A collection of mathematical formulas involving π

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abstract

In this note we give a collection of mathematical formulas involving π :

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1}$$

keywords: number Pi , integrals , series

I. Introduction

Recall that

$$\pi = 4 \int_0^1 \frac{1}{1+x^2}\, dx \tag{1}$$

In this note we give some formulas involving π.

II. Formulas

Entry 1.

$$\pi = \int_0^1 \frac{\sqrt{\sqrt{1+x^2}+1} + \sqrt{\sqrt{1+x^2}-1}}{\sqrt{(1-x)(1+x^2)}}\, dx \tag{2}$$

$$\frac{\pi}{2} = \int_0^1 \frac{\sqrt{1-x}}{(1-x)^2+x^2} \left(\sqrt{\sqrt{1+x^2}+1} - (1-2\,x) \sqrt{\sqrt{1+x^2}-1} \right) dx \tag{3}$$

Entry 2.

$$\frac{\pi}{2} = \int_0^1 \frac{f(x) - (1-2\,x)\, g(x)}{(1-x)^2+x^2}\, dx \tag{4}$$

where

$$f(x) = \cosh(\sin(1-x)\cosh(x)) \cos(\cos(1-x)\sinh(x)) \tag{5}$$

$$g(x) = \sinh(\sin(1-x)\cosh(x)) \sin(\cos(1-x)\sinh(x)) \tag{6}$$

Entry 3.

$$\frac{\pi\, e}{2} = \int_0^1 \frac{e^{\cos(1-x)\cosh(x)} (\cos(\sin(1-x)\sinh(x)) + (1-2\,x)\sin(\sin(1-x)\sinh(x)))}{(1-x)^2+x^2}\, dx \tag{7}$$

Entry 4.

$$\frac{\pi}{4\cos(1)} = \int_0^1 \frac{\sec(x)}{1+x^2}\,dx + \int_0^1 \frac{\sin(x)\tan^{-1}(x)}{(\cos(x))^2}\,dx \tag{8}$$

$$\frac{\pi}{4\cosh(1)} = \int_0^1 \frac{\operatorname{sech}(x)}{1+x^2}\,dx - \int_0^1 \frac{\sinh(x)\tan^{-1}(x)}{(\cosh(x))^2}\,dx \tag{9}$$

Entry 5.

$$\frac{\pi}{4\cos(1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=0}^{n}\frac{E_k}{(2k)!} + \sum_{n=1}^{\infty}\frac{E_n}{(2n)!}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1} \tag{10}$$

$$\frac{\pi}{4\cosh(1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=0}^{n}\frac{(-1)^k E_k}{(2k)!} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}E_n}{(2n)!}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1} \tag{11}$$

where

$$E_n = \{1,\ 1,\ 5,\ 61,\ 1385,\ 50\,521,\ 2\,702\,765,\ \ldots\},\ \text{Euler numbers} \tag{12}$$

Entry 6.

$$\pi = 2\sqrt{2} + 2\int_0^{\left(\sqrt{2}-1\right)/2}\left(\sqrt{1+\sqrt{1-4x-4x^2}} - \sqrt{1-\sqrt{1-4x-4x^2}}\right)\frac{1}{\sqrt{1+x}}\,dx \tag{13}$$

Entry 7.

$$\frac{\pi}{4} = \frac{5\ln 2}{4} + \sum_{n=1}^{\infty}\left(\tanh\left(\frac{n\pi}{2}\right) - \tanh\left(\frac{(n+1)\pi}{2}\right)\right)\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} \tag{14}$$

$$\left(\frac{5\ln 2}{4} - \frac{\pi}{4}\right)\coth\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty}\left(1 - \tanh\left(\frac{n\pi}{2}\right)\tanh\left(\frac{(n+1)\pi}{2}\right)\right)\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} \tag{15}$$

$$\frac{5\ln 2}{4} - \frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{-n\pi/2}}{n\cosh(n\pi/2)} \tag{16}$$

$$\frac{\pi}{6} - \frac{3\ln 2}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{-n\pi}}{n\sinh(n\pi)} \tag{17}$$

$$\frac{\pi}{12} - \frac{\ln 2}{4} = \sum_{n=1}^{\infty}\frac{1}{n+1}\left(\frac{n+1}{\sinh(n\pi)} - \frac{n}{\sinh((n+1)\pi)}\right) \tag{18}$$

$$\frac{\pi}{12} - \frac{\ln 2}{4} = \sum_{n=1}^{\infty}\frac{1}{\sinh(n\pi)} - \sum_{n=1}^{\infty}\frac{n}{(n+1)\sinh((n+1)\pi)} \tag{19}$$

$$\frac{\pi}{12} - \frac{\ln 2}{4} = \sum_{n=1}^{\infty}\frac{2}{e^{(2n+1)\pi}-1} - \sum_{n=1}^{\infty}\frac{n}{(n+1)\sinh((n+1)\pi)} \tag{20}$$

Entry 8. If $H_n = \sum_{k=1}^{n}\frac{1}{k}$, then

$$\frac{\pi}{12} - \frac{\ln 2}{4} = \sum_{n=1}^{\infty}\left(\frac{1}{\sinh(n\pi)} - \frac{1}{\sinh((n+1)\pi)}\right)H_n \tag{21}$$

$$\left(\frac{\pi}{24} - \frac{\ln 2}{8}\right)\frac{1}{\sinh(\pi/2)} = \sum_{n=1}^{\infty}\frac{\cosh((2n+1)\pi/2)}{\sinh(n\pi)\sinh((n+1)\pi)}H_n \tag{22}$$

$$\left(\frac{\pi}{48} - \frac{\ln 2}{16}\right) \frac{1}{\sinh(\pi/2)} = \sum_{n=1}^{\infty} \frac{\cosh((2n+1)\pi/2)}{\cosh((2n+1)\pi) - \cosh(\pi)} H_n \tag{23}$$

Entry 9.

$$\frac{\pi}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{5 e^{-n\pi}}{\sinh(n\pi)} + \frac{3 e^{-n\pi/2}}{\cosh(n\pi/2)}\right) \tag{24}$$

Entry 10.

$$\frac{\pi}{2e} = \int_0^1 \frac{e^{-2x}(\cos(2x(1-x)) - (1-2x)\sin(2x(1-x)))}{(1-x)^2 + x^2} dx \tag{25}$$

$$\frac{\pi e}{2} = \int_0^1 \frac{e^{2x}(\cos(2x(1-x)) + (1-2x)\sin(2x(1-x)))}{(1-x)^2 + x^2} dx \tag{26}$$

Entry 11. If $a = \frac{1}{6}\left(108 + 12\sqrt{177}\right)^{1/3} - 4\left(108 + 12\sqrt{177}\right)^{-1/3}$, then

$$a = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \ldots\right)^3\right)^3\right)^3 \tag{27}$$

$$\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(n\pi a)}{n^2} = \sum_{n=1}^{\infty} \frac{\sin(n\pi a)}{n^3} \tag{28}$$

$$\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{2n+1} \operatorname{Re}\left(\left(\frac{1 + i\sqrt{3a-1}}{\sqrt{3}}\right)^{2n+1}\right), \quad i = \sqrt{-1} \tag{29}$$

$$\pi = 12 \sum_{n=0}^{\infty} \frac{a^{2n+1}}{2n+1} \operatorname{Im}\left(\left(\frac{\sqrt{3a-1} + i}{\sqrt{3}}\right)^{2n+1}\right), \quad i = \sqrt{-1} \tag{30}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{c_n}{2n+1} a^{2n+1} \tag{31}$$

where

$$c_{n+3} = -4c_{n+2} - 4c_{n+1} - c_n, \quad c_0 = 2, \quad c_1 = -5, \quad c_2 = 12 \tag{32}$$

$$\pi = 4\tan^{-1}\left(a^3\right) + 4 \sum_{n=0}^{\infty} a^{n+1} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} \binom{n}{n-2k} \tag{33}$$

Entry 12. If $e = \sum_{n=0}^{\infty} \frac{1}{n!}$, then

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-4n-1}}{(2n+1)^2} \left(\left(\sqrt{e}\right)^{2n+1} + \frac{1}{2}\left(\sqrt{4e - e^2}\right)^{2n+1} - \left(\sqrt{4-e}\right)^{2n+1}\right) \tag{34}$$

Entry 13.

$$\frac{\pi}{6} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{-2n}}{(2n+1)^3} F\left(n + \frac{1}{2}, n + \frac{1}{2}, n + \frac{3}{2}, \frac{1}{4}\right) = \frac{\sqrt{3}}{4} + \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}(2n+1)^3} F\left(-n - \frac{1}{2}, n + \frac{1}{2}, n + \frac{3}{2}, \frac{1}{4}\right) \tag{35}$$

$$\frac{\pi}{3\sqrt{3}} + \frac{4}{3} \sum_{n=1}^{\infty} \frac{(-1)^n 3^{-n}}{(2n+1)^3} F\left(1, n + \frac{1}{2}, n + \frac{3}{2}, -\frac{1}{3}\right) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{3^n}{\binom{2n}{n}(2n+1)^3} F\left(-n - \frac{1}{2}, 1, n + \frac{3}{2}, -\frac{1}{3}\right) \tag{36}$$

where $F(a, b, c, x)$ is the Gauss hypergeometric function.

Entry 14.

$$\pi \sin(1) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \sum_{k=0}^{n} \frac{1}{(2k+1)!} + 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} I_n \tag{37}$$

where

$$I_n = A_n + B_n \sin(1) + C_n \cos(1) \ , \ \ n = 0, 1, 2, 3, \ldots \tag{38}$$

$$A_{n+1} = -(2n+2)(2n+3) A_n \ , \ \ A_0 = -1 \tag{39}$$

$$B_{n+1} = 1 - (2n+2)(2n+3) B_n \ , \ \ B_0 = 1 \tag{40}$$

$$C_{n+1} = (2n+3) - (2n+2)(2n+3) C_n \ , \ \ C_0 = 1 \tag{41}$$

Entry 15.

$$\pi \sinh(1) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} + 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} I_n \tag{42}$$

where

$$I_n = A_n + B_n \sinh(1) + C_n \cosh(1) \ , \ \ n = 0, 1, 2, 3, \ldots \tag{43}$$

$$A_{n+1} = (2n+2)(2n+3) A_n \ , \ \ A_0 = 1 \tag{44}$$

$$B_{n+1} = 1 + (2n+2)(2n+3) B_n \ , \ \ B_0 = 1 \tag{45}$$

$$C_{n+1} = -(2n+3) + (2n+2)(2n+3) C_n \ , \ \ C_0 = -1 \tag{46}$$

Entry 16. If $E_n = \{1, 1, 5, 61, 1385, \ldots\}$ are Euler numbers , then

$$\frac{2-\sqrt{3}}{16} + \frac{\pi}{48} = \sum_{k=0}^{\infty} (-1)^k \sum_{n=0}^{k} \frac{2^{-2n} E_n}{(k-n+1)^{2n}} \left(\sum_{m=0}^{2n} \frac{1}{m!} \left(3^{-k+n-1} \left((k-n+1) \ln 3 \right)^m - \left(1+\sqrt{2}\right)^{-2n+2n-2} \left((2k-2n+2) \ln\left(1+\sqrt{2}\right) \right)^m \right) \right) \tag{47}$$

Entry 17.

$$\frac{\pi^2 \sqrt{2}}{6} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-3n}}{(n+1)(2n+1)} \left(F\left(\frac{1}{2}, n+1, n+2, \frac{1}{4}\right) + 2^{-n} F\left(\frac{1}{2}, n+1, n+2, \frac{1}{2}\right) \right) \tag{48}$$

where $F(a, b, c, x)$ is the Gauss hypergeometric function.

Entry 18.

$$\frac{\pi}{2} = \int_0^{\infty} \sin^{-1}(2^{-x}) \, dx \tag{49}$$

$$\pi = \int_0^{\infty} \sin^{-1}\left(2^{-x/2}\right) dx \tag{50}$$

$$\pi = \int_0^{\infty} \cos^{-1}\left(1 - 2^{1-2x}\right) dx \tag{51}$$

Entry 19.

$$\frac{\pi}{4} = 1 - \int_0^{1/2} \ln\left(\frac{1}{6} \left(\frac{108 + 12\sqrt{12x^2 + 81}}{x} \right)^{1/3} - 2 \left(\frac{108 + 12\sqrt{12x^2 + 81}}{x} \right)^{-1/3} \right) dx \tag{52}$$

Entry 20. If $t = \tan 1$, then

$$\pi = 3 + \left(1 - \frac{t^2}{3}\right)\sum_{n=0}^{\infty} 3^{-n}\, t^{-2n-1} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n+k}{n-k} \left(\frac{(3-t^2)^2}{3}\right)^k \tag{53}$$

Entry 21. If $t = \tan 3$, then

$$\pi = 3 - t + \frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7} - \frac{t^9}{9} + \ldots \tag{54}$$

$$\pi = 3 - \left(\frac{t}{1+}\,\frac{t^2}{3+}\,\frac{4t^2}{5+}\,\frac{9t^2}{7+}\,\frac{16t^2}{9+}\ldots\right) \tag{55}$$

Entry 22.

$$\frac{\pi}{20} + \frac{\ln 2}{10} = \int_0^1 \frac{(1-2x^2)\tan^{-1}(x)}{((1-x)^2+x^2)^2}\, dx \tag{56}$$

$$\frac{11\pi}{20} + \frac{\ln 2}{10} = \int_0^1 \frac{(1-2x^2)}{((1-x)^2+x^2)^2} \ln\left(\frac{\sqrt{2-2x+x^2}}{x}\right) dx \tag{57}$$

Entry 23.

$$\pi\sqrt{2} = \int_0^1 \frac{\sqrt{1-\sqrt{1-x^2}} + \sqrt{1+\sqrt{1-x^2}}}{\sqrt{x(1-x^2)}}\, dx \tag{58}$$

Entry 24. If $a > 0$, then

$$\frac{\pi}{2\sqrt{2}} = 1 + 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(1+a)^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-a)^{n-k}}{(4k+3)(4k+5)} \tag{59}$$

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{4}{3}\sum_{n=0}^{\infty} 3^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-2)^k}{(4k+3)(4k+5)} \tag{60}$$

Entry 25.

$$\frac{\pi}{2\sqrt{2}} = \int_0^1 \left(2\cosh^{-1}\sqrt{\frac{1+2x+\sqrt{1+4x-4x^2}}{4x}} - \cosh^{-1}\sqrt{\frac{x+\sqrt{2x-x^2}}{2x}}\right) dx \tag{61}$$

Entry 26. If $0 < a \le 1$, $b \ge e = 2.718\ldots$, then

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}\left(a^{2n+1} - \left(\frac{a}{e}\right)^{2n+1} + \left(\frac{e}{b}\right)^{2n+1} - \left(\frac{1}{b}\right)^{2n+1}\right) + \int_a^b \frac{\tan^{-1} x - \tan^{-1}(x/e)}{x}\, dx \tag{62}$$

Entry 27.

$$\pi = \sin 1 \int_{-\infty}^{\infty} \frac{x \sinh x}{\sinh^2 x + \cos^2 1}\, dx \tag{63}$$

$$\pi = \sin 1 \int_{-\infty}^{\infty} \frac{x \sinh x}{\cosh^2 x - \sin^2 1}\, dx \tag{64}$$

$$\pi = 2\sin 1 \int_{-\infty}^{\infty} \frac{x \sinh x}{\cosh(2x) + \cos 2}\, dx \tag{65}$$

Entry 28. If $0 \le a \le \pi/2$, $u = \frac{(\sin a)^4}{(\sin a)^4 + (\cos a)^4}$, then

$$\pi\left(\frac{1}{4}-\frac{u}{2}\right)+a\,u=\int_0^{\sin a}\frac{x^4}{(1-2x^2+2x^4)\sqrt{1-x^2}}\,dx+\int_u^1\tan^{-1}\left(\left(\frac{1-x}{x}\right)^{1/4}\right)dx \tag{66}$$

Entry 29. If $a \geq 0$, then

$$\frac{\pi}{2}=1+e^{-a}\ln 2-\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-2n}e^{-(2n+1)a}}{2n(2n+1)}+\int_0^a e^{-x}\ln\left(1+\sqrt{1-e^{-2x}}\right)dx \tag{67}$$

Entry 30. If $0 < a < 1$, then

$$\pi a(1-\ln a)=4-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k(1-a^2)^{n-k}}{(2k+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a^{2n}}{n(2n-1)^2} \tag{68}$$

Entry 31.

$$\frac{\pi}{4}=\int_{-1}^{1}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{2+x}\right)dx \tag{69}$$

Entry 32.

$$\frac{\pi}{4}=\int_0^1\sqrt{-\frac{1}{3}+\sqrt[3]{\sqrt{\frac{4}{27}+\frac{1}{4x^2}-\frac{10}{27x}}+\frac{1}{2x}-\frac{10}{27}}-\sqrt[3]{\sqrt{\frac{4}{27}+\frac{1}{4x^2}-\frac{10}{27x}}-\frac{1}{2x}+\frac{10}{27}}}\;dx \tag{70}$$

Entry 33.

$$\frac{(\Gamma(1/4))^2}{2\sqrt{\pi}}=\int_0^{\infty}\left(\frac{\left(12\sqrt{3}\sqrt{27x^4+4}+108x^2\right)^{1/3}}{6x^2}-\frac{2\left(12\sqrt{3}\sqrt{27x^4+4}+108x^2\right)^{-1/3}}{x^2}\right)dx \tag{71}$$

where $\Gamma(x)$ is the Gamma function.

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