

Pappus chain and division by zero

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Abstract. We consider a Pappus chain using division by zero.

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1. INTRODUCTION

Let O be a point on the segment AB such that $|AO| = 2a$ and $|BO| = 2b$ (see Figure 1). For an arbelos configuration formed by three circles α , β and γ with diameters AO , BO and AB , respectively, we consider circles touching two of the three circles by division by zero [2]:

$$(1) \quad \frac{z}{0} = 0 \text{ for any real number } z.$$

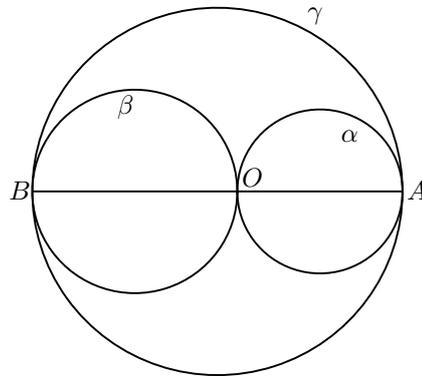


Figure 1.

2. CIRCLES TOUCHING TWO OF α , β AND γ

If a circle touches one of given two circles internally and the other externally, we say that the circle touches the two circles in the opposite sense, otherwise in the same sense. Let $c = a + b$.

Theorem 1. *The following statements hold.*

(i) *A circle touches the circles β and γ in the opposite sense if and only if its has radius and center of coordinates*

$$r_{\alpha z} = \frac{abc}{a^2z^2 + bc} \text{ and } (x_{\alpha z}, y_{\alpha z}) = \left(-2b + \frac{bc(b+c)}{a^2z^2 + bc}, 2zr_{\alpha z} \right)$$

of coordinates $(x_{\gamma_z}, y_{\gamma_z})$ by γ_z . Notice that $\alpha_1 = \beta_1 = \gamma_1$ is the incircle of the arbelos in the region $y \geq 0$. We have $\gamma_0 = \gamma$, and denote the point O and the line AB by γ_∞ and γ_∞ , respectively. Let $d = \sqrt{ab}/c$. The external common tangent of α and β has an equation [16, 17]

$$(2) \quad (a-b)x \mp 2\sqrt{aby} + 2ab = 0,$$

which is denoted by $\gamma_{\pm d}$.

3. PAPPUS CHAIN

Let $r_A = ab/(a+b)$.

Theorem 2. *Let w and z be real numbers. Then each of the two circles of the three pairs $\alpha_z, \alpha_w; \beta_z, \beta_w; \gamma_z, \gamma_w$ touch if and only if $|w - z| = 1$.*

Proof. If $|w| \neq d$ and $|z| \neq d$, we get

$$(x_{\gamma_w} - x_{\gamma_z})^2 + (y_{\gamma_w} - y_{\gamma_z})^2 - (r_{\gamma_w} + r_{\gamma_z})^2 = \frac{4a^2b^2c^2((w-z)^2 - 1)}{(c^2w^2 - ab)(c^2z^2 - ab)}.$$

Hence γ_w and γ_z touch if and only if $|w - z| = 1$. While γ_d and $\gamma_{d\pm 1}$ have only one point in common, whose coordinates equal

$$(3) \quad \left(2r_A \frac{(\sqrt{a} \mp \sqrt{b})^2}{-(a-b)}, \pm 2r_A \right).$$

Therefore they touch. Since the figure is symmetric in AB , γ_{-d} and $\gamma_{-d\pm 1}$ also touch. The rest of the theorem is proved in a similar way. \square

Circles of radius r_A are said to be Archimedean. The next corollary is given by (3).

Corollary 1. *If $z = d$ or $z = -d$, then the smallest circle passing through the point of tangency of γ_z and $\gamma_{z\pm 1}$ and touching AB is Archimedean.*

4. DIVISION BY ZERO CALCULUS

For the Laurent expansion of a function $f(z)$ around $z = a$:

$$f(z) = \sum_{n=-1}^{-\infty} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n,$$

the definition $f(a) = C_0$ is called the division by zero calculus [13], [18].

Let

$$\gamma(z) = (x - x_{\gamma_z})^2 + (y - y_{\gamma_z})^2 - r_{\gamma_z}^2.$$

If

$$\gamma(z) = \cdots + C_{-2}(z-d)^{-2} + C_{-1}(z-d)^{-1} + C_0 + \cdots$$

is the Laurent expansion of $\gamma(z)$ around $z = d$, then

$$C_{-1} = \frac{\sqrt{ab}}{a+b}((a-b)x - 2\sqrt{aby} + 2ab).$$

Therefore we get half part of (2). Also from the Laurent expansion of $\gamma(z)$ around $z = -d$:

$$\gamma(z) = \cdots + C_{-2}(z+d)^{-2} + C_{-1}(z+d)^{-1} + C_0 + \cdots,$$

we get

$$C_{-1} = -\frac{\sqrt{ab}}{a+b}((a-b)x + 2\sqrt{aby} + 2ab).$$

Therefore we get the rest part of (2).

For more applications of division by zero to circle geometry, see [1], [3], [4, 5, 6, 7, 8, 9, 10, 11, 12] [13, 14, 15, 16], and an extensive referece see [18].

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