Letter N°4 : On Pi

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ABSTRACT. We give a formula involving Pi

Keywords: number Pi, radicals .

I. Introduction .

Recall that:

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.14159265 \ldots \tag{1}$$

Notation: Define

$$s_k = \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{k-radicals} \quad , \quad k = 1, 2, 3, \ldots \tag{2}$$

examples

$$s_1 = \sqrt{2} \quad , \quad s_2 = \sqrt{2 - \sqrt{2}} \quad , \quad s_3 = \sqrt{2 - \sqrt{2 + \sqrt{2}}} \quad , \quad \ldots \tag{3}$$

Remark:

$$\pi = \lim_{n \to \infty} 2^{n+1} \sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n}} \tag{4}$$

II. Formula for Pi

Entry 1. If $k = 1, 2, 3, \ldots$, then

$$\frac{\pi}{2^k s_k} = 1 + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1} \sum_{m=0}^{n} \binom{n}{m} (-1)^m \left(\frac{1}{2}\right)^{n-m} F(k, m) + \sum_{n=0}^{\infty} (-1)^n e^{-2^{k+1}(n+1)} \left(\frac{e}{2^{k+1}(n+1) - 1} - \frac{e^{-1}}{2^{k+1}(n+1) + 1} \right) \tag{5}$$

where

$$F(k,\,m) = \frac{1 - e^{-\left(2^{k+1}\,(m+1)-1\right)}}{2^{k+1}\,(m+1) - 1} - \frac{1 - e^{-\left(2^{k+1}\,(m+1)+1\right)}}{2^{k+1}\,(m+1) + 1} \tag{6}$$

Entry 2. example $k = 3$

$$\frac{\pi}{8\sqrt{2 - \sqrt{2 + \sqrt{2}}}} = 1 + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1} \sum_{m=0}^{n} \binom{n}{m} (-1)^m \left(\frac{1}{2}\right)^{n-m} F(3,\,m) + \sum_{n=0}^{\infty} (-1)^n\, e^{-16\,(n+1)} \left(\frac{e}{16\,n+15} - \frac{e^{-1}}{16\,n+17}\right) \tag{7}$$

where

$$F(3,\,m) = \frac{1 - e^{-16\,m-15}}{16\,m+15} - \frac{1 - e^{-16\,m-17}}{16\,m+17} \tag{8}$$

Entry 3.

$$\frac{\pi}{2} = 1 + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1} \sum_{m=0}^{n} \binom{n}{m} (-1)^m \left(\frac{1}{2}\right)^{n-m} F(0,\,m) + \sum_{n=0}^{\infty} (-1)^n\, e^{-2\,(n+1)} \left(\frac{e}{2\,n+1} - \frac{e^{-1}}{2\,n+3}\right) \tag{9}$$

where

$$F(0,\,m) = \frac{1 - e^{-2\,m-1}}{2\,m+1} - \frac{1 - e^{-2\,m-3}}{2\,m+3} \tag{10}$$

Remark: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

III. References .

[1] Arndt, J., and Haenel, C.: π unleashed. Springer - Verlag , 2001.

[2] Beckmann, P.: A History of π. 2nd ed. Golem Press, Boulder, CO, 1971.

[3] Berndt, B.: Ramanujan's Notebooks. Part I. Springer - Verlag , 1985.