$\pi(n)$ and the sum of consecutive prime numbers

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Abstract

In this paper it is proved that the sum of consecutive prime numbers up to the square root of a given natural number $S(n)$ is asymptotically equivalent to the prime counting function $\pi(n)$. Also, they are found some solutions such that $\pi(n) = S(n)$. Finally, they are listed the prime numbers p_k such that $\pi(p_k) = S(p_k)$, and exposed some conjectures regarding this type of prime numbers.

1 Introduction

We define the prime counting function up to a given natural number n as

$$\pi(n) = \#\{p \in P \mid p \leq n\}$$

We define the sum of consecutive prime numbers up to the integer part of the square root of a given natural number n as

$$S(n) = \sum_{p \leq \sqrt{n}} p \tag{1}$$

We define p_k as the last prime number which is a term of $S(n)$.

We define set Q as the set of values of n such that $\pi(n) = S(n)$.

We define set M as the set of values of n such that $\pi(n) = S(n)$and n is some prime number.

2 Asymptotic equivalence of $\pi(n)$ and $S(n)$

It can be stated the following

Theorem.

$$S(n) \sim \pi(n) \tag{2}$$

Proof.

By partial summation

$$S(n) = \left(\lfloor\sqrt{n}\rfloor \, \pi(\sqrt{n})\right) - \sum_{m=2}^{\lfloor\sqrt{n}\rfloor - 1} \pi(m) \tag{3}$$

Where $\lfloor\sqrt{n}\rfloor$ denotes the integer part of $\sqrt{n}$.

By the Prime Number Theorem with error term, there exists a constant C such that

$$\left|\pi(x) - \frac{x}{\log x}\right| \le C \frac{x}{\log^2 x} \qquad \textbf{for } x \ge 2 \tag{4}$$

Therefore, substituting $\pi(\sqrt{n})$ and $\pi(m)$ by the application of the Prime Number Theorem on (3)

$$S(n) = \left(\lfloor\sqrt{n}\rfloor \frac{\sqrt{n}}{\log(\sqrt{n})}\right) - \sum_{m=2}^{\lfloor\sqrt{n}\rfloor - 1} \frac{m}{\log(m)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right) \tag{5}$$

Applying Riemman Sums theory to the sum on the right of (3)

$$\sum_{m=2}^{\lfloor\sqrt{n}\rfloor - 1} \frac{m}{\log(m)} = \int_2^{\lfloor\sqrt{n}\rfloor} \frac{x}{\log(x)} dx + O\left(\frac{n}{\log^2(\sqrt{n})}\right) \tag{6}$$

Solving the integral by partial integration, we have that

$$\int_2^{\lfloor\sqrt{n}\rfloor} \frac{x}{\log(x)} dx = \left[\frac{x^2}{2\log(x)}\right]_2^{\lfloor\sqrt{n}\rfloor} + \int_2^{\lfloor\sqrt{n}\rfloor} \frac{x}{2\log^2(x)} =$$

$$= \frac{n}{2\ln(\lfloor\sqrt{n}\rfloor)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right) \tag{7}$$

It is easy to see that

$$\frac{n}{2\log\left(\lfloor\sqrt{n}\rfloor\right)} \sim \frac{n}{\log n} \tag{8}$$

Thus

$$\sum_{m=2}^{\lfloor\sqrt{n}\rfloor-1} \frac{m}{\log(m)} \sim \frac{n}{\log(n)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right) \tag{9}$$

Regarding the left product on (3) it can be seen that

$$\lfloor\sqrt{n}\rfloor \frac{\sqrt{n}}{\log(\sqrt{n})} \sim \frac{n}{\log(\sqrt{n})} = \frac{n}{\frac{1}{2}\log(n)} = \frac{2n}{\log(n)} \tag{10}$$

Substituting (9) and (10) on (3), we have that

$$S(n) \sim \frac{2n}{\log(n)} - \frac{n}{\log(n)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right) \tag{11}$$

As

$$\frac{2n}{\log(n)} - \frac{n}{\log(n)} = \frac{n}{\log(n)} \tag{12}$$

Thus

$$S(n) \sim \frac{n}{\log(n)} \tag{13}$$

And subsequently, as by the Prime Number Theorem,

$$\pi(n) \sim \frac{n}{\log(n)} \tag{14}$$

It can be stated that

$$S(n) \sim \pi(n) \tag{15}$$

3 The existence of solutions $\pi(n) = S(n)$

After noticing the Theorem exposed at the Introduction Section, it has been studied the set Q of solutions such that $\pi(n) = S(n)$.

As a result, it has been found that Q is non empty, and that the first solutions are

$\{Q\}$	n	$\pi(n) = S(n)$
q_1	11	5
q_2	12	5
q_3	29	10
q_4	30	10
q_5	59	17
q_6	60	17
q_7	179	41
q_8	180	41
q_9	389	77
...	...	...

It can be easily noticed that the first value of n with a concrete $\pi(n) = S(n)$ seems to be always a prime number. As the prime counting function up to some composite number equals the prime counting function up to the inmediate prior prime number, considering the set $M = \{m_1, m_2, ..., m_k\}$ as the set of values of n such that $\pi(n) = S(n)$and n is some prime number, if $\pi(m_k = p_n) = S(m_k = p_n)$, then, as $\pi(m_k) = \pi(m_k + 1) = \pi(m_k + 2) = ... = \pi(p_{n+1} - 1)$, it follows that all the composite numbers between m_k and p_{n+1} are intersection points.

4 Some conjectures regarding the solutions $\pi(n) = S(n)$

It can be conjectured that the first value of n with a concrete $\pi(n) = S(n)$ will be always a prime number. This conjecture assumes the truth of the following

Conjecture. *It does not exist any squared prime number p^2 such that $\pi\left(p^2\right) = S\left(p^2\right)$except of $p_1 = 2$. That is,*

$$\pi\left(p_n^2\right) \neq \sum_{k=1}^{n} p_k$$

If the Conjecture were false, then it could happen that $S\left(p_n < p^2\right) = S\left(p^2\right) - p$, so it would imply that $S\left(p_n\right) = S\left(p_n + 1\right) = S\left(p_n + 2\right) = ... = S\left(p^2 - 1\right) = S\left(p^2\right) - p$, and if $\pi\left(p_n\right) = S\left(p^2\right)$, then suddenly $\pi\left(p^2\right) = S\left(p^2\right)$, and $p^2 \in Q$, whereas p_n does not, and p^2 would be the first of a series of consecutive elements of Q until p_{n+1}.

The conjecture has been tested and found to be true for the first thousands of primes.

If we focus only on set M, we get the following table

$\{M\}$	n	$\pi(n) = S(n)$	p_k	k
m_1	11	5	3	2
m_2	29	10	5	3
m_3	59	17	7	4
m_4	179	41	13	6
m_5	389	77	19	8
m_6	541	100	23	9
m_7	5399	712	73	21
m_8	12401	1480	109	29
m_9	13441	1593	113	30
m_{10}	40241	4227	199	46
m_{11}	81619	7982	283	61
m_{12}	219647	19580	467	91
m_{13}	439367	36888	661	121
m_{14}	1231547	95165	1109	186
m_{15}	1263173	97405	1123	188
m_{16}	1279021	98534	1129	189
m_{17}	1699627	128112	1303	213
m_{18}	1718471	129419	1307	214
m_{19}	1756397	132059	1321	216
...	...	...	...	...

It can be seen that the set of k values is dense enough to formulate the following

Conjecture. *Set M has infinitely many elements.*

As $M \subset Q$, the Conjecture implies that $\pi(n)$ intersects $S(n)$ infinitely many times, so $S(n)$ is not only asymptotically equivalent to $\pi(n)$: it is infinitely many times equal to $\pi(n)$.

Finally, it implies also that the number of primes between p_n^2 and p_{n+1}^2 , on average, do not differ much from p_{n+1}.

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