On the Metric Coefficients

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Abstract

The article investigates the metric coefficients in the light of the Field Equations of General Relativity. We drive results connecting the metric coefficients(exclusively). These results do not apply where the energy density tensor has zero values. They follow from the consistent application of mathematics on the existing theory of General Relativity is quite starling in view of the fact that these results impose a heavy constraint on the theory.

Introduction

The metric coefficients are considered in the light of the Einstein's Field Equations. Resultsconnecting the metric coefficients(exclusively) are derived. These results which follow by a consistent application of mathematics on existing theory is quite starling in view of the fact they impose a heavy constraint on the existing theory. The derived results ,incidentally, do not apply where the energy density tensor has zero values.

A Mathematical Result

For arbitrary real numbers $a_1, a_2, b_1$and b_2

Result: $(a_1b_1 - a_2b_2)^2 \geq |a_1{}^2 - a_2{}^2|\left|b_1{}^2 - b_2{}^2\right| \Rightarrow |a_1b_1 - a_2b_2| \geq \sqrt{|a_1{}^2 - a_2{}^2|}\sqrt{\left|b_1{}^2 - b_2{}^2\right|}$ (1)

Derivation:

$$(a_1b_1 - a_2b_2)^2 - (a_1{}^2 - a_2{}^2)\left(b_1{}^2 - b_2{}^2\right)$$

$$= a_1{}^2b_1{}^2 + a_2{}^2b_2{}^2 - 2a_1b_1a_2b_2 - a_1{}^2b_1{}^2 - a_2{}^2b_2{}^2 + a_1{}^2b_2{}^2 + a_2{}^2b_1{}^2$$

$$= a_1{}^2b_2{}^2 + a_2{}^2b_1{}^2 - 2a_1b_1a_2b_2$$

$$= (a_1b_2 - a_2b_2)^2 \geq 0$$

Therefore

$$(a_1b_1 - a_2b_2)^2 \geq ({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2) \ (1')$$

$$|a_1b_1 - a_2b_2| \geq \sqrt{|{a_1}^2 - {a_2}^2|}\sqrt{|{b_1}^2 - {b_2}^2|}$$

$a_1b_1 - a_2b_2 \geq \sqrt{|{a_1}^2 - {a_2}^2|}\sqrt{|{b_1}^2 - {b_2}^2|}$ (1'')

or $a_1b_1 - a_2b_2 \leq -\sqrt{|{a_1}^2 - {a_2}^2|}\sqrt{|{b_1}^2 - {b_2}^2|}$ $(1''')$

For (1'') and (1''') equation (1') will hold irrespective of how ${a_1}^2$ compares with ${a_2}^2$ or how ${b_1}^2$ compares with ${b_2}^2$

For

$$-|({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2)| < a_1b_1 - a_2b_2 < +|({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2)|$$

equation (1') will fail irrespective of how ${a_1}^2$ compares with ${a_2}^2$ or how ${b_1}^2$ compares with ${b_2}^2$

If $({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2) < 0$ then

$|a_1b_1 - a_2b_2| \geq \sqrt{|{a_1}^2 - {a_2}^2|}\sqrt{|{b_1}^2 - {b_2}^2|}$; $|a_1b_1 - a_2b_2| \leq \sqrt{|{a_1}^2 - {a_2}^2|}\sqrt{|{b_1}^2 - {b_2}^2|}$

are both inadmissible options since the right side in each case is a pure imaginary number and it cannot be compared with the left side which is a real number, $|a_1b_1 - a_2b_2|$.

The square roots imply positive roots[as considered by convention]

In any event $({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2) < 0$ or $({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2) \geq 0$ then (1'') or (1''') are the only possible options. Other conceivable options are not valid ones.

From the Field Equations

We start with the field equations[1]

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = \frac{8\pi G}{c^4}T^{\alpha\beta} \ (2)$$

$$\Rightarrow R^{\alpha\beta} - \frac{8\pi G}{c^4}T^{\alpha\beta} = \frac{1}{2}Rg^{\alpha\beta}$$

$$\Rightarrow g_{\alpha\beta}\left(R^{\alpha\beta} - \frac{8\pi G}{c^4}T^{\alpha\beta}\right) = \frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta}$$

$$\Rightarrow g_{\alpha\beta}\left(R^{\alpha\beta}-\frac{8\pi G}{c^4}T^{\alpha\beta}\right)=2R \quad (3)$$

In the orthogonal system (3) reduces to

$$\Rightarrow g_{\alpha\alpha}\left(R^{\alpha\alpha}-\frac{8\pi G}{c^4}T^{\alpha\alpha}\right)=2R \quad (4)$$

We use the metric signature (+,-,-,-).

$$\Rightarrow g_{00}\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)-g_{11}\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)-g_{22}\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)-g_{33}\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right) = 2R \ ; g_{ii}>0 (5)$$

Further Calculations

Applying the Cauchy Schwarz[2] inequality

$$\left[g_{11}\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)+g_{22}\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)+g_{33}\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)\right]^2 > = ({g_{11}}^2+{g_{22}}^2+{g_{33}}^2)\left(\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2 + \left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2\right)$$

$$\Rightarrow \frac{\left[g_{11}\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)+g_{22}\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)+g_{33}\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)\right]^2}{({g_{11}}^2+{g_{22}}^2+{g_{33}}^2)\left(\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2\right)}\le 1$$

$$\Rightarrow -1\le \frac{g_{11}\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)+g_{22}\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)+g_{33}\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)}{\sqrt{{g_{11}}^2+{g_{22}}^2+{g_{33}}^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}}\le 1$$

$$\frac{g_{11}\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)+g_{22}\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)+g_{33}\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)}{\sqrt{{g_{11}}^2+{g_{22}}^2+{g_{33}}^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}} = Cos\theta \quad (6)$$

Since both the numerator and the denominator of the left side of (6) are positive, $Cos\theta$in the current context should be positive. $2R = g_{00}\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)-\sqrt{{g_{11}}^2+{g_{22}}^2+{g_{33}}^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}\ Cos\theta \ge 0 \quad (7)$

$$2R$$
$$= g_{00}\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right)$$
$$- \sqrt{{g_{11}}^2 + {g_{22}}^2 + {g_{33}}^2}\sqrt{\left(R^{11} - \frac{8\pi G}{c^4}T^{11}\right)^2 + \left(R^{22} - \frac{8\pi G}{c^4}T^{22}\right)^2 + \left(R^{33} - \frac{8\pi G}{c^4}T^{33}\right)^2}\, Cos\theta$$

$$\geq g_{00}\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right)$$
$$- \sqrt{{g_{11}}^2 + {g_{22}}^2 + {g_{33}}^2}\sqrt{\left(R^{11} - \frac{8\pi G}{c^4}T^{11}\right)^2 + \left(R^{22} - \frac{8\pi G}{c^4}T^{22}\right)^2 + \left(R^{33} - \frac{8\pi G}{c^4}T^{33}\right)^2} \quad (8)$$

[For (8) we should note that $g_{00}\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right) \geq 0$]

Now

$$R^{\alpha\alpha} - \frac{8\pi G}{c^4}T^{\alpha\alpha} = \frac{1}{2}Rg^{\alpha\alpha}$$

Therefore

$$\left(R^{11} - \frac{8\pi G}{c^4}T^{11}\right)^2 + \left(R^{22} - \frac{8\pi G}{c^4}T^{22}\right)^2 + \left(R^{33} - \frac{8\pi G}{c^4}T^{33}\right)^2 = \left(\frac{1}{2}Rg^{11}\right)^2 + \left(\frac{1}{2}Rg^{22}\right)^2 + \left(\frac{1}{2}Rg^{33}\right)^2$$
$$= \frac{1}{4}R^2\left[{g^{11}}^2 + {g^{22}}^2 + {g^{33}}^2\right] = \frac{1}{4}R^2\left[\frac{1}{{g^{11}}^2} + \frac{1}{{g^{22}}^2} + \frac{1}{{g^{33}}^2}\right] \quad (9.1)$$

In the above line we have applied the standard formula[3] [considering an orthogonal system] $g^{jj} = \frac{1}{g^{jj}}$[no summation on j]

$$\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right)^2 - \left(R^{11} - \frac{8\pi G}{c^4}T^{11}\right)^2 - \left(R^{22} - \frac{8\pi G}{c^4}T^{22}\right)^2 - \left(R^{33} - \frac{8\pi G}{c^4}T^{33}\right)^2$$
$$= \left(\frac{1}{2}Rg^{00}\right)^2 - \left(\frac{1}{2}Rg^{11}\right)^2 - \left(\frac{1}{2}Rg^{22}\right)^2 - \left(\frac{1}{2}Rg^{33}\right)^2$$
$$= \frac{1}{4}R^2\left[{g^{00}}^2 - {g^{11}}^2 - {g^{22}}^2 - {g^{33}}^2\right]$$

$$= \frac{1}{4}R^2\left[\frac{1}{{g^{00}}^2} - \frac{1}{{g^{11}}^2} - \frac{1}{{g^{22}}^2} - \frac{1}{{g^{33}}^2}\right] \quad (9.2)$$

Again in the above line we have applied [considering an orthogonal system] $g^{jj} = \frac{1}{g^{jj}}$[no summation on j]

Option 1[following from (1'')]

We have

$$2R \geq g_{00}\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right) - \sqrt{{g_{11}}^2 + {g_{22}}^2 + {g_{33}}^2}\sqrt{\left(R^{11} - \frac{8\pi G}{c^4}T^{11}\right)^2 + \left(R^{22} - \frac{8\pi G}{c^4}T^{22}\right)^2 + \left(R^{33} - \frac{8\pi G}{c^4}T^{33}\right)^2}$$
$$\geq \sqrt{\left|{g_{11}}^2 - {g_{11}}^2 - {g_{22}}^2 - {g_{33}}^2\right|}\sqrt{\frac{1}{4}R^2\left|\frac{1}{{g^{00}}^2} - \frac{1}{{g^{11}}^2} - \frac{1}{{g^{22}}^2} - \frac{1}{{g^{33}}^2}\right|} \quad (10)$$

The last inequality in (10) follows (1’). Equation (9.2) has been applied in the last line

$$2R \geq \sqrt{\frac{1}{4}R^2\left|{g_{00}}^2 - {g_{11}}^2 - {g_{22}}^2 - {g_{33}}^2\right|\left|\frac{1}{{g^{00}}^2} - \frac{1}{{g^{11}}^2} - \frac{1}{{g^{22}}^2} - \frac{1}{{g^{33}}^2}\right|}$$

$$16 \geq \sqrt{\left|{g_{00}}^2 - {g_{11}}^2 - {g_{22}}^2 - {g_{33}}^2\right|\left|\frac{1}{{g^{00}}^2} - \frac{1}{{g^{11}}^2} - \frac{1}{{g^{22}}^2} - \frac{1}{{g^{33}}^2}\right|}$$

$$-4 \leq \sqrt{\left|{g_{00}}^2 - {g_{11}}^2 - {g_{22}}^2 - {g_{33}}^2\right|\left|\frac{1}{{g^{00}}^2} - \frac{1}{{g^{11}}^2} - \frac{1}{{g^{22}}^2} - \frac{1}{{g^{33}}^2}\right|} \leq 4 \quad (11)$$

E

Option 2

$$2R = g_{00}\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right) - \sqrt{{g_{11}}^2 + {g_{22}}^2 + {g_{33}}^2}\sqrt{\left(R^{11} - \frac{8\pi G}{c^4}T^{11}\right)^2 + \left(R^{22} - \frac{8\pi G}{c^4}T^{22}\right)^2 + \left(R^{33} - \frac{8\pi G}{c^4}T^{33}\right)^2}\, Cos\theta$$
$$\geq g_{00}\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right) - \sqrt{{g_{11}}^2 + {g_{22}}^2 + {g_{33}}^2}\sqrt{\left(R^{11} - \frac{8\pi G}{c^4}T^{11}\right)^2 + \left(R^{22} - \frac{8\pi G}{c^4}T^{22}\right)^2 + \left(R^{33} - \frac{8\pi G}{c^4}T^{33}\right)^2}$$

The next inequality follows from (1’’’).

We have,

$$g_{00}\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)$$
$$-\sqrt{{g_{11}}^2+{g_{22}}^2+{g_{33}}^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}$$
$$\leq -\sqrt{\left|{g_{00}}^2-{g_{11}}^2-{g_{22}}^2-{g_{33}}^2\right|}$$
$$\times\sqrt{\left|\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)^2-\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2-\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2-\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2\right|}$$

We multiply both sides of the above by $Cos\,\theta > 0$[we apply this physical insight now that $Cos\,\theta > 0$ so that the inequality sign does not reverse itself]

$$\left[g_{00}\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)\right.$$
$$\left.-\sqrt{{g_{11}}^2+{g_{22}}^2+{g_{33}}^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}\right]Cos\theta$$
$$\leq -\sqrt{\left|{g_{00}}^2-{g_{11}}^2-{g_{22}}^2-{g_{33}}^2\right|}$$
$$\times\sqrt{\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)^2-\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2-\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2-\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}\;cos\theta$$

Since Cos theta>0 the inequality sign does not reverse itself in the above inequality

Now

$$\left[g_{00}\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)\right.$$
$$\left.-\sqrt{{g_{11}}^2+{g_{22}}^2+{g_{33}}^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}\right]Cos\theta$$

$$= \Bigg[g_{00}\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right)$$

$$- \sqrt{{g_{11}}^2 + {g_{22}}^2 + {g_{33}}^2}\sqrt{\left(R^{11} - \frac{8\pi G}{c^4}T^{11}\right)^2 + \left(R^{22} - \frac{8\pi G}{c^4}T^{22}\right)^2 + \left(R^{33} - \frac{8\pi G}{c^4}T^{33}\right)^2}\, Cos\theta \Bigg]$$

$$- g_{00}\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right)(1 - Cos\theta)$$

$$\le - \sqrt{\frac{1}{4}R^2 |{g_{00}}^2 - {g_{11}}^2 - {g_{22}}^2 - {g_{33}}^2| \left|\frac{1}{{g^{00}}^2} - \frac{1}{{g^{11}}^2} - \frac{1}{{g^{22}}^2} - \frac{1}{{g^{33}}^2}\right|}\, Cos\theta \ (13)$$

Equation (9.2) has been applied in the last line

$$2R - g_{00}\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right)(1 - Cos\theta)$$

$$\le \sqrt{\frac{1}{4}R^2 |{g_{00}}^2 - {g_{11}}^2 - {g_{22}}^2 - {g_{33}}^2| \left|\frac{1}{{g^{00}}^2} - \frac{1}{{g^{11}}^2} - \frac{1}{{g^{22}}^2} - \frac{1}{{g^{33}}^2}\right|}\, Cos\theta$$

$g_{00}\frac{1}{2}g^{00}R = \frac{1}{2}R; noting\ g_{ii}g^{ii} = 1$ in orthogonal systems[even curvilinear] (important:no summation on i)

$$2R - \frac{R}{2}(1 - Cos\theta) \le - \sqrt{\frac{1}{4}R^2 |{g_{00}}^2 - {g_{11}}^2 - {g_{22}}^2 - {g_{33}}^2| \left|\frac{1}{{g^{00}}^2} - \frac{1}{{g^{11}}^2} - \frac{1}{{g^{22}}^2} - \frac{1}{{g^{33}}^2}\right|}\, Cos\theta 14)$$

But the above relation is not possible since $\frac{3}{2}R + \frac{R}{2}Cos\theta > 0$. [We take note of the fact as stated earlier that $Cos\theta$ from equation (6) positive.]

Therefore (14) is an impossible relation.

If $T^{\alpha\beta} = 0$ [implying $R^{\alpha\beta} = 0, R = 0$]the issues considered in this article may be considered by a limiting process with $T^{\alpha\beta} \to 0$.

Same results as obtained earlier[for example equation (11)] should persist in the absence of discontinuities. Discontinuities could alter the whole subject maintaining(expectedly) observed results. We seem to be having a quantum spectacle hiding in the scenario of General Relativity.

Conclusions

As claimed at the outset we have arrived at surprising results by applying mathematical tools on formal theory. The impact of these results may have far reaching consequences.

References

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