Theorem 1 If ϕ is a homomorphism of G into $\bar{G}$ with kernel K, then K is a normal subgroup of G.
Proof.
This is Lemma 2.7.3 in [1].

Theorem 2 Let ϕ be a homomorphism of G *onto* $\bar{G}$ with kernel K, and let $\bar{N}$ be a normal subgroup of $\bar{G}$, $N = \{x \in G \mid \phi(x) \in \bar{N}\}$. Then $G/N \approx \bar{G}/\bar{N}$. Equivalently, $G/N \approx (G/K)/(N/K)$.
Proof.
The first part of the proof can be found in [1]. By theorem 1, K is a normal subgroup of G. Define the mapping $\psi : G \to G/K$ by $\psi(g) = Kg$. Then ψ is a homomorphism of G *onto* G/K with kernel K. Clearly N is a subgroup of G and $N \supset K$. Since $\bar{N}$ is a normal subgroup of $\bar{G}$, N is a normal subgroup of G. It follows that N/K is a normal subgroup of G/K. It remains to be shown that $N = \{x \in G \mid \psi(x) \in N/K\}$. The first inclusion is obvious. To prove the second inclusion, let $x \in G$ such that $\psi(x) \in N/K$. So $Kx \in N/K$ and thus $Kx = Kn$ for some $n \in N$. Since $x \in Kx$ and $Kx \subset Kn$, $x \in Kn$. Hence $x = kn$ for some $k \in K$. To conclude, $x \in N$ because $N \supset K$. To sum it up, ψ is a homomorphism of G *onto* G/K with kernel K such that N/K is a normal subgroup of G/K and $N = \{x \in G \mid \psi(x) \in N/K\}$. By the first part of the theorem, $G/N \approx (G/K)/(N/K)$.

References

[1] I. N. Herstein, *Topics in Algebra*, John Wiley & Sons, New York, 1975.