On Four Velocity and Four Momentum

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Abstract

We derive in this article that the four dot product between two arbitrary velocities is less than the square of the speed of light in vacuum and the product of two four momenta is less than the product of the two masses involved and the square of the speed of light[in vacuuo]. Finally we find a contradiction to these formulas.

Introduction

The writing concerns itself with a derivation that the four dot product of two arbitrary velocities is less than c^2, c being the speed of light in vacuum and that the four dot product of two arbitrary four momenta is less than $m_1 m_2 c^2$. Finally we find a contradiction to these formulas.

A Mathematical Result

For arbitrary real numbers $a_1, a_2, b_1$and b_2

Result: $(a_1 b_1 - a_2 b_2)^2 \geq |{a_1}^2 - {a_2}^2|\left|{b_1}^2 - {b_2}^2\right| \Rightarrow |a_1 b_1 - a_2 b_2| \geq \sqrt{|{a_1}^2 - {a_2}^2|}\sqrt{\left|{b_1}^2 - {b_2}^2\right|}$ (1)

Derivation:

$$(a_1 b_1 - a_2 b_2)^2 - ({a_1}^2 - {a_2}^2)\left({b_1}^2 - {b_2}^2\right)$$

$$= {a_1}^2{b_1}^2 + {a_2}^2{b_2}^2 - 2a_1 b_1 a_2 b_2 - {a_1}^2{b_1}^2 - {a_2}^2{b_2}^2 + {a_1}^2{b_2}^2 + {a_2}^2{b_1}^2$$

$$= {a_1}^2{b_2}^2 + {a_2}^2{b_1}^2 - 2a_1 b_1 a_2 b_2$$

$$= (a_1 b_2 - a_2 b_2)^2 \geq 0$$

Therefore

$$(a_1 b_1 - a_2 b_2)^2 \geq ({a_1}^2 - {a_2}^2)\left({b_1}^2 - {b_2}^2\right) \quad (1')$$

$$|a_1b_1 - a_2b_2| \geq \sqrt{|a_1{}^2 - a_2{}^2|}\sqrt{|b_1{}^2 - b_2{}^2|}$$

$a_1b_1 - a_2b_2 \geq \sqrt{|a_1{}^2 - a_2{}^2|}\sqrt{|b_1{}^2 - b_2{}^2|}$ (1'')

or $a_1b_1 - a_2b_2 \leq -\sqrt{|a_1{}^2 - a_2{}^2|}\sqrt{|b_1{}^2 - b_2{}^2|}$ (1''')

For (1'') and (1''') equation (1') will hold irrespective of how $a_1{}^2$ compares with $a_2{}^2$ or how $b_1{}^2$ compares with $b_2{}^2$

For

$$-|(a_1{}^2 - a_2{}^2)(b_1{}^2 - b_2{}^2)| < a_1b_1 - a_2b_2 < +|(a_1{}^2 - a_2{}^2)(b_1{}^2 - b_2{}^2)|$$

equation (1') will fail irrespective of how $a_1{}^2$ compares with $a_2{}^2$ or how $b_1{}^2$ compares with $b_2{}^2$

If $(a_1{}^2 - a_2{}^2)(b_1{}^2 - b_2{}^2) < 0$ then

$|a_1b_1 - a_2b_2| \geq \sqrt{|a_1{}^2 - a_2{}^2|}\sqrt{|b_1{}^2 - b_2{}^2|}$; $|a_1b_1 - a_2b_2| \leq \sqrt{|a_1{}^2 - a_2{}^2|}\sqrt{|b_1{}^2 - b_2{}^2|}$

are both inadmissible options since the right side in each case is a pure imaginary number and it cannot be compared with the left side which is a real number, $|a_1b_1 - a_2b_2|$.

The square roots imply positive roots[as considered by convention]

In any event $(a_1{}^2 - a_2{}^2)(b_1{}^2 - b_2{}^2) < 0$ or $(a_1{}^2 - a_2{}^2)(b_1{}^2 - b_2{}^2) \geq 0$ then (1'') or (1''') are the only possible options. Other conceivable options are not valid ones.

Dot Product of Two Four Velocities

Using the above result we will prove that the four velocity[1] dot product[2],

$$v_1.v_2 >= c^2 \quad (2)$$

Proof: First we consider the relations

$$1) c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| \geq \sqrt{c^2 v_{1t}{}^2 - |\vec{v}_1|^2}\sqrt{c^2 v_{12t}{}^2 - |\vec{v}_2|^2}$$

$$2) c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| \leq -\sqrt{c^2 v_{1t}{}^2 - |\vec{v}_1|^2}\sqrt{c^2 v_{12t}{}^2 - |\vec{v}_2|^2}$$

The above two relations have been written from (1) where we have considered the following,

$a_1 = cv_{1t}, a_2 = |\vec{v}_1|, b_1 = cv_{2t}, , b_2 = |\vec{v}_2|$

[suffix 't' denotes the time component]

But $c^2 {v_{1t}}^2 - {v_{1t}}^2 {v_{2t}}^2 = c^2; c^2 {v_{12t}}^2 - {v_{1t}}^2 {v_{2t}}^2 = c^2$

[We are working with the metric signature $(+,-,-,-)$

The first inequality(1'') gives us

$$c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| \geq c^2$$

Where $\vec{v}_1 = (v_{1x}, v_{1y}, v_{1z}); \vec{v}_2 = (v_{2x}, v_{2y}, v_{2z})$ where v_{1i} and v_{2i} are the spatial components of proper velocity and not of coordinate velocity

$$v_1.v_2 = c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| Cos\theta$$

One should take cognizance of the fact that by Cauchy Schwarz inequality

$$(v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z})^2 \leq ({v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2)({v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2)$$

$$\Rightarrow \frac{(v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z})^2}{({v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2)({v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2)} \leq 1$$

$$\Rightarrow -1 \leq \frac{v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}}{\sqrt{{v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2}\sqrt{{v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2}} \leq 1$$

Therefore with some suitable θ we may write

$$\frac{v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}}{\sqrt{{v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2}\sqrt{{v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2}} = Cos\theta$$

$$\Rightarrow v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = \sqrt{{v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2}\sqrt{{v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2}\, Cos\theta$$

Therefore

$$\Rightarrow v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = |\vec{v}_1||\vec{v}_2|\, Cos\theta$$

We do have the above relation for some suitable θ[at any relativistic speed notwithstanding thefasct that we are not considering coordinate speed components but that we have taken the spatial part of proper velocity[celertity]

For $0 < Cos\theta < 1$,

$$c^2v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2|Cos\theta > c^2v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| > c^2$$

$$c^2v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2|Cos\theta > c^2$$

$$v_1.v_2 > c^2$$

For $-1 < Cos\theta < 0$,

$c^2v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2|Cos\theta > c^2v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| > c^2$[since $Cos\theta$ is negative]

$$c^2v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2|Cos\theta > c^2$$

$$v_1.v_2 > c^2$$

If $Cos\theta = 1$,

$$c^2v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| = c^2$$

Therefore in general:

$$v_1.v_2 \geq c^2 \ (3)$$

Let us consider the second inequality

$$2)c^2v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| \leq -\sqrt{c^2{v_{1t}}^2 - |\vec{v}_1|^2}\sqrt{c^2{v_{12t}}^2 - |\vec{v}_2|^2}$$

If $v_1 = v_2$ this inequality reduces to $v^2 \leq -c^2$ which is not true.But we have used the signature $(+,-,-,-)$ So we may dismiss the second inequality

Alternative considerations:

If possible let

$$v_1.v_2 < c^2$$

Or,

$$c^2v_{1t}.v_{2t} - \vec{v}_1.\vec{v}_2 < c^2$$

We transform to a frame of reference where $\vec{v}_1 = 0$

We now have,

$$c^2v_{1t}.v_{2t} < c^2$$

Now

$$v_{1t} = \frac{dt_1}{d\tau} = \gamma_1; v_{2t} = \frac{dt_2}{d\tau} = \gamma_2$$

Therefore

$$c^2 v_{1t}.v_{2t} = c^2\gamma_1\gamma_2$$

$$c^2 v_{1t}.v_{2t} < c^2 \Rightarrow c^2\gamma_1\gamma_2 < c^2$$

$$\gamma_1\gamma_2 < 1$$

But $\gamma_1 = 1$since $\vec{v}_1 = 0$

$$\Rightarrow \gamma_2 < 1$$

The above relation is an impossible relation since γcannot be less than unity.

Therefore,

$$v_1.v_2 \geq c^2$$

Dot Product of Two Four Momenta

Next we pass on to the dot product of four momenta[3]. We prove

$$p_1.p_2 = \frac{E_1}{c}.\frac{E_2}{c} - \vec{p}_1.\vec{p}_2 \geq m_1c^2 \quad (4)$$

that is

$$E_1.E_2 - c^2|\vec{p}|^2 \geq m_1m_2c^4$$

If possible let

$$E_1.E_2 - c^2|\vec{p}|^2 < m_1m_2c^4$$

where m_1 and m_2 are the rest masses of the two particles.

We transform to a frame of reference where the velocity of the first particle, $\vec{v}_1 = 0$:$\Rightarrow$ $\vec{p}_1 = 0$

Therefore

$$E_1.E_1 < m_1m_2c^4$$

$$\Rightarrow m_1\gamma_1c^2m_2\gamma_2c^2 < m_1m_2c^4$$

$$\Rightarrow \gamma_1\gamma_2 < 1$$

But the product of the gammas cannot be less than unity

Therefore

$$p_1.p_2 \nless {m_1}^2c^2$$

Alternative considerations:

Let $a_1 = E_1; a_2 = c|\vec{p}_1|; b_1 = E_2; b_2 = c|\vec{p}_2|$

Considring (1) we have

$$\left(E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|\right)^2 \geq \left({E_1}^2 - c^2|\vec{p}_1|\right)\left({E_2}^2 - c^2|\vec{p}_2|\right) = {m_1}^2c^4{m_2}^2c^4$$

Therefore

$\left(E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|\right) \geq m_1m_2c^4$ or $\left(E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|\right) \leq -m_1m_2c^4$

If $p_1 = p_2$ we have for the second inequality

$${E_1}^2 - c^2|\vec{p}_1|^2 \leq -m_1m_2c^4$$

So we discard the second inequality

$$E_1E_2 - c^2|\vec{p}_1||\vec{p}_2| \geq \sqrt{{E_1}^2 - c^2|\vec{p}_1|}\sqrt{{E_2}^2 - c^2|\vec{p}_2|} = m_1m_2c^4$$

We have

$$E_1E_2 - c^2|\vec{p}_1||\vec{p}_2| \geq m_1m_2c^4$$

$$p_1.p_2 = E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|Cos\theta$$

$$E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|Cos\theta \geq E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|$$

Therefore

$$p_1.p_2 \geq m_1m_2c^2 \ (6)$$

$$E_1E_2 - c^2\vec{p}_1.\vec{p}_2 \geq m_1m_2c^4 \ (6')$$

The Extra Bit

[Recollected from one of my earlier writings]

Flat space time metric

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \ (7)$$

$$c^2 = c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2$$

$$c^2 = c^2 {v_t}^2 - {v_x}^2 - {v_y}^2 - {v_z}^2 \ (8)$$

v_i: proper velocity component

We consider two proper velocities on the same manifold at the same point

$$c^2 = c^2 {v_{1t}}^2 - {v_{1x}}^2 - {v_{1y}}^2 - {v_{1z}}^2 \ (9)$$

$$c^2 = c^2 {v_{2t}}^2 - {v_{2x}}^2 - {v_{2y}}^2 - {v_{2z}}^2 \ (10)$$

Proper velocities being tensors we can add them [to obtain other proper velocities]

$$c^2 = c^2 (v_{1t} + v_{2t})^2 - (v_{1x} + v_{2x})^2 - \left(v_{1y} + v_{2y}\right)^2 - (v_{1z} + v_{2z})^2 \ (11)$$

[NB: Any quadruplet that satisfies (8) is a certified four velocity. Therefore v_1, v_2 and $v = v_1 + v_2$ are all certified four velocities on the manifold if each satisfies an equation of the type (8): there are eight unknowns v_{1i}, v_{2j} while here are three equations:(9),(10) and (11)]

Now from equation (11) we have

$$c^2 = c^2 {v_{1t}}^2 - {v_{1x}}^2 - {v_{1y}}^2 - {v_{1z}}^2 + c^2 {v_{2t}}^2 - {v_{2x}}^2 - {v_{2y}}^2 - {v_{2z}}^2 + 2v_1 . v_2 \ (12)$$

$$c^2 = 2c^2 + 2v_1 . v_2 \ (13)$$

$$2v_1 . v_2 = -c^2 \ (14)$$

If $v_1 = v_2$ equation(14) stands as a contradiction to theory .It also stands as a contradiction to $v_1 . v_2 \geq c^2$

Given

$$c^2 = c^2 {v_t}^2 - {v_x}^2 - {v_y}^2 - {v_z}^2 \ (15)$$

In order to break up a proper velocity into two parts we have to solve the following equations

$$c^2 = c^2 {v_{1t}}^2 - {v_{1x}}^2 - {v_{1y}}^2 - {v_{1z}}^2 \ (16.1)$$

$$c^2 = c^2 {v_{2t}}^2 - {v_{2x}}^2 - {v_{2y}}^2 - {v_{2z}}^2 \ (16.2)$$

$$v_{1t} + v_{2t} = v_t (16.3)$$

$$v_{1x} + v_{2x} = v_x \ (16.4)$$

$$v_{1y} + v_{2y} = v_y \ (16.5)$$

$$v_{1z} + v_{2z} = v_z (16.6)$$

We have eight variables[unknowns],v_{1i} and ,v_{1i};i.j=1,2,3,4; and six equations (15) through (16.6) . There exists an infinitude of solutions each complying with equation of the type (10) or (15)

Each v_1, v_2 and $v = v_1 + v_2$ all satisfy equation (7) and hence they are certified proper velocities on the manifold. Interestingly from (7) we can pass to the metric. There is only one set of transformations corresponding to this metric: the Lorentz Transformations. This may be proved theoretically[4]. From the Lorentz Transformation we may deduce the metric. It is true the other way round also.]

Conclusion

As claimed we have that the four dot product between two arbitrary velocities is greater than the square of the speed of light in vacuum and the product of two four momenta is greater than the product of the two masses involved and the square of the speed of light[in vacuuo]. Finally we find a contradiction that was mentioned right at the outset

References

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