On the Metric Coefficients

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Abstract

The article considers a simple algebraic result to bring out an interesting mathematical property of the general relativity metric . This property creates limitation on its application in that it may not apply to all space time points.

Introduction

A simple mathematical result is formulated in the beginning of the article. It is applied to bring out a limitation on the general relativity metric in that it may not apply to all space time points.

A Mathematical Result

For arbitrary real numbers a_1, a_2, b_1 and b_2,

$$(a_1b_1 - a_2b_2)^2 \geq (a_1{}^2 - a_2{}^2)(b_1{}^2 - b_2{}^2) \quad (1)$$

Proof: $(a_1b_1 - a_2b_2)^2 - (a_1{}^2 - a_2{}^2)(b_1{}^2 - b_2{}^2)$

$$= a_1{}^2b_1{}^2 + a_2{}^2b_2{}^2 - 2a_1a_2b_1b_2 - \left(a_1{}^2b_1{}^2 + a_2{}^2b_2{}^2 - a_1{}^2b_2{}^2 - a_2{}^2b_1{}^2\right)$$

$$= a_1{}^2b_2{}^2 + a_2{}^2b_1{}^2 - 2a_1a_2b_1b_2$$

$$= (a_1b_2 - a_2b_1)^2 \geq 0$$

Therefore$(a_1b_1 - a_2b_2)^2 - (a_1{}^2 - a_2{}^2)(b_1{}^2 - b_2{}^2) \geq 0$

$$(a_1b_1 - a_2b_2)^2 \geq (a_1{}^2 - a_2{}^2)(b_1{}^2 - b_2{}^2)$$

From the Field Equations

We start with the Field Equations[1]

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = \frac{8\pi G}{c^4} g_{\alpha\beta} T^{\alpha\beta} \ (2)$$

$$\Rightarrow R^{\alpha\beta} - \frac{8\pi G}{c^4} T^{\alpha\beta} = \frac{1}{2} R g^{\alpha\beta}$$

$$g_{\alpha\beta}\left(R^{\alpha\beta} - \frac{8\pi G}{c^4} T^{\alpha\beta}\right) = \frac{1}{2} R g_{\alpha\beta} g^{\alpha\beta}$$

$$g_{\alpha\beta}\left(R^{\alpha\beta} + \frac{8\pi G}{c^4} T^{\alpha\beta}\right) = 2R \ \ (3)$$

Each term of the sum on the left side has the same dimension. But all $g_{\alpha\beta}$ may not have the same dimension[example: Schwarzchild Gometry[2]]. We maintain each $g_{\alpha\beta}$ dimensionless transferring its dimension, if any, to the other factor [that is , to $R^{\alpha\beta} + \frac{8\pi G}{c^4} T^{\alpha\beta}$]We do this in order to consider terms like $(|g_{00}|^2 - |g_{11}|^2 - |g_{22}|^2 - |g_{33}|^2)$ which will come up shortly in this writing.

In the orthogonal coordinates (3) reduces to

$$g_{\alpha\alpha}\left(R^{\alpha\alpha} - \frac{8\pi G}{c^4} T^{\alpha\alpha}\right) = 2R \ (4)$$

$$g_{00}\left(R^{00} - \frac{8\pi G}{c^4} T^{00}\right) + g_{11}\left(R^{11} - \frac{8\pi G}{c^4} T^{11}\right) + g_{22}\left(R^{22} - \frac{8\pi G}{c^4} T^{22}\right) + g_{33}\left(R^{33} - \frac{8\pi G}{c^4} T^{33}\right) = 2R$$

With the $(+,-,-,-)$ signature of the metric we have,

$$|g_{00}|\left(R^{00} - \frac{8\pi G}{c^4} T^{00}\right) - |g_{11}|\left(R^{11} - \frac{8\pi G}{c^4} T^{11}\right) - |g_{22}|\left(R^{22} - \frac{8\pi G}{c^4} T^{22}\right) - |g_{33}|\left(R^{33} - \frac{8\pi G}{c^4} T^{33}\right) = 2R (5)$$

By applying the Cauchy Schwarz inequality[3] to (5) we have,

$$\left[|g_{11}|\left(R^{11} - \frac{8\pi G}{c^4} T^{11}\right) + |g_{22}|\left(R^{22} - \frac{8\pi G}{c^4} T^{22}\right) + |g_{33}|\left(R^{33} - \frac{8\pi G}{c^4} T^{33}\right)\right]^2 \le (|g_{11}|^2 + |g_{22}|^2 + |g_{33}|^2)\left[\left(R^{11} - \frac{8\pi G}{c^4} T^{11}\right)^2 + \left(R^{22} - \frac{8\pi G}{c^4} T^{22}\right)^2 + \left(R^{33} - \frac{8\pi G}{c^4} T^{33}\right)^2\right] \ (6)$$

$$\Rightarrow \frac{\left[|g_{11}|\left(R^{11} + \frac{8\pi G}{c^4} T^{11}\right) + |g_{22}|\left(R^{22} + \frac{8\pi G}{c^4} T^{22}\right) + |g_{33}|\left(R^{33} + \frac{8\pi G}{c^4} T^{33}\right)\right]^2}{(|g_{11}|^2 + |g_{22}|^2 + |g_{33}|^2)\left[\left(R^{11} - \frac{8\pi G}{c^4} T^{11}\right)^2 + \left(R^{22} - \frac{8\pi G}{c^4} T^{22}\right)^2 + \left(R^{33} - \frac{8\pi G}{c^4} T^{33}\right)^2\right]} \le \mathbf{1}$$

$$\Rightarrow -1 \le \frac{|g_{11}|\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)+|g_{22}|\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)+|g_{33}|\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)}{\sqrt{|g_{11}|^2+|g_{22}|^2+|g_{33}|^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}} \le 1 \quad (7)$$

Therefore we my write

$$\frac{|g_{11}|\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)+|g_{22}|\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)+|g_{33}|\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)}{\sqrt{|g_{11}|^2+|g_{22}|^2+|g_{33}|^2}\sqrt{\left(R^{11}+\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}+\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}+\frac{8\pi G}{c^4}T^{33}\right)^2}} = Cos\,\theta \quad (8)$$

Therefore,

$$|g_{11}|\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)+|g_{22}|\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)+|g_{33}|\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right) = \sqrt{|g_{11}|^2+|g_{22}|^2+|g_{33}|^2}\sqrt{\left(R^{11}+\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}+\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}+\frac{8\pi G}{c^4}T^{33}\right)^2} \times \mathrm{Cos}\,\theta \quad (9)$$

Considering (9) we may rewrite (5) as

$$|g_{00}|\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)-|g_{11}|\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)+|g_{22}|\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)+|g_{33}|\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)$$
$$= |g_{00}|\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)$$
$$-\sqrt{|g_{11}|^2+|g_{22}|^2+|g_{33}|^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}\,Cos\theta$$
$$= 2R \quad (10)$$

Now,

$$|g_{00}|\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)-|g_{11}|\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)+|g_{22}|\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)+|g_{33}|\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)$$
$$= |g_{00}|\left(R^{00}+\frac{8\pi G}{c^4}T^{00}\right)$$
$$-\sqrt{|g_{11}|^2+|g_{22}|^2+|g_{33}|^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}\,Cos\theta$$
$$\ge |g_{00}|\left(R^{00}+\frac{8\pi G}{c^4}T^{00}\right)$$
$$-\sqrt{|g_{11}|^2+|g_{22}|^2+|g_{33}|^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2} \quad (11)$$

By (1) we have

$$\left(|g_{00}|\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)\right.$$
$$\left.-\sqrt{|g_{11}|^2+|g_{22}|^2+|g_{33}|^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}\right)^2$$
$$\geq (|g_{00}|-|g_{11}|^2-|g_{22}|^2-|g_{33}|^2)\left[\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)^2-\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2-\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2\right.$$
$$\left.-\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2\right] \quad (12)$$

Therefore from (11) and (12) we have,

$$2R$$
$$=\left(|g_{00}|\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)\right.$$
$$\left.-\sqrt{|g_{11}|^2+|g_{22}|^2+|g_{33}|^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}\,Cos\theta\right)^2$$
$$\geq\left(|g_{00}|\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)\right.$$
$$\left.-\sqrt{|g_{11}|^2+|g_{22}|^2+|g_{33}|^2}\sqrt{\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2+\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2+\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2}\right)$$
$$\geq (|g_{00}|-|g_{11}|^2-|g_{22}|^2-|g_{33}|^2)\left[\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)^2-\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2-\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2\right.$$
$$\left.-\left(R^{33}-\frac{8\pi G}{c^4}T^{33}\right)^2\right]$$

$$\Rightarrow 4R^2=\left[|g_{00}|\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)-|g_{11}|\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)-|g_{22}|\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)\right.$$
$$\left.-|g_{33}|\left(R^{33}+\frac{8\pi G}{c^4}T^{33}\right)\right]^2\geq$$
$$(|g_{00}|^2-|g_{11}|^2-|g_{22}|^2-|g_{33}|^2)\left[\left(R^{00}-\frac{8\pi G}{c^4}T^{00}\right)^2-\left(R^{11}-\frac{8\pi G}{c^4}T^{11}\right)^2-\left(R^{22}-\frac{8\pi G}{c^4}T^{22}\right)^2\right.$$
$$\left.-\left(R^{33}+\frac{8\pi G}{c^4}T^{33}\right)^2\right] \quad (13)$$

$$\Rightarrow (|g_{00}|^2 - |g_{11}|^2 - |g_{22}|^2 - |g_{33}|^2)\left[\left(R^{00} - \frac{8\pi G}{c^4}T^{00}\right)^2 - \left(R^{11} - \frac{8\pi G}{c^4}T^{11}\right)^2 - \left(R^{22} - \frac{8\pi G}{c^4}T^{22}\right)^2 - \left(R^{33} - \frac{8\pi G}{c^4}T^{33}\right)^2\right] \leq 4R^2$$

Since by he field equations, $R^{\alpha\beta} + \frac{8\pi G}{c^4}T^{\alpha\beta} = \frac{1}{2}Rg^{\alpha\beta}$, we obtain

$$(|g_{00}|^2 - |g_{11}|^2 - |g_{22}|^2 - |g_{33}|^2)\left[\left(\frac{1}{2}Rg^{00}\right)^2 - \left(\frac{1}{2}Rg^{11}\right)^2 - \left(\frac{1}{2}Rg^{22}\right)^2 - \left(\frac{1}{2}Rg^{33}\right)^2\right] \leq 4R^2$$

$$(|g_{00}|^2 - |g_{11}|^2 - |g_{22}|^2 - |g_{33}|^2)^2 \leq 16$$

$-4 \leq |g_{00}|^2 - |g_{11}|^2 - |g_{22}|^2 - |g_{33}|^2 \leq 4$ (14)

The above relation may not hold true for all points[example Schwarzschild metric]

Conclusion

As claimed we have establish from a simple mathematical result the general relativity metric may not apply to all space time points.

References

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