Theorem 1 If V is finite-dimensional vector space over a field F and T is a homomorphism of V *onto* V, prove that T must be one-to-one, and so an isomorphism.

Proof.

Let $n = \dim V$ and $v_1, \ldots, v_n$ be a basis of V. Since T is *onto*, $v_i = w_i T$ for some $w_i \in V$ and $i = 1, \ldots, n$. To show the linear independence of the w_i, consider $\alpha_1 w_1 + \cdots + \alpha_n w_n = 0$ with $\alpha_1, \ldots, \alpha_n$ in F. It follows that

$$\begin{aligned}
\alpha_1 v_1 + \cdots + \alpha_n v_n &= \alpha_1(w_1 T) + \cdots + \alpha_n(w_n T) \\
&= (\alpha_1 w_1)T + \cdots + (\alpha_n w_n)T \\
&= (\alpha_1 w_1 + \cdots + \alpha_n w_n)T \\
&= 0T \\
&= 0
\end{aligned}$$

and hence by the linear independence of $v_1, \ldots, v_n$ forces $\alpha_i = 0$ for $i = 1, \ldots, n$. Since V is of dimension n, any set of n linearly independent vectors in V forms a basis of V. Therefore $w_1, \ldots, w_n$ is a basis of V. Now suppose $vT = 0$ for some $v \in V$. Thus $v = \lambda_1 w_1 + \cdots + \lambda_n w_n$ with $\lambda_1, \ldots, \lambda_n$ in F. Moreover

$$\begin{aligned}
\lambda_1 v_1 + \cdots + \lambda_n v_n &= \lambda_1(w_1 T) + \cdots + \lambda_n(w_n T) \\
&= (\lambda_1 w_1)T + \cdots + (\lambda_n w_n)T \\
&= (\lambda_1 w_1 + \cdots + \lambda_n w_n)T \\
&= vT \\
&= 0
\end{aligned}$$

and hence by the linear independence of $v_1, \ldots, v_n$ forces $\lambda_i = 0$ for $i = 1, \ldots, n$. So $v = 0$. Since its kernel is (0), T is an isomorphism.

References

[1] I. N. Herstein, *Topics in Algebra*, John Wiley & Sons, New York, 1975.