The Navier–Stokes problem

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A proposed solution to the millennium problem on the existence and smoothness of the Navier–Stokes equations.

1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^d$ where $d \in \mathbb{N}$, see [1,3,7]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^d$ be the fluid velocity and let $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the fluid pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^d$ and time $t \geqslant 0$. I take the externally applied force acting on the fluid to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of $\mathbb{R}^d$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \tag{3}$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^d$. In these equations

$$\nabla = (\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \dots, \frac{\partial}{\partial \mathbf{x}_d}) \tag{4}$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^{d} \frac{\partial^2}{\partial {\mathbf{x}_i}^2} \tag{5}$$

is the Laplacian operator. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + Le_i) = \mathbf{u}_0(\mathbf{x}) \tag{6}$$

for $1 \leqslant i \leqslant d$ where e_i is the i^{th} unit vector in $\mathbb{R}^d$ and $L > 0$ is a constant [7]. The initial condition $\mathbf{u}_0$ is a given C^∞ divergence-free vector field on $\mathbb{R}^d$. A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

$$\mathbf{u}(\mathbf{x} + Le_i, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + Le_i, t) = p(\mathbf{x}, t) \tag{7}$$

on $\mathbb{R}^d \times [0, \infty)$ for $1 \leqslant i \leqslant d$ and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^d \times [0, \infty)). \tag{8}$$

2. Solution to the Navier–Stokes problem

I provide a proof of the following theorem [2,3,6,7].
Theorem. Let $\mathbf{u}_0$ be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions $\mathbf{u}, p$ on $\mathbb{R}^d \times [0, \infty)$ that satisfy (1), (2), (3), (7), (8).
Proof. Let the Galerkin approximation of $\mathbf{u}, p$ be

$$\tilde{\mathbf{u}} = \sum_{\mathbf{L}=-\infty}^{\infty} \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}, \tag{9}$$

$$\tilde{p} = \sum_{\mathbf{L}=-\infty}^{\infty} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

respectively. Here $\mathbf{u}_{\mathbf{L}} = \mathbf{u}_{\mathbf{L}}(t) \in \mathbb{C}^d$, $p_{\mathbf{L}} = p_{\mathbf{L}}(t) \in \mathbb{C}$, $i = \sqrt{-1}$, $k = 2\pi/L$, and $\sum_{\mathbf{L}=-\infty}^{\infty}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^d$. The initial condition $\mathbf{u}_0$ is a Fourier series [2] of which is convergent for all $\mathbf{x} \in \mathbb{R}^d$. Substituting $\mathbf{u} = \tilde{\mathbf{u}}$, $p = \tilde{p}$ into (1) gives

$$\begin{aligned} &\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} e^{ik\mathbf{L}\cdot\mathbf{x}} + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}} e^{ik(\mathbf{L}+\mathbf{M})\cdot\mathbf{x}} \\ &= -\sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\infty}^{\infty} ik\mathbf{L} p_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}. \end{aligned} \tag{11}$$

Equating like powers of the exponentials in (11) yields

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_{\mathbf{M}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u}_{\mathbf{L}} - ik\mathbf{L} p_{\mathbf{L}} \tag{12}$$

on using the Cauchy product type formula [4]

$$\sum_{l=-\infty}^{\infty} a_l x^l \sum_{m=-\infty}^{\infty} b_m x^m = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{l-m} b_m x^l. \tag{13}$$

Substituting $\mathbf{u} = \tilde{\mathbf{u}}$ into (2) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} ik\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} = 0. \tag{14}$$

Equating like powers of the exponentials in (14) yields

$$\mathbf{L} \cdot \mathbf{u}_{\mathbf{L}} = 0. \tag{15}$$

Applying $\mathbf{L}\cdot$ to (12) and noting (15) leads to

$$p_{\mathbf{L}} = -\sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{16}$$

where $p_{\mathbf{0}}$ is arbitrary and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Then substituting (16) into (12) gives

$$\frac{\partial \mathbf{u}_{\mathbf{L}}}{\partial t} = -\sum_{\mathbf{M}=-\infty}^{\infty} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \mathrm{i}k\mathbf{M})\mathbf{u}_{\mathbf{M}} - \nu k^2|\mathbf{L}|^2\mathbf{u}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} \mathrm{i}k\mathbf{L}(\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_{\mathbf{M}} \cdot \hat{\mathbf{L}}) \tag{17}$$

where $\mathbf{u}_{\mathbf{0}} = \mathbf{u}_{\mathbf{0}}(0)$. Without loss of generality [2], I take $\mathbf{u}_{\mathbf{0}} = \mathbf{0}$. This is due to the Galilean invariance property of solutions to the Navier–Stokes equations. The equations for $\mathbf{u}_{\mathbf{L}}$ are to be solved for all $\mathbf{L} \in \mathbb{Z}^d$.
Let

$$\mathbf{u}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}} + \mathrm{i}\mathbf{b}_{\mathbf{L}}, \tag{18}$$

$$p_{\mathbf{L}} = c_{\mathbf{L}} + \mathrm{i}d_{\mathbf{L}} \tag{19}$$

where $\mathbf{a}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}}(t) \in \mathbb{R}^d$, $\mathbf{b}_{\mathbf{L}} = \mathbf{b}_{\mathbf{L}}(t) \in \mathbb{R}^d$, $c_{\mathbf{L}} = c_{\mathbf{L}}(t) \in \mathbb{R}$, and $d_{\mathbf{L}} = d_{\mathbf{L}}(t) \in \mathbb{R}$. Substituting (18), (19) into (12) gives

$$\begin{aligned}&\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + \mathrm{i}\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} + \mathrm{i}\mathbf{b}_{\mathbf{L}-\mathbf{M}}) \cdot \mathrm{i}k\mathbf{M})(\mathbf{a}_{\mathbf{M}} + \mathrm{i}\mathbf{b}_{\mathbf{M}})\\ &= -\nu k^2|\mathbf{L}|^2(\mathbf{a}_{\mathbf{L}} + \mathrm{i}\mathbf{b}_{\mathbf{L}}) - \mathrm{i}k\mathbf{L}(c_{\mathbf{L}} + \mathrm{i}d_{\mathbf{L}}).\end{aligned} \tag{20}$$

Equating real and imaginary parts in (20) gives

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (-(\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}}) = -\nu k^2|\mathbf{L}|^2\mathbf{a}_{\mathbf{L}} + k\mathbf{L}d_{\mathbf{L}}, \tag{21}$$

$$\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) = -\nu k^2|\mathbf{L}|^2\mathbf{b}_{\mathbf{L}} - k\mathbf{L}c_{\mathbf{L}}. \tag{22}$$

Substituting (18) into (15) gives

$$\mathbf{L} \cdot (\mathbf{a}_{\mathbf{L}} + \mathrm{i}\mathbf{b}_{\mathbf{L}}) = 0. \tag{23}$$

Equating real and imaginary parts in (23) gives

$$\mathbf{L} \cdot \mathbf{a}_{\mathbf{L}} = 0, \tag{24}$$

$$\mathbf{L} \cdot \mathbf{b}_{\mathbf{L}} = 0. \tag{25}$$

From (21) and in light of (24) it is possible to write

$$\frac{\partial \mathbf{a_L}}{\partial t} \cdot \hat{\mathbf{a}}_\mathbf{L} + \sum_{\mathbf{M}=-\infty}^{\infty} (-(\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{b_M} - (\mathbf{b_{L-M}} \cdot k\mathbf{M})\mathbf{a_M}) \cdot \hat{\mathbf{a}}_\mathbf{L} = -\nu k^2 |\mathbf{L}|^2 \mathbf{a_L} \cdot \hat{\mathbf{a}}_\mathbf{L} \quad (26)$$

where $\hat{\mathbf{a}}_\mathbf{L} = \mathbf{a_L}/|\mathbf{a_L}|$ is the unit vector in the direction of $\mathbf{a_L}$. Then (26) implies

$$\frac{\partial |\mathbf{a_L}|}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} (-(\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{b_M} - (\mathbf{b_{L-M}} \cdot k\mathbf{M})\mathbf{a_M}) \cdot \hat{\mathbf{a}}_\mathbf{L} = -\nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}|. \quad (27)$$

From (27) it is possible to write

$$\frac{\partial |\mathbf{a_L}|}{\partial t} \leqslant \sum_{\mathbf{M}=-\infty}^{\infty} (|\mathbf{a_{L-M}}| k |\mathbf{M}| |\mathbf{b_M}| + |\mathbf{b_{L-M}}| k |\mathbf{M}| |\mathbf{a_M}|) - \nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}| \quad (28)$$

on using the Cauchy–Schwarz inequality [5]

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|. \quad (29)$$

It then follows from (28) that

$$\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial |\mathbf{a_L}|}{\partial t} e^{k|\mathbf{L}|X} \leqslant \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a_{L-M}}| k |\mathbf{M}| |\mathbf{b_M}| e^{k|\mathbf{L}|X} + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b_{L-M}}| k |\mathbf{M}| |\mathbf{a_M}| e^{k|\mathbf{L}|X} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}| e^{k|\mathbf{L}|X} \quad (30)$$

where $0 \leqslant X \ll 1$, implying that

$$\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial |\mathbf{a_L}|}{\partial t} e^{k|\mathbf{L}|X} \leqslant \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a_L}| k |\mathbf{M}| |\mathbf{b_M}| e^{k|\mathbf{L}+\mathbf{M}|X} + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b_L}| k |\mathbf{M}| |\mathbf{a_M}| e^{k|\mathbf{L}+\mathbf{M}|X} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}| e^{k|\mathbf{L}|X} \quad (31)$$

in light of (13), which yields

$$\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial |\mathbf{a_L}|}{\partial t} e^{k|\mathbf{L}|X} \leqslant \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a_L}| k |\mathbf{M}| |\mathbf{b_M}| e^{k(|\mathbf{L}|+|\mathbf{M}|)X} + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b_L}| k |\mathbf{M}| |\mathbf{a_M}| e^{k(|\mathbf{L}|+|\mathbf{M}|)X} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}| e^{k|\mathbf{L}|X} \quad (32)$$

on using the triangle inequality [5]

$$|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|. \quad (33)$$

From (22) and in light of (25) it is possible to write

$$\frac{\partial \mathbf{b}_{\mathbf{L}}}{\partial t} \cdot \hat{\mathbf{b}}_{\mathbf{L}} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) \cdot \hat{\mathbf{b}}_{\mathbf{L}} = -\nu k^2 |\mathbf{L}|^2 \mathbf{b}_{\mathbf{L}} \cdot \hat{\mathbf{b}}_{\mathbf{L}} \quad (34)$$

where $\hat{\mathbf{b}}_{\mathbf{L}} = \mathbf{b}_{\mathbf{L}}/|\mathbf{b}_{\mathbf{L}}|$ is the unit vector in the direction of $\mathbf{b}_{\mathbf{L}}$. Then (34) implies

$$\frac{\partial |\mathbf{b}_{\mathbf{L}}|}{\partial t} + \sum_{\mathbf{M}=-\infty}^{\infty} ((\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{a}_{\mathbf{M}} - (\mathbf{b}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M})\mathbf{b}_{\mathbf{M}}) \cdot \hat{\mathbf{b}}_{\mathbf{L}} = -\nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}|. \quad (35)$$

From (35) it is possible to write

$$\frac{\partial |\mathbf{b}_{\mathbf{L}}|}{\partial t} \leqslant \sum_{\mathbf{M}=-\infty}^{\infty} (|\mathbf{a}_{\mathbf{L}-\mathbf{M}}| k |\mathbf{M}| |\mathbf{a}_{\mathbf{M}}| + |\mathbf{b}_{\mathbf{L}-\mathbf{M}}| k |\mathbf{M}| |\mathbf{b}_{\mathbf{M}}|) - \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}| \quad (36)$$

on using the Cauchy–Schwarz inequality. It then follows from (36) that

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial |\mathbf{b}_{\mathbf{L}}|}{\partial t} e^{k|\mathbf{L}|X} \leqslant & \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}-\mathbf{M}}| k |\mathbf{M}| |\mathbf{a}_{\mathbf{M}}| e^{k|\mathbf{L}|X} \\ & + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}-\mathbf{M}}| k |\mathbf{M}| |\mathbf{b}_{\mathbf{M}}| e^{k|\mathbf{L}|X} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}| e^{k|\mathbf{L}|X} \end{aligned} \quad (37)$$

implying that

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial |\mathbf{b}_{\mathbf{L}}|}{\partial t} e^{k|\mathbf{L}|X} \leqslant & \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}}| k |\mathbf{M}| |\mathbf{a}_{\mathbf{M}}| e^{k|\mathbf{L}+\mathbf{M}|X} \\ & + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}}| k |\mathbf{M}| |\mathbf{b}_{\mathbf{M}}| e^{k|\mathbf{L}+\mathbf{M}|X} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}| e^{k|\mathbf{L}|X} \end{aligned} \quad (38)$$

in light of (13), which yields

$$\begin{aligned}\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial |\mathbf{b}_{\mathbf{L}}|}{\partial t} e^{k|\mathbf{L}|X} \leqslant & \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}}| k |\mathbf{M}| |\mathbf{a}_{\mathbf{M}}| e^{k(|\mathbf{L}|+|\mathbf{M}|)X} \\ & + \sum_{\mathbf{L}=-\infty}^{\infty} \sum_{\mathbf{M}=-\infty}^{\infty} |\mathbf{b}_{\mathbf{L}}| k |\mathbf{M}| |\mathbf{b}_{\mathbf{M}}| e^{k(|\mathbf{L}|+|\mathbf{M}|)X} - \sum_{\mathbf{L}=-\infty}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{b}_{\mathbf{L}}| e^{k|\mathbf{L}|X} \end{aligned} \quad (39)$$

on using the triangle inequality.
Let

$$\psi = \sum_{\mathbf{L}=-\infty}^{\infty} |\mathbf{a}_{\mathbf{L}}| e^{k|\mathbf{L}|X}, \quad (40)$$

$$\phi = \sum_{\mathbf{L}=-\infty}^{\infty} |\mathbf{b_L}| e^{k|\mathbf{L}|X} \tag{41}$$

and note that $|\tilde{\mathbf{u}}| \leqslant Q$ where $Q = \psi + \phi$. Then (32) can be written as

$$\frac{\partial \psi}{\partial t} \leqslant \psi \frac{\partial \phi}{\partial X} + \phi \frac{\partial \psi}{\partial X} - \nu \frac{\partial^2 \psi}{\partial X^2} \tag{42}$$

and (39) can be written as

$$\frac{\partial \phi}{\partial t} \leqslant \psi \frac{\partial \psi}{\partial X} + \phi \frac{\partial \phi}{\partial X} - \nu \frac{\partial^2 \phi}{\partial X^2}. \tag{43}$$

Adding (42) and (43) yields

$$\frac{\partial Q}{\partial t} \leqslant Q \frac{\partial Q}{\partial X} - \nu \frac{\partial^2 Q}{\partial X^2}. \tag{44}$$

Now both

$$\tilde{\mathbf{a}}|_{t=0} = \sum_{\mathbf{L}=-\infty}^{\infty} \mathbf{a_L}(0) e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{45}$$

and

$$\tilde{\mathbf{b}}|_{t=0} = \sum_{\mathbf{L}=-\infty}^{\infty} \mathbf{b_L}(0) e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{46}$$

converge for all $\mathbf{x} \in \mathbb{R}^d$ since $\mathbf{u}_0 = \tilde{\mathbf{u}}|_{t=0} = \tilde{\mathbf{a}}|_{t=0} + i\tilde{\mathbf{b}}|_{t=0}$ is smooth. Then

$$\sum_{\mathbf{L}=-\infty}^{\infty} \{\mathbf{a_L}(0)\}_i^2 < \infty \tag{47}$$

for $1 \leqslant i \leqslant d$ in light of Theorem 3.5-2 of [5] which yields

$$\sum_{\mathbf{L}=-\infty}^{\infty} |\mathbf{a_L}(0)|^2 < \infty. \tag{48}$$

Then with $n \in \mathbb{N}$

$$\begin{aligned} \sum_{\mathbf{L}\neq\mathbf{0}} |\mathbf{a_L}(0)| &= \sum_{\mathbf{L}\neq\mathbf{0}} |\mathbf{a_L}(0)||\mathbf{L}|^n|\mathbf{L}|^{-n} \\ &\leqslant (\sum_{\mathbf{L}\neq\mathbf{0}} |\mathbf{a_L}(0)|^2|\mathbf{L}|^{2n})^{1/2}(\sum_{\mathbf{L}\neq\mathbf{0}} |\mathbf{L}|^{-2n})^{1/2} \end{aligned} \tag{49}$$

on using the Cauchy–Schwarz inequality. It can be found that there are less than cq^d vectors $\mathbf{L}$ with $|\mathbf{L}|^2 = q$ where $c = c(d)$. Then

$$\sum_{\mathbf{L}\neq\mathbf{0}} |\mathbf{L}|^{-2n} = \sum_{q=1}^{\infty} h_q q^{-n} \tag{50}$$

where h_q is the number of vectors $\mathbf{L}$ with $|\mathbf{L}|^2 = q$. Therefore

$$\sum_{\mathbf{L}\neq\mathbf{0}} |\mathbf{L}|^{-2n} < \sum_{q=1}^{\infty} c q^d q^{-n}. \tag{51}$$

It is possible to choose $n > d + 1 \in \mathbb{N}$ so that (50) converges. It then follows that

$$\sum_{\mathbf{L}\neq\mathbf{0}} |\mathbf{a}_{\mathbf{L}}(0)| < \infty \tag{52}$$

since $\mathbf{u}_0 = \tilde{\mathbf{u}}|_{t=0}$ is smooth. Likewise,

$$\sum_{\mathbf{L}\neq\mathbf{0}} |\mathbf{b}_{\mathbf{L}}(0)| < \infty. \tag{53}$$

Therefore $Q|_{X=t=0}$ converges. Similarly $\frac{\partial^2 Q}{\partial X^2}|_{X=t=0}$ converges and therefore $Q|_{t=0}$ converges for $0 \leqslant X \ll 1$. Note also that

$$\frac{\partial^s Q}{\partial X^s} \geqslant 0 \ \text{ for } \ s \geqslant 0. \tag{54}$$

At points where Q is a maximum,

$$\frac{\partial Q}{\partial t} \geqslant 0. \tag{55}$$

Equation (44) can be written as

$$\frac{\partial Q}{\partial t} - Q\frac{\partial Q}{\partial X} + \nu\frac{\partial^2 Q}{\partial X^2} = H \tag{56}$$

where $H = H(X,t) \leqslant 0$ can be thought of as a force. The extreme case is then $Q = \Omega$ where

$$\frac{\partial \Omega}{\partial t} = \Omega\frac{\partial \Omega}{\partial X} - \nu\frac{\partial^2 \Omega}{\partial X^2}. \tag{57}$$

Let

$$\Omega = \lambda\frac{\partial A}{\partial X}/A = \lambda\frac{\partial}{\partial X}\log_{\mathrm{e}} A \tag{58}$$

where λ is a constant. Substituting (58) into (57) gives

$$\lambda\frac{\partial}{\partial X}(\frac{\partial A}{\partial t}/A) = \lambda^2\frac{1}{2}\frac{\partial}{\partial X}((\frac{\partial A}{\partial X}/A)^2) - \lambda\nu\frac{\partial}{\partial X}((\frac{\partial^2 A}{\partial X^2}A - (\frac{\partial A}{\partial X})^2)/A^2). \tag{59}$$

Then with $\lambda = -2\nu$, equation (59) gives

$$\frac{\partial}{\partial X}(\frac{\partial A}{\partial t}/A) = -\nu\frac{\partial}{\partial X}(\frac{\partial^2 A}{\partial X^2}/A) \tag{60}$$

which leads to

$$\frac{\partial A}{\partial t} = -\nu \frac{\partial^2 A}{\partial X^2} + hA \tag{61}$$

where $h = h(t)$ is arbitrary.
The separation of variables method and the form of Q necessitates to let

$$A = \sum_{\mathbf{L}=-\infty}^{\infty} A_{\mathbf{L}} \mathrm{e}^{k|\mathbf{L}|X} \tag{62}$$

where $A_{\mathbf{L}} = A_{\mathbf{L}}(t)$. Substituting (62) into (61) gives

$$\sum_{\mathbf{L}=-\infty}^{\infty} \frac{\partial A_{\mathbf{L}}}{\partial t} \mathrm{e}^{k|\mathbf{L}|X} = -\nu \sum_{\mathbf{L}=-\infty}^{\infty} k^2 |\mathbf{L}|^2 A_{\mathbf{L}} \mathrm{e}^{k|\mathbf{L}|X} + h \sum_{\mathbf{L}=-\infty}^{\infty} A_{\mathbf{L}} \mathrm{e}^{k|\mathbf{L}|X}. \tag{63}$$

Equating like powers of the exponentials in (63) leads to

$$\frac{\partial A_{\mathbf{L}}}{\partial t} = -\nu k^2 |\mathbf{L}|^2 A_{\mathbf{L}} + A_{\mathbf{L}} h. \tag{64}$$

Equation (64) is easily solved to find

$$A_{\mathbf{L}} = A_{\mathbf{L}}(0) \mathrm{e}^{-\nu k^2 |\mathbf{L}|^2 t + \int_0^t h(\tau)\, d\tau}. \tag{65}$$

It then follows that

$$\Omega = \frac{\partial}{\partial X} \log_{\mathrm{e}}\Big(\big(\sum_{\mathbf{L}=-\infty}^{\infty} A_{\mathbf{L}}(0) \mathrm{e}^{-\nu k^2 |\mathbf{L}|^2 t} \mathrm{e}^{k|\mathbf{L}|X}\big)^{-2\nu}\Big). \tag{66}$$

Now with

$$\Omega = \sum_{\mathbf{L}=-\infty}^{\infty} \Omega_{\mathbf{L}} \mathrm{e}^{k|\mathbf{L}|X}, \;\; \Omega_{\mathbf{0}} = 0 \tag{67}$$

where $\Omega_{\mathbf{L}} = \Omega_{\mathbf{L}}(t) \geqslant 0$ it follows that

$$\begin{aligned} A|_{t=0} &= \mathrm{e}^{\int^X \frac{\Omega}{\lambda}\, dX}|_{t=0} \\ &= \mathrm{e}^{\frac{1}{\lambda} \sum_{\mathbf{L}\neq\mathbf{0}} \frac{\Omega_{\mathbf{L}}(0) \mathrm{e}^{k|\mathbf{L}|X}}{k|\mathbf{L}|}} \\ &= 1 + \frac{1}{\lambda} \sum_{\mathbf{L}\neq\mathbf{0}} \frac{\Omega_{\mathbf{L}}(0) \mathrm{e}^{k|\mathbf{L}|X}}{k|\mathbf{L}|} + \frac{1}{2}\Big(\frac{1}{\lambda} \sum_{\mathbf{L}\neq\mathbf{0}} \frac{\Omega_{\mathbf{L}}(0) \mathrm{e}^{k|\mathbf{L}|X}}{k|\mathbf{L}|}\Big)^2 + \ldots \end{aligned} \tag{68}$$

which is valid since $Q|_{t=0}$ converges for $0 \leqslant X \ll 1$. For consistency, matching (62) with (68) yields

$$A_{\mathbf{0}}(0) = 1, \;\; A_{\mathbf{L}}(0) = \frac{\Omega_{\mathbf{L}}(0)}{\lambda k |\mathbf{L}|} + O\Big(\frac{1}{\lambda^2 k^2}\Big) \;\text{ for }\; \mathbf{L} \neq \mathbf{0}. \tag{69}$$

Then (66) becomes

$$\Omega = \frac{\partial}{\partial X}\log_{\mathrm{e}}(A^{\lambda}) \tag{70}$$

where

$$A = 1 + \sum_{\mathbf{L}\neq\mathbf{0}}\left(\frac{\Omega_{\mathbf{L}}(0)}{\lambda k|\mathbf{L}|} + O\left(\frac{1}{\lambda^2 k^2}\right)\right)\mathrm{e}^{-\nu k^2|\mathbf{L}|^2 t}\mathrm{e}^{k|\mathbf{L}|X} = \mathrm{e}^{\int^X \frac{\Omega}{\lambda}\,dX}. \tag{71}$$

Equation (71) can be written as

$$\begin{aligned} A \;=\; & 1 + \Big\{\frac{1}{2}\left(\frac{1}{\lambda k}\right)^2 \sum_{\mathbf{L}\neq\mathbf{0}}\sum_{\mathbf{M}\neq\mathbf{0}} \frac{\Omega_{\mathbf{L}}(0)\Omega_{\mathbf{M}}(0)}{|\mathbf{L}||\mathbf{M}|}\mathrm{e}^{k(|\mathbf{L}|+|\mathbf{M}|)X}\mathrm{e}^{-\nu k^2(|\mathbf{L}|+|\mathbf{M}|)^2 t} \\ & + \frac{1}{24}\left(\frac{1}{\lambda k}\right)^4 \sum_{\mathbf{L}\neq\mathbf{0}}\sum_{\mathbf{M}\neq\mathbf{0}}\sum_{\mathbf{N}\neq\mathbf{0}}\sum_{\mathbf{P}\neq\mathbf{0}} \frac{\Omega_{\mathbf{L}}(0)\Omega_{\mathbf{M}}(0)\Omega_{\mathbf{N}}(0)\Omega_{\mathbf{P}}(0)}{|\mathbf{L}||\mathbf{M}||\mathbf{N}||\mathbf{P}|} \\ & \times \mathrm{e}^{k(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|+|\mathbf{P}|)X}\mathrm{e}^{-\nu k^2(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|+|\mathbf{P}|)^2 t} + \ldots\Big\} + \Big\{\left(\frac{1}{\lambda k}\right)\sum_{\mathbf{L}\neq\mathbf{0}}\frac{\Omega_{\mathbf{L}}(0)}{|\mathbf{L}|}\mathrm{e}^{k|\mathbf{L}|X}\mathrm{e}^{-\nu k^2|\mathbf{L}|^2 t} \\ & + \frac{1}{6}\left(\frac{1}{\lambda k}\right)^3 \sum_{\mathbf{L}\neq\mathbf{0}}\sum_{\mathbf{M}\neq\mathbf{0}}\sum_{\mathbf{N}\neq\mathbf{0}} \frac{\Omega_{\mathbf{L}}(0)\Omega_{\mathbf{M}}(0)\Omega_{\mathbf{N}}(0)}{|\mathbf{L}||\mathbf{M}||\mathbf{N}|}\mathrm{e}^{k(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|)X}\mathrm{e}^{-\nu k^2(|\mathbf{L}|+|\mathbf{M}|+|\mathbf{N}|)^2 t} \\ & + \ldots\Big\}. \end{aligned} \tag{72}$$

In light of (72) and due to $A \in [0, 1]$ from (71) it is then clear that A increases with increasing $t \geqslant 0$. This is more easily seen to be the case by applying the Cauchy product type formula to (72). It then follows that Ω has no finite-time singularity at $X = 0$ and $|\tilde{\mathbf{u}}| \leqslant \Omega|_{X=0}$. Similarly it can be shown that $\frac{\partial^2\Omega}{\partial X^2}$ has no finite-time singularity at $X = 0$ and $|\nabla^2\tilde{\mathbf{u}}| \leqslant \frac{\partial^2\Omega}{\partial X^2}|_{X=0}$. Then $\sum_{\mathbf{L}=-\infty}^{\infty} k^4|\mathbf{L}|^4|\mathbf{u}_{\mathbf{L}}|^2$ converges for all $t \geqslant 0$ in light of Theorem 3.5-2 of [5]. It then follows that [2]

$$\sup_{0\leqslant t\leqslant T} \nu \sum_{\mathbf{L}=-\infty}^{\infty} |\mathbf{L}|^2|\mathbf{u}_{\mathbf{L}}|^2 < \infty \tag{73}$$

for all $T \geqslant 0$. Therefore the theorem is true. □

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