Decomposition of the Riemann and Riccci curvatures

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November 5, 2019

Abstract

We propose a decomposition of the Riemann curvature due to the Bianchi identities. We deduce a decomposition of the Ricci curvature.

1 The Riemann and Ricci curvatures

In the case of a riemannian manifold (M,g), the Riemann curvature R can be defined with help of the Levi-Civita connection. The Ricci curvature Ric is then computed as a contraction of the Riemann curvature with respect of the metric. The Einstein equations are [Be]:

$$Ric = \lambda g$$

2 Decomposition of the Riemann curvature

The Riemann curvature R possesses symmetries like the Bianchi identities. The first Bianchi identity seems to imply that:

$$g(R(x,y)z,t) = \sum_i \pm[g(S_i(x),z)g(S_i(y),t) - g(S_i(x),t)g(S_i(y),z)]$$

with S_i symmetric endomorphisms of the tangent bundle $S_i^* = S_i$. The second identity of Bianchi is implied if the endomorphisms S_i are parallel with respect of the Levi-Civita connection $\nabla S_i = 0$.

3 Decomposition of the Ricci curvature

The Ricci curvature is decomposed following the decomposition of the Riemann curvature:

$$Ric = \sum_i \pm[tr(S_i)S_i - S_i^2]$$

in the case of an Einstein manifold, it can be supposed that $\nabla S_i = 0$.

References

[Be] A.Besse, "Einstein Manifolds", Springer Verlag, Berlin, 1987.