Minimal Fractal Manifold as Foundation of Quantum Information Theory

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Abstract

Derived from the mathematics of the Renormalization Group, the minimal fractal manifold (MFM) represents a spacetime continuum endowed with arbitrarily small deviations from four dimensions ($\varepsilon = 4 - D << 1$). The geometrical structure of the MFM can be conveniently formulated using the concept of *dimensional quaternion*, a vector-like entity built from component deviations along the four spacetime coordinates. Our analysis shows that dimensional quaternions form a natural basis for qubit systems and Quantum Information Theory.

Key words: minimal fractal manifold, quaternion, qubit, entanglement, Quantum Information Theory.

1. Dimensional quaternions from the topological entropy

It was shown in [1] that the topological entropy of a geometrical object of normalized size r covered by M measuring boxes is given by

$$S_0(r) = \ln M \tag{1}$$

By definition, the box-counting dimension of the same object is

$$D_0 \approx \frac{\ln M}{\ln r} \Rightarrow M \approx r^{D_0} = \varepsilon^{-D_0} \tag{2}$$

in which $\varepsilon = r^{-1}$ stands for the normalized size of the box. The dimension of ordinary Euclidean space corresponds to integer and positive-definite values of the box-counting dimension, $D_0 = k,\ k = 0,1,2\dots$. By contrast, the box-counting dimension of non-smooth objects assumes non-integer values, an often-cited hallmark of fractal geometry and nonlinear dynamics.

Comparing (1) to (2) leads to the connection between the box-counting dimension and topological entropy via

$$\varepsilon^{-D_0} = \exp[S_0(r)] \tag{3}$$

Two straightforward conclusions may be drawn from (3):

- Maximal topological entropy ($S_0(r) \to \infty$) matches the limit $\varepsilon \to 0$ and corresponds to the four-dimensional continuum of both General Relativity and Quantum Field Theory.
- The steady growth of topological entropy along the Renormalization Group flow implies that, near or above the Fermi scale, spacetime exhibits a *continuous* spectrum of dimensions, described through $\varepsilon = 4 - D_0 << 1$ and asymptotically reaching $D_0 = 4$ as $\varepsilon \to 0$ [2].

The link between topological entropy and fractal dimension $D_0 = 4 - \varepsilon$ means that the deviation ε acts as an *inherent carrier of information,* an attribute which may be readily translated to qubits and Quantum Information Theory. To uncover this connection, we proceed with the straightforward assumption that component deviations along the four spacetime coordinates are independent from each other. The overall deviation ε amounts to the sum [3]

$$\varepsilon = 4 - D_0 = \sum_{\mu} \varepsilon_{\mu} = \sum_{\mu} (1 - D_{\mu}),\ \ \mu = 0,1,2,3 \tag{4}$$

Using the change of variables,

$$\varepsilon^{1/2} = \xi, \ \ \varepsilon_\mu^{1/2} = \xi_\mu \tag{5}$$

opens the possibility of casting the MFM formalism in the language of quaternions. The *dimensional quaternion* (q) and its conjugate $(\bar{q})$ are accordingly defined as

$$\boxed{q = \xi_0 + i\xi_1 + j\xi_2 + k\xi_3} \tag{6a}$$

$$\boxed{\bar{q} = \xi_0 - i\xi_1 - j\xi_2 - k\xi_3} \tag{6b}$$

with magnitude and norm given by, respectively,

$$|q| = \sqrt{\varepsilon} = \sqrt{\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2} = \sqrt{\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3} \tag{7}$$

$$|q|^2 = q\bar{q} = \varepsilon = \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{8}$$

The next two sections delve into the construction of single and two-qubit systems from the dimensional quaternions (6).

2. Single qubit representation from dimensional quaternions

Since unit quaternions form a group that is isomorphic to the $SU(2)$ group, they naturally reflect the behavior of pure spin-$1/2$ quantum states and qubits [4]. It is known, in this context, that the *Bloch sphere* provides the appropriate geometrical representation of qubits. This is because qubits are arbitrary points along the surface of the Bloch sphere, whose "north" and "south" poles correspond to the binary states $|0\rangle$ and $|1\rangle$, respectively. The representation of qubits in spherical coordinates (φ,θ) is given by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \ , \quad \alpha,\beta \in \mathbf{C} \tag{9}$$

where

$$\alpha = e^{-i\varphi/2}\cos(\theta/2) \tag{10}$$

$$\beta = e^{i\varphi/2}\sin(\theta/2) \tag{11}$$

$$|\alpha|^2 + |\beta|^2 = 1 \tag{12}$$

The so-called *Hopf map* enables conversion of the complex coefficients α,β to a triplet of coordinates on the Bloch sphere, namely [5]

$$x_1 = 2\,\mathrm{Re}(\overline{\alpha}\beta) \tag{13a}$$

$$x_2 = 2\,\mathrm{Im}(\overline{\alpha}\beta) \tag{13b}$$

$$x_3 = |\alpha|^2 - |\beta|^2 \tag{13c}$$

The Hopf map works by taking points on the 3-sphere

$$S^3 = \{(X_1, X_2, X_3, X_4\} : X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1\} \tag{14}$$

to points on a 2-sphere

$$S^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\} \tag{15}$$

according to the following prescription [6]

$$x_1 = 2(X_1 X_2 + X_3 X_4) \tag{16a}$$

$$x_2 = 2(X_1 X_4 - X_2 X_3) \tag{16b}$$

$$x_3 = (X_1^2 + X_3^2) - (X_2^2 + X_4^2) \tag{16c}$$

As any other quaternions, (6) represent points on the unit 3-sphere and are subject to the normalization constraint (8) for $|q|^2 = 1$. Combined use of (13) and (14-16), along with the identification $X_{\mu+1} = \xi_\mu$, yields the connection between qubits and the dimensional quaternions (6), expressed symbolically as

$$\boxed{(\alpha, \overline{\alpha}) \Leftrightarrow (\xi_\mu)} \tag{17a}$$

$$\boxed{(\beta, \overline{\beta}) \Leftrightarrow (\xi_\mu)} \tag{17b}$$

3. Quantum entanglement from dimensional quaternions

Following [7] in detail, consider now the case of a two-qubit system whose state is defined as

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \chi|10\rangle + \delta|11\rangle , \quad \alpha, \beta, \chi, \delta \in \mathbf{C} \tag{18}$$

where

$$|\alpha|^2 + |\beta|^2 + |\chi|^2 + |\delta|^2 = 1 \tag{19}$$

and

$$\alpha = \alpha_R + i\alpha_I\,, \quad \beta = \beta_R + i\beta_I\,, \quad \chi = \chi_R + i\chi_I\,, \quad \delta = \delta_R + i\delta_I \tag{20}$$

The qubits are entangled if $\alpha\delta - \beta\chi = 0$ and disentangled otherwise. We introduce the following pair of dimensional quaternions

$$\xi_A = \alpha_R + \alpha_I\mathbf{i} + \beta_R\mathbf{j} + \beta_I\mathbf{k} \tag{21a}$$

$$\xi_B = \chi_R + \chi_I\mathbf{i} + \delta_R\mathbf{j} + \delta_I\mathbf{k} \tag{21b}$$

subject to

$$\left|\xi_A\right|^2 + \left|\xi_B\right|^2 = 1 \tag{22}$$

The explicit expression for the quaternion equivalent to the two-qubit (18) can be shown to assume the form

$$Q(\xi_A, \xi_B) = 2(\xi_A \overline{\xi_B}) = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k} \tag{23}$$

in which

$$Q_0 = 2\,\mathrm{Re}(\overline{\alpha}\chi + \overline{\beta}\delta) \tag{24a}$$

$$Q_1 = 2\,\mathrm{Im}(\overline{\alpha}\chi + \overline{\beta}\delta) \tag{24b}$$

$$Q_2 = 2\,\mathrm{Re}(\alpha\delta - \beta\chi) \tag{24c}$$

$$Q_3 = 2\,\mathrm{Im}(\alpha\delta - \beta\chi) \tag{24d}$$

and

$$\sqrt{1-|Q|^2} = |\xi_A|^2 - |\xi_B|^2 \tag{25}$$

These considerations suggest that *quantum entanglement* – a fundamental aspect of Quantum Mechanics – is deeply related to the non-local and scale-invariant properties of the MFM, as embodied in the dimensional deviation $\varepsilon = 4 - D$ [2].

References

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