Catalan's constant and Pi: Integrals

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ABSTRACT. Double integrals

I. Introduction

Catalan's constant is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\,n+1)^2} \tag{1}$$

the number Pi is defined by

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1} \tag{2}$$

this note presents some double integrals involving G and π .

II. Integrals

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}}{\cosh\left(x+y-\frac{i\,\pi}{4}\right)}\,dx\,dy = \frac{G}{2\sqrt{2}} + \frac{\pi^2}{96\sqrt{2}} + i\left(\frac{G}{2\sqrt{2}} - \frac{\pi^2}{96\sqrt{2}}\right) \tag{3}$$

$$\int_0^\infty \int_0^\infty \frac{e^{-x-y}}{\cosh\left(x+y+\frac{i\,\pi}{4}\right)}\,dx\,dy = \frac{G}{2\sqrt{2}} + \frac{\pi^2}{96\sqrt{2}} - i\left(\frac{G}{2\sqrt{2}} - \frac{\pi^2}{96\sqrt{2}}\right) \tag{4}$$

$$\int_0^\infty \int_0^\infty (e^{-x-y}/(\cosh(x+y) - i\sinh(x+y)))\,dx\,dy = \frac{G}{4} + \frac{\pi^2}{192} + i\left(\frac{G}{4} - \frac{\pi^2}{192}\right) \tag{5}$$

$$\int_0^\infty \int_0^\infty (e^{-x-y}/(\cosh(x+y) + i\sinh(x+y)))\,dx\,dy = \frac{G}{4} + \frac{\pi^2}{192} - i\left(\frac{G}{4} - \frac{\pi^2}{192}\right) \tag{6}$$

$$\int_0^\infty \int_0^\infty \frac{(x+y)\,e^{-x}}{\cosh(x+y)}\,dx\,dy = 2\,G - \frac{\pi^2}{24} \tag{7}$$

$$\int_0^\infty \int_0^\infty \frac{(x+y)\,(e^{-x}+e^{-y})}{\cosh(x+y)}\,dx\,dy = 4\,G - \frac{\pi^2}{12} \tag{8}$$

$$\int_0^\infty\int_0^\infty \frac{(x+y)\cosh(x-y)\, e^{-x-y}}{\cosh(2x+2y)}\, dx\, dy = \frac{G}{4} - \frac{\pi^2}{192} \tag{9}$$

$$\int_0^\infty\int_0^\infty \frac{e^{-x-y}\cosh(x+y)}{\cosh(2x+2y)}\, dx\, dy = \frac{G}{4} + \frac{\pi^2}{192} \tag{10}$$

$$\int_0^\infty\int_0^\infty \frac{e^{-x-y}\sinh(x+y)}{\cosh(2x+2y)}\, dx\, dy = \frac{G}{4} - \frac{\pi^2}{192} \tag{11}$$

$$\int_0^\infty\int_0^\infty \frac{\tanh(x)\tanh(y)}{\cosh(x)\cosh(y)+i}\, dx\, dy = G - \frac{\pi^2}{48}\, i \tag{12}$$

$$\int_0^\infty\int_0^\infty \frac{\tanh(x)\tanh(y)}{\cosh(x)\cosh(y)-i}\, dx\, dy = G + \frac{\pi^2}{48}\, i \tag{13}$$

$$\int_0^\infty\int_0^\infty \frac{e^{x+y}}{\sinh(2x+2y)}\, dx\, dy = \frac{\pi^2}{8} + G \tag{14}$$

$$\int_0^\infty\int_0^\infty \frac{e^{-x-y}}{\sinh(2x+2y)}\, dx\, dy = \frac{\pi^2}{8} - G \tag{15}$$

$$\int_0^\infty\int_0^\infty \left(\frac{1}{\sinh(x+y)} + \frac{1}{\cosh(x+y)}\right) dx\, dy = \frac{\pi^2}{4} + 2G \tag{16}$$

$$\int_0^\infty\int_0^\infty \left(\frac{1}{\sinh(x+y)} - \frac{1}{\cosh(x+y)}\right) dx\, dy = \frac{\pi^2}{4} - 2G \tag{17}$$

Remark: $i = \sqrt{-1}$.

Final integrals:

$$\frac{\pi^2}{4} = \int_0^\infty\int_0^\infty \frac{1}{\sinh(x+y)}\, dx\, dy \;\; ; \;\; 2G = \int_0^\infty\int_0^\infty \frac{1}{\cosh(x+y)}\, dx\, dy \tag{18}$$

$$\frac{G}{4} - \frac{\pi^2}{192} = \int_0^\infty \frac{x\, e^{-x}\sinh(x)}{\cosh(2x)}\, dx = \int_0^\infty \frac{x\,(1-e^{-2x})}{2\cosh(2x)}\, dx \tag{19}$$

III. References

[1] Guillera, J. and Sondow, J.: Double Integrals and Infinite Products for Some Classical Constants Via Analytic Continuations of Lerch's Transcendent. 16 June 2005. http//arxiv.org/abs/math.NT/0506319 .