch the following language in that regard.

Definition 16.1. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S})$ be an expansion with corresponding boundary $\mathcal{B}^m$. Then we say the map Λ is a rotation of the boundary $\mathcal{B}^m$ if

$$\Lambda : \mathcal{B}^m \longrightarrow \mathcal{B}^m.$$

We say an expansion admits a rotation if there exist such a map. In other words, we say the map Λ induces a rotation on the expansion. We say the boundary is stable under the rotation if $||\Lambda(\mathcal{S}_a)|| \approx ||\mathcal{S}_a||$ for all $\mathcal{S}_a \in \mathcal{B}^m$. Otherwise we say it is unstable.

Proposition 16.1. *Let $f(x) := a_nx^n + \cdots + a_1x + a_0 \in \mathbb{R}[x]$, a polynomial of degree $n \geq 3$. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)$ be an expansion with corresponding boundary $\mathcal{B}^m$ admits the rotation Λ. If*

$$\left|\left| \int\limits_{\substack{\mathcal{B}^m \\ m<n}} f(t)dt \right|\right| < 1$$

then the boundary $\mathcal{B}^m$ is stable.

Proof. Let $f(x) := a_nx^n + \cdots + a_1x + a_0 \in \mathbb{R}[x]$, a polynomial of degree $n \geq 3$ and suppose $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S}_f)$ is an expansion with corresponding boundary $\mathcal{B}^m$ admitting the rotation Λ. Suppose also that

$$\left|\left| \int\limits_{\substack{\mathcal{B}^m \\ m<n}} f(t)dt \right|\right| < 1$$

then it follows from Theorem 13.3 that $||\mathcal{S}_j|| \approx ||\mathcal{S}_{j+1}||$ for $1 \leq j \leq \#\mathcal{B}^m - 1$ with $\mathcal{S}_j, \mathcal{S}_{j+1} \in \mathcal{B}^m$. It follows that for the rotation $\Lambda : \mathcal{B}^m \longrightarrow \mathcal{B}^m$, we have that for any $\mathcal{S}_j \in \mathcal{B}^m$, then

$$\Lambda(\mathcal{S}_j) = \mathcal{S}_k$$

for some $\mathcal{S}_k \in \mathcal{B}^m$. It follows that $||\Lambda(\mathcal{S}_j)|| = ||\mathcal{S}_k|| \approx ||\mathcal{S}_j||$, thereby ending the proof. $\square$

17. **Simple expansions**

In this section we study a particular type of expansion. The main tool in the classification of these types of expansion is the concept of rotation of the boundary of an expansion. We launch more formally the following language:

Definition 17.1. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{m+1}(\mathcal{S})$ be an expansion with corresponding boundary $\mathcal{B}^{m+1}$. We say the expansion is simple if any rotation of $\mathcal{B}^{m+1}$ given by $\Lambda : \mathcal{B}^{m+1} \longrightarrow \mathcal{B}^{m+1}$ is not a rotation of $\mathcal{B}^m$.

18. **Compact expansions**

In this section we introduce the notion of compactness of an expansion. We launch the following language in that regard.

Definition 18.1. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^m(\mathcal{S})$ be an expansion. Let $\epsilon > 0$ be small, then we say the expansion is compact there exist some $\mathcal{S}_l \in \mathcal{B}^{m+1}$ such that $\mathcal{S}_l \in \mathcal{E}_\epsilon(\mathcal{S}_j)$ for each $\mathcal{S}_j \in \mathcal{B}^m$ for all $1 \leq m \leq deg(\mathcal{S}_j) - 1$.

Remark 18.2. Next we prove that the mass of an expansion diminishes uniformly for these types of expansions, thereby satisfying the sendov conjecture.

Theorem 18.3. *Let* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})$ *be a compact expansion, then* $\mathcal{H}(\mathcal{S}^{m+1}) < \mathcal{H}(\mathcal{S}^m)$ *for all* $1 \leq m \leq deg(\mathcal{S}) - 1$.

Proof. Suppose $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m(\mathcal{S})$ is a compact expansion, then it follows from definition 18.1 that for some small $\epsilon > 0$, there exist some $\mathcal{S}_l \in \mathcal{B}^{m+1}$ such that $\mathcal{S}_l \in \mathcal{E}_\epsilon(\mathcal{S}_j)$. Then it follows that

$$||\mathcal{S}_l - \mathcal{S}_j|| < \epsilon$$

for some small $\epsilon > 0$ for all $\mathcal{S}_j \in \mathcal{B}^m$. Since boundary points decrease with expansions, the result follows immediately. □

19. End remarks and future works

In this paper we put a premium on inverse problems; in particular, inverse problems for higher - extremely higher - phase expansions, eventhough understanding higher phase inverse problems requires understanding the higher phase expansions. Simply put we would desire some very nice formula that represents the n copies of the recovery map $\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma$, that is, can we write

$$(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma)^n = F\circ(\Delta\circ\gamma^{-1}\circ\beta^{-1}\circ\gamma),$$

where F is some smooth map depending on n? The theory as developed is still opened to further development. One area that may seem fertile is to study this theory in the case our polynomial has not just one but several indeterminates, which one may consider as several variables **expansivity theory**. This inevitably comes with as many applications and connections with other areas of mathematics.

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