Relations for Massive Spinors

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Recently introduced massive spinors are written as 2-vectors consisting of two massless spinors with opposite helicities. With this notation a couple of relations between them can be derived easily, entirely avoiding the spinor indices. The high energy limit of three point amplitudes is discussed shortly. Finally we add some comments on recursion relations with massive particles.

1. Introduction

The spinor helicity formalism, see for example the reviews in [1],[2],[3],[4], has boosted the calculation of amplitudes in particle physics. Amplitudes that could not be done even with computers can now be calculated with much less effort. But the advantage is not only on the side of faster calculations. Feynman diagrams relying on manifest Lorentz invariance of the Lagrangian, describe massless spin one bosons like photons or gluons by a vector with four components and the massless spin two graviton by a symmetric rank two tensor with ten components. Massless states however have only two helicities, positive and negative. This redundancy in the description necessarily requires gauge and diffeomorphism invariance of the gauge and graviton field. The redundancy appears already at the level of scalar fields in the form of field redefinitions [4]. Graviton physics becomes very complicated with the Lagrangian formalism, see for example the complicated term for the interaction between fermions and gravitons in [5] or the infinitely many terms for graviton selfinteractions. Compare this with the simple expressions for gravity amplitudes in literature.

The spinor helicity formalism had one limitation, it was only valid for massless particles and thus could only serve as an approximation for massive particles in the high energy regime, where their mass can be neglected. Massive spinor helicity variables were first introduced by several authors, see for example [6] and related work. In their seminal work Arkani-Hamed, Huang and Huang extended the spinor helicity formalism to amplitudes for all masses and spins [7]. For massless particles with momentum $p^\mu = \begin{pmatrix} P & 0 & 0 & P \end{pmatrix}$ the little group is $U(1)$, while for massive particles with momentum $p^\mu = \begin{pmatrix} m & 0 & 0 & 0 \end{pmatrix}$ in the restframe the little group is $SU(2)$. Massive particles are described as 2x2 matrices $\lambda_\alpha^I, \tilde{\lambda}_\alpha^I$ with $\alpha, \dot{\alpha}$ denoting the $SL(2,\mathbb{C})$ indices and I, J denoting the $SU(2)$ spin indices. Many following papers have investigated amplitudes within this new formalism, see for example [8],[9],[10],[11] and many others.

Here we make a minor step in formulating massive spinors as 2-vectors consisting of two massless spinors with opposite helicities. Of course this is already implicit in [7] and was also suggested in [12]. We shall find that many relations between massive spinors can be derived easily with this. The high energy limit of three particle amplitudes is discussed. Finally some comments on recursion relations are made.

2. Relations between massive spinors by 2-vectors

We use the representation of massive spinors given in [10],[11] with mostly minus metric and the four momentum given by

$$p_\mu = \begin{pmatrix} E & P\sin(\theta)\cos(\phi) & P\sin(\theta)\sin(\phi) & P\cos(\theta) \end{pmatrix} \tag{1}$$

Using the Pauli matrices, the momentum can be written in spinor notation $p = p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu$ and $\overline{p} = p^{\dot{\alpha}\alpha} = p_\mu \overline{\sigma}^\mu$,

$$p = p_{\alpha\dot{\alpha}} = \begin{pmatrix} E + P(cc - ss^*) & 2Pcs^* \\ 2Pcs & E - P(cc - ss^*) \end{pmatrix}, \; \overline{p} = p^{\dot{\alpha}\alpha} = \begin{pmatrix} E - P(cc - ss^*) & -2Pcs^* \\ -2Pcs & E + P(cc - ss^*) \end{pmatrix} \tag{2}$$

where as usual $c=\cos(\theta/2)$, $s=\sin(\theta/2)\mathrm{e}^{i\phi}$, $s^*=\sin(\theta/2)\mathrm{e}^{-i\phi}$. Now we write the massive spinors given in [7],[10],[11] as 2-vectors for example $|\,\mathrm{i}^{\mathrm{I}}\rangle=|\mathrm{p}_{\mathrm{i}}^{\mathrm{I}}\rangle=(|\,\mathrm{i}\,\rangle\ \ |\mathrm{n}_{\mathrm{i}}\rangle)$. The massless spinor $|\,\mathrm{i}\,\rangle$ scales with $\sqrt{\mathrm{E}_{\mathrm{i}}+\mathrm{P}_{\mathrm{i}}}$ and is denoted in the same way as the corresponding spinor for massless particles scaling with $\sqrt{2\mathrm{E}_{\mathrm{i}}}$ because they are equal in the high energy limit. This should in general not generate any confusion, since one knows for any amplitude which particles are massive and which are massless. One could attach an index 0 for massless particles if necessary. The second massless spinor $|\mathrm{n}_{\mathrm{i}}\rangle$, (memo n = nullvector) was denoted as $|\eta_{\mathrm{i}}\rangle$ in [7],[10],[11], and scales with $\sqrt{\mathrm{E}_{\mathrm{i}}-\mathrm{P}_{\mathrm{i}}}$ and therefore vanishes in the high energy limit. We now write down all possible massive spinors in the 2-vector notation,

$$|\,\mathrm{i}^{\mathrm{I}}\rangle=(|\,\mathrm{i}\,\rangle\ \ |\,\mathrm{n}_{\mathrm{i}}\rangle)\ ,\ \langle\mathrm{i}^{\mathrm{I}}|=(\langle\,\mathrm{i}\,|\ \ \langle\,\mathrm{n}_{\mathrm{i}}|),\ |\mathrm{i}^{\mathrm{I}}]=(-|\,\mathrm{n}_{\mathrm{i}}]\ \ |\,\mathrm{i}\,])\ ,\ [\mathrm{i}^{\mathrm{I}}|=(-[\,\mathrm{n}_{\mathrm{i}}|\ \ [\,\mathrm{i}\,|) \tag{3}$$
$$|\,\mathrm{i}_{\mathrm{I}}\rangle=(|\,\mathrm{n}_{\mathrm{i}}\rangle\ \ -|\,\mathrm{i}\,\rangle)\ ,\ \langle\,\mathrm{i}_{\mathrm{I}}|=(\langle\,\mathrm{n}_{\mathrm{i}}|\ \ -\langle\,\mathrm{i}\,|),\ |\,\mathrm{i}_{\mathrm{I}}]=(|\,\mathrm{i}\,]\ \ |\,\mathrm{n}_{\mathrm{i}}])\ ,\ [\,\mathrm{i}_{\mathrm{I}}|=([\,\mathrm{i}\,|\ \ [\,\mathrm{n}_{\mathrm{i}}|)$$

where the spinors i and n_{i} are explicitly given as

$$|\,\mathrm{i}\,\rangle=\sqrt{\mathrm{E}_{\mathrm{i}}+\mathrm{P}_{\mathrm{i}}}\begin{pmatrix}\mathrm{c}_{\mathrm{i}}\\ \mathrm{s}_{\mathrm{i}}\end{pmatrix}\ ,\ |\,\mathrm{n}_{\mathrm{i}}\rangle=\sqrt{\mathrm{E}_{\mathrm{i}}-\mathrm{P}_{\mathrm{i}}}\begin{pmatrix}-\mathrm{s}_{\mathrm{i}}^*\\ \mathrm{c}_{\mathrm{i}}\end{pmatrix}\ ,\ \langle\,\mathrm{i}\,|=\sqrt{\mathrm{E}_{\mathrm{i}}+\mathrm{P}_{\mathrm{i}}}\begin{pmatrix}\mathrm{s}_{\mathrm{i}}\\ -\mathrm{c}_{\mathrm{i}}\end{pmatrix}\ ,\ \langle\,\mathrm{n}_{\mathrm{i}}|=\sqrt{\mathrm{E}_{\mathrm{i}}-\mathrm{P}_{\mathrm{i}}}\begin{pmatrix}\mathrm{c}_{\mathrm{i}}\\ \mathrm{s}_{\mathrm{i}}^*\end{pmatrix} \tag{4}$$
$$|\,\mathrm{i}\,]=\sqrt{\mathrm{E}_{\mathrm{i}}+\mathrm{P}_{\mathrm{i}}}\begin{pmatrix}\mathrm{s}_{\mathrm{i}}^*\\ -\mathrm{c}_{\mathrm{i}}\end{pmatrix},\ |\,\mathrm{n}_{\mathrm{i}}]=\sqrt{\mathrm{E}_{\mathrm{i}}-\mathrm{P}_{\mathrm{i}}}\begin{pmatrix}\mathrm{c}_{\mathrm{i}}\\ \mathrm{s}_{\mathrm{i}}\end{pmatrix}\ ,\ [\,\mathrm{i}\,|=\sqrt{\mathrm{E}_{\mathrm{i}}+\mathrm{P}_{\mathrm{i}}}\begin{pmatrix}\mathrm{c}_{\mathrm{i}}\\ \mathrm{s}_{\mathrm{i}}^*\end{pmatrix}\ ,\ [\,\mathrm{n}_{\mathrm{i}}\,|=\sqrt{\mathrm{E}_{\mathrm{i}}-\mathrm{P}_{\mathrm{i}}}\begin{pmatrix}-\mathrm{s}_{\mathrm{i}}\\ \mathrm{c}_{\mathrm{i}}\end{pmatrix}$$

One doesn't need to write the explicit $SL(2,\mathbb{C})$ indices anymore, which simplifies many formulas. They can be reinserted easily by recalling that in angle brackets $\langle\mathrm{i\ j}\rangle$ the index α is descending from left to right, while for square brackets $[\mathrm{i\ j}]$ the index $\dot\alpha$ is ascending from left to right. In Lorentzinvariant amplitudes these indices are always contracted. From the explicit representation in (4) one can derive two important relations
(memo: negative/positive helicity spinors give a minus/plus sign).

$$\langle\mathrm{p}_{\mathrm{i}}\ \mathrm{n}_{\mathrm{i}}\rangle=-\mathrm{m}_{\mathrm{i}}\ ,\ [\mathrm{p}_{\mathrm{i}}\ \mathrm{n}_{\mathrm{i}}]=+\mathrm{m}_{\mathrm{i}} \tag{5}$$

Therefore in rest of this paper we don't need the explicit representation given in (4) any more. A further explicit representation was provided in [8], [9]. In appendix A still another representation with the standard momentum p^μ given by (1) is written down. The momentum in spinor form (2) can be written in the following form, as can be checked with the explicit spinors in (4)

$$\mathrm{p}_{\mathrm{i}}=|\,\mathrm{i}^{\mathrm{I}}\rangle[\,\mathrm{i}_{\mathrm{I}}\,|=-|\,\mathrm{i}_{\mathrm{I}}]\langle\,\mathrm{i}^{\mathrm{I}}\,|=|\,\mathrm{i}\,\rangle[\,\mathrm{i}\,|+|\mathrm{n}_{\mathrm{i}}\rangle[\mathrm{n}_{\mathrm{i}}|\ ,\ \bar{\mathrm{p}}_{\mathrm{i}}=|\,\mathrm{i}_{\mathrm{I}}]\langle\,\mathrm{i}^{\mathrm{I}}\,|=-|\,\mathrm{i}^{\mathrm{I}}]\langle\,\mathrm{i}_{\mathrm{I}}\,|=|\,\mathrm{i}\,]\langle\,\mathrm{i}\,|+|\mathrm{n}_{\mathrm{i}}]\langle\mathrm{n}_{\mathrm{i}}\,| \tag{6}$$

With the 2-vector notation we can derive this much easier by using a dot product between the vectors:
$\mathrm{p}_{\mathrm{i}}=|\,\mathrm{i}^{\mathrm{I}}\rangle[\mathrm{i}_{\mathrm{I}}\,|=(|\,\mathrm{i}\,\rangle\ \ |\,\mathrm{n}_{\mathrm{i}}\rangle)\cdot([\,\mathrm{i}\,|\ \ [\mathrm{n}_{\mathrm{i}}\,|)=|\,\mathrm{i}\,\rangle[\,\mathrm{i}\,|+|\,\mathrm{n}_{\mathrm{i}}\rangle[\,\mathrm{n}_{\mathrm{i}}\,|$. The square of a momentum can be obtained using (5):
$\mathrm{p}_{\mathrm{i}}\cdot\mathrm{p}_{\mathrm{i}}=\frac{1}{2}\mathrm{Tr}\{\mathrm{p}_{\mathrm{i}}\cdot\bar{\mathrm{p}}_{\mathrm{i}}\}=\frac{1}{2}\mathrm{Tr}\{(|\,\mathrm{i}\,\rangle[\,\mathrm{i}\,|+|\,\mathrm{n}_{\mathrm{i}}\rangle[\,\mathrm{n}_{\mathrm{i}}|)(|\,\mathrm{i}\,]\langle\,\mathrm{i}\,|+|\mathrm{n}_{\mathrm{i}}]\langle\,\mathrm{n}_{\mathrm{i}}\,|)=\frac{1}{2}([\mathrm{i\ n}_{\mathrm{i}}]\langle\mathrm{n}_{\mathrm{i}}\ \mathrm{i}\rangle+[\mathrm{n}_{\mathrm{i}}\ \mathrm{i}]\langle\mathrm{i\ n}_{\mathrm{i}}\rangle)=\mathrm{m}_{\mathrm{i}}^2$. The action of momentum on a spinor now goes as: $\mathrm{p}_{\mathrm{i}}|\,\mathrm{i}^{\mathrm{I}}]=(|\,\mathrm{i}\,\rangle[\,\mathrm{i}\,|+|\,\mathrm{n}_{\mathrm{i}}\,\rangle[\,\mathrm{n}_{\mathrm{i}}|)(-|\,\mathrm{n}_{\mathrm{i}}]\ \ |\,\mathrm{i}\,])=-\mathrm{m}_{\mathrm{i}}(|\,\mathrm{i}\,\rangle\ \ |\mathrm{n}_{\mathrm{i}}\rangle)=-\mathrm{m}_{\mathrm{i}}|\,\mathrm{i}^{\mathrm{I}}\rangle$. Square or angle brackets require a tensor product between 2-vectors, for example:
$$\langle\mathrm{i}^{\mathrm{J}}\ \mathrm{i}^{\mathrm{K}}\rangle=(\langle\,\mathrm{i}\,|\ \ \langle\,\mathrm{n}_{\mathrm{i}}|)(|\,\mathrm{i}\,\rangle\ \ |\,\mathrm{n}_{\mathrm{i}}\rangle)=\begin{pmatrix}\langle\mathrm{i\ i}\rangle & \langle\mathrm{i\ n}_{\mathrm{i}}\rangle\\ \langle\mathrm{n}_{\mathrm{i}}\ \mathrm{i}\rangle & \langle\mathrm{n}_{\mathrm{i}}\ \mathrm{n}_{\mathrm{i}}\rangle\end{pmatrix}=\begin{pmatrix}0 & -\mathrm{m}_{\mathrm{i}}\\ \mathrm{m}_{\mathrm{i}} & 0\end{pmatrix}=-\mathrm{m}_{\mathrm{i}}\,\varepsilon^{\mathrm{JK}}$$
In the same manner using (3) and (5), the following relations can be obtained:

$$\langle\mathrm{i}^{\mathrm{J}}\ \mathrm{i}^{\mathrm{K}}\rangle=-\mathrm{m}_{\mathrm{i}}\,\varepsilon^{\mathrm{JK}}\ ,\ \langle\mathrm{i}_{\mathrm{J}}\ \mathrm{i}_{\mathrm{K}}\rangle=+\mathrm{m}_{\mathrm{i}}\,\varepsilon^{\mathrm{JK}},\ [\mathrm{i}^{\mathrm{J}}\ \mathrm{i}^{\mathrm{K}}]=+\mathrm{m}_{\mathrm{i}}\,\varepsilon^{\mathrm{JK}}\ ,\ [\mathrm{i}_{\mathrm{J}}\ \mathrm{i}_{\mathrm{K}}]=-\mathrm{m}_{\mathrm{i}}\,\varepsilon_{\mathrm{JK}} \tag{7}$$
$$\langle\mathrm{i}_{\mathrm{J}}\ \mathrm{i}^{\mathrm{K}}\rangle=+\mathrm{m}_{\mathrm{i}}\,\delta_{\mathrm{J}}^{\mathrm{K}}\ ,\ \langle\mathrm{i}^{\mathrm{J}}\ \mathrm{i}_{\mathrm{K}}\rangle=-\mathrm{m}_{\mathrm{i}}\,\delta_{\mathrm{K}}^{\mathrm{J}},\ [\mathrm{i}_{\mathrm{J}}\ \mathrm{i}^{\mathrm{K}}]=-\mathrm{m}_{\mathrm{i}}\,\delta_{\mathrm{J}}^{\mathrm{K}}\ ,\ [\mathrm{i}^{\mathrm{J}}\ \mathrm{i}_{\mathrm{K}}]=+\mathrm{m}_{\mathrm{i}}\,\delta_{\mathrm{K}}^{\mathrm{J}}$$
$$\langle\mathrm{i}^{\mathrm{J}}\ \mathrm{i}_{\mathrm{K}}\rangle=-2\mathrm{m}_{\mathrm{i}}=-\langle\mathrm{i}_{\mathrm{J}}\ \mathrm{i}^{\mathrm{J}}\rangle\ ,\ [\mathrm{i}^{\mathrm{J}}\ \mathrm{i}_{\mathrm{J}}]=+2\mathrm{m}_{\mathrm{i}}=-[\mathrm{i}_{\mathrm{J}}\ \mathrm{i}^{\mathrm{J}}]$$
$$\mathrm{p}_{\mathrm{i}}|\,\mathrm{i}^{\mathrm{I}}]=-\mathrm{m}_{\mathrm{i}}|\,\mathrm{i}^{\mathrm{I}}\rangle,\ \bar{\mathrm{p}}_{\mathrm{i}}|\,\mathrm{i}^{\mathrm{I}}\rangle=-\mathrm{m}_{\mathrm{i}}|\,\mathrm{i}^{\mathrm{I}}]\ ,\ \langle\,\mathrm{i}^{\mathrm{I}}\,|\mathrm{p}_{\mathrm{i}}=[\,\mathrm{i}^{\mathrm{I}}\,|\mathrm{m}_{\mathrm{i}}\ ,\ [\mathrm{i}^{\mathrm{I}}|\bar{\mathrm{p}}_{\mathrm{i}}=\langle\,\mathrm{i}^{\mathrm{I}}\,|\mathrm{m}_{\mathrm{i}}$$

These relations were of course already described in literature [7-11], but derived here in a simple way using 2-vectors. We note some properties of the ε tensor and of $SU(2)$ vectors, which will be crucial later. The ε tensor is defined as $\varepsilon^{JK} = -\varepsilon_{JK} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Raising and lowering of indices as well as products between $SU(2)$ vectors goes as follows:

$$a_I = a^J \varepsilon_{JI} \ , \ a^I = a_J \varepsilon^{JI} \ , \ a^J b_J = -a_J b^J \ , \ a^J a_J = 0 = a_J a^J \tag{8}$$

Note also, if $a^I = \begin{pmatrix} a^1 & a^2 \end{pmatrix}$ then $a_I = \begin{pmatrix} a^2 & -a^1 \end{pmatrix}$, if $a_I = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$ then $a^I = \begin{pmatrix} -a_2 & a_1 \end{pmatrix}$.

Now consider the helicity operator defined as $h = \dfrac{\vec{p}\cdot\vec{\sigma}}{2|\vec{p}|} = \begin{pmatrix} -(cc-ss^*) & -2cs^* \\ -2cs & (cc-ss^*) \end{pmatrix}$. Acting on the explicit spinors in (4) gives the result, that $|\, i\,\rangle, |\, i\,]$ have the same helicity as their massless counterparts, but the spinors $|\, n_i\,\rangle, |\, n_i\,]$ just have the opposite helicities. It can also be seen from the explicit form in (4) that for example $|\, n_i\,] \sim |\, i\,\rangle$ and therefore these spinors should have the same helicity.

$h|\, i\,\rangle = -\frac{1}{2}|\, i\,\rangle$, $h|\, i\,] = +\frac{1}{2}|\, i\,]$, $h|\, n_i\,\rangle = +\frac{1}{2}|\, n_i\,\rangle$, $h|\, n_i\,] = -\frac{1}{2}|\, n_i\,]$.

3. Three particle amplitudes and high energy limit

In this section we consider three point vertices for particles with mass, the three legs are called i, j, k. Momentum conservation demands $p_i + p_j + p_k = 0$ or explicitly: $|\,\mathrm{i}\,\rangle[\,\mathrm{i}\,| + |\,\mathrm{n_i}\rangle[\,\mathrm{n_i}\,| + |\,\mathrm{j}\rangle[\,\mathrm{j}\,| + |\,\mathrm{n_j}\,\rangle[\,\mathrm{n_j}\,| + |\,\mathrm{k}\,\rangle[\,\mathrm{k}\,| + |\,\mathrm{n_k}\,\rangle[\,\mathrm{n_k}\,| = 0$. Multiplying from left with $\langle\,\mathrm{j}\,|, \langle\,\mathrm{n_j}\,|, \langle\,\xi\,|$ (ξ = arbitrary spinor) and from right with $|\,\mathrm{k}\,]$ gives the following equations, using that $\mathrm{n_i}$ scales with $\mathrm{m_i}$ and therefore can be neglected at first order in the high energy limit.

$$\langle \mathrm{j\ i}\rangle[\mathrm{i\ k}] \approx 0 + \mathrm{O(m^2)} \tag{9}$$

$$\langle \mathrm{n_j\ i}\rangle[\mathrm{i\ k}] \approx -\mathrm{m_j}[\mathrm{j\ k}] + \mathrm{O(m^3)} \tag{10}$$

$$\langle \xi\ \mathrm{j}\rangle[\mathrm{j\ k}] \approx -\langle \xi\ \mathrm{i}\rangle[\mathrm{i\ k}] + \mathrm{O(m^2)} \tag{11}$$

Starting from $\bar{p}_i + \bar{p}_j + \bar{p}_k = 0$ or $|\,\mathrm{i}\,]\langle\,\mathrm{i}\,| + |\,\mathrm{n_i}\,]\langle\,\mathrm{n_i}\,| + |\,\mathrm{j}\,]\langle\,\mathrm{j}\,| + |\,\mathrm{n_j}\,]\langle\,\mathrm{n_j}\,| + |\,\mathrm{k}\,]\langle\,\mathrm{k}\,| + |\,\mathrm{n_k}\,]\langle\,\mathrm{n_k}\,| = 0$ similar relations can be obtained. Multiplying with $[\,\mathrm{j}\,|, [\,\mathrm{n_j}\,|, [\,\xi\,|$ from left and $|\,\mathrm{k}\,\rangle$ from right one obtains using the scaling of $\mathrm{n_i}$:

$$[\mathrm{j\ i}]\langle \mathrm{i\ k}\rangle \approx 0 + \mathrm{O(m^2)} \tag{12}$$

$$[\mathrm{n_j\ i}]\langle \mathrm{i\ k}\rangle \approx -\mathrm{m_j}\langle \mathrm{j\ k}\rangle + \mathrm{O(m^3)} \tag{13}$$

$$[\xi\ \mathrm{j}]\langle \mathrm{j\ k}\rangle \approx -[\xi\ \mathrm{i}]\langle \mathrm{i\ k}\rangle + \mathrm{O(m^2)} \tag{14}$$

The high energy limit of factors in (4) is $\sqrt{\mathrm{E+P}} \approx \sqrt{2\mathrm{E}}\left(1 - \dfrac{\mathrm{m^2}}{8\mathrm{E^2}}\right)$ and $\sqrt{\mathrm{E-P}} = \dfrac{\mathrm{m}}{\sqrt{\mathrm{E+P}}} \approx \sqrt{2\mathrm{E}}\dfrac{\mathrm{m}}{2\mathrm{E}}\left(1 + \dfrac{\mathrm{m^2}}{8\mathrm{E^2}}\right) \approx \dfrac{\mathrm{m}}{\sqrt{2\mathrm{E}}}$, where we expanded $\mathrm{P} = \sqrt{\mathrm{E^2 - m^2}}$. Therefore the spinors $|\,\mathrm{i}\,\rangle$, $|\,\mathrm{i}\,]$ go into their massless counterparts, while the spinors $|\,\mathrm{n_i}\,\rangle$, $|\,\mathrm{n_i}\,]$ vanish.

If two masses are equal, say $\mathrm{m_i = m_j = m}$ and the third mass is zero, i.e. $\mathrm{m_k} = 0$, as is the case when two massive fermions interact with a massless boson, then one needs the so called x-factor, which can be obtained contracting

$p_j|k] = m x|k\rangle$ with $\langle\xi|$. Here leg $|k]$ has positive helicity, for leg $|k\rangle$ with negative helicity one contracts $p_j|k\rangle = m\tilde{x}|k]$ with $[\xi|$.

The x-factor is in the high energy limit using (11):

$$x = \frac{\langle\xi|p_j|k]}{m\langle\xi\, k\rangle} \approx \frac{\langle\xi\, j\rangle[j\, k][k\, j]}{m\langle\xi\, k\rangle[k\, j]} \approx \frac{-\langle\xi\, i\rangle[i\, k][k\, j]}{-m\langle\xi\, i\rangle[i\, j]} = \frac{[i\, k][k\, j]}{m[i\, j]} \tag{15}$$

The amplitude then becomes with (9)-(11):

$$x\langle i^J\ j^K\rangle \approx \frac{[i\, k][k\, j]}{m[i\, j]}\begin{pmatrix}\langle i\, j\rangle & \langle i\, n_j\rangle \\ \langle n_i\, j\rangle & \langle n_i\, n_j\rangle\end{pmatrix} \approx \begin{pmatrix} 0 & \dfrac{-[k\, j]^2}{[i\, j]} \\ \dfrac{[i\, k]^2}{[i\, j]} & 0\end{pmatrix}$$

This example shows that amplitude calculations go faster without using the explixcit spinors in (4) and employing only $|i\rangle$ and $|n_i\rangle$.

4. Comments on recursion relations

In this section we comment on recursion relations, which in spinor helicity with massless particles is a important tool for calculating higher tree amplitudes, see [1-4] and [13].
We follow the discussion in [3] and [4]. In the soft limit of the propagator $P \to 0$ any amplitude can be factorized in smaller amplitudes. One deforms at least two momenta p_i and p_j by a complex variable z in a way, that momentum conservation and onshell condition is guaranteed. This is the case if the following equations are satisfied:

$$\hat{p}_i = p_i - z q\,,\ \hat{p}_j = p_j + z q \tag{16}$$

$$q^2 = p_i \cdot q = p_j \cdot q = 0 \tag{17}$$

With the first equations (16) momentum conservation is satisfied due to $\hat{p}_i + \hat{p}_j = p_i + p_j$. With the next equations (17) the onshell condition is satisfied due to $\hat{p}_i^2 = (p_i + zq)^2 = p_i^2 + 2z p_i \cdot q + q^2 = p_i^2$ and similar for p_j. q must be a nullvector and orthogonal to p_i and p_j. If both particles are massless one can choose $q = |i\rangle[j|$ satisfying the equations in (17). The momentum spinors then change according to

$$|\hat{i}\rangle = |i\rangle,\ |\hat{i}] = |i] - z|j],\ |\hat{j}\rangle = |j\rangle + z|i\rangle,\ |\hat{j}] = |j] \tag{18}$$

One then has $\hat{p}_i = |\hat{i}\rangle[\hat{i}| = |i\rangle[i| - z|i\rangle[j|$ and $\hat{p}_j = |\hat{j}\rangle[\hat{j}| = |j\rangle[j| + z|i\rangle[j|$ realizing (16). This is the famous BCFW recursion [13]. The amplitude now becomes complex and can be calculated with the residue theorem, for details see [3] and [4]. The poles contributing to the residues are from keeping the propagator momentum $P_I(z) = P_I - zq$ onshell: $P_I(z)^2 = P_I^2 - 2zP_I \cdot q = 0 \Rightarrow z = z_I = \dfrac{P_I^2}{2q \cdot P_I}$. For a massive propagator P_I^2 should be replaced by $P_I^2 - M^2$. If the boundary contribution is zero, the amplitude can be written as $\mathcal{A}(0) = -\sum_{z_I,h} \mathcal{A}_L(z_I)\frac{1}{P_I^2}\mathcal{A}_R(z_I)$.
The conditions (16) and (17) are not easy to satisfy if one or two particles have mass, the simple generalization of (18) does not work as discussed in [11]. We first discuss the cases, when one particle is massive and the other massless.

Case I: $m_i = 0, m_j \neq 0$

The momenta are given as $p_i = |i\rangle[i|$ and $p_j = |j\rangle[j| + |n_j\rangle[n_j|$. From inspecting (18) it is clear, that one needs an $SU(2)$ vector a^I, to implement a shift of $|i]$ with $|j_I]$. We make an ansatz for the shifts analogue to (18)

$$\left|\hat{i}\right\rangle=\left|i\right\rangle,\ \left|\hat{i}\right]=\left|i\right]-za^{I}\left|j_{I}\right],\ \left|\hat{j}^{I}\right\rangle=\left|j^{I}\right\rangle+za^{I}\left|i\right\rangle,\ \left|\hat{j}_{I}\right]=\left|j_{I}\right] \tag{19}$$

One sees that momentum conservation is satisfied $\hat{p}_i=\left|\hat{i}\right\rangle\left[\hat{i}\right|=\left|i\right\rangle\left[i\right|-za^{I}\left|i\right\rangle\left[j_I\right|,\ \hat{p}_j=\left|\hat{j}^{I}\right\rangle\left[\hat{j}_I\right|=\left|j^{I}\right\rangle\left[j_I\right|+za^{I}\left|i\right\rangle\left[j_I\right|$ and one obtains for the vector q, trivially satisfying $q^2=0=q\cdot p_i$.

$$q=\left|i\right\rangle a^{I}\left[j_I\right| \tag{20}$$

From $q\cdot p_j=0$ we get a condition for the vector $a^I: 2q\cdot p_j=\left\langle j^J\ i\right\rangle a^I\left[j_I\ j_J\right]=\left\langle j^J\ i\right\rangle a^I\cdot -m_j\varepsilon_{IJ}=+m_j\left\langle j_I\ i\right\rangle a^I=0$. Using (3) and $a^I=\left(a^1\quad a^2\right)$ one obtains $\left\langle n_j\ i\right\rangle a^1-\left\langle j\ i\right\rangle a^2=0$. In order to get as correct limit the BCFW recursion for $\left\langle n_j\right|=0$ one puts $a^1=1,\ a^2=\left\langle n_j\ i\right\rangle/\left\langle j\ i\right\rangle=a$.

$$a^I=\left(1\quad a\right),\ a=\frac{\left\langle n_j\ i\right\rangle}{\left\langle j\ i\right\rangle},\ a_I=\left(a\quad -1\right),\ a^I\left\langle j_I\ i\right\rangle=0 \tag{21}$$

Case II: $m_i\neq 0, m_j=0$

This case is not entirely trivial, as one would think first, so we discuss it separately. The momenta are now given as $p_i=\left|i\right\rangle\left[i\right|+\left|n_i\right\rangle\left[n_i\right|$ and $p_j=\left|j\right\rangle\left[j\right|$. We make an similar ansatz for the shifts and use another vector b_I, which will turn out to be different from a_I.

$$\left|\hat{i}^{I}\right\rangle=\left|i^{I}\right\rangle,\ \left|\hat{i}_I\right]=\left|i_I\right]-zb_I\left|j\right],\ \left|\hat{j}\right\rangle=\left|j\right\rangle+zb_I\left|i^{I}\right\rangle,\ \left|\hat{j}\right]=\left|j\right] \tag{22}$$

First we check momentum conservation: $\hat{p}_i=\left|\hat{i}^{I}\right\rangle\left[\hat{i}_I\right|=\left|i^{I}\right\rangle\left[i_I\right|-zb_I\left|i^{I}\right\rangle\left[j\right|,\ \hat{p}_j=\left|\hat{j}\right\rangle\left[\hat{j}\right|=\left|j\right\rangle\left[j\right|+zb_I\left|i^{I}\right\rangle\left[j\right|$. Thereby one sees that vector q is now defined as

$$q=b_I\left|i^{I}\right\rangle\left[j\right|=-b^{I}\left|i_I\right\rangle\left[j\right| \tag{23}$$

$q^2=0=q\cdot p_j$ are trivially valid, from $q\cdot p_i$ we can determine the vector b_I: $2q\cdot p_i=b_I\left\langle i^K\ i^I\right\rangle\left[j\ i_K\right]=b_I\cdot -m_i\varepsilon^{KI}\left[j\ i_K\right]=+m_ib^K\left[j\ i_K\right]=0$. This gives $b^1\left[j\ i\right]+b^2\left[j\ n_i\right]=0$, now we put $b^2=1, b^1=-\left[j\ n_i\right]/\left[j\ i\right]=b$ again in order to get the correct limit for $\left|n_i\right]=0$. In summary we have

$$b^I=\left(b\quad 1\right),\ b=\frac{-\left[j\ n_i\right]}{\left[j\ i\right]},\ b_I=\left(1\quad -b\right) \tag{24}$$

Case III: $m_i\neq 0, m_j\neq 0$

Since we have obtained different vectors a^I and b^I in the two previous cases and we have seen the combinations $a^J\left|j_J\right]$ and $b_J\left|i^J\right]$ in (19) and (22) we try to retain them in the ansatz for two massive spinors:

$$\left|\hat{i}^{I}\right\rangle=\left|i^{I}\right\rangle,\ \left|\hat{i}_I\right]=\left|i_I\right]-zb_Ia^J\left|j_J\right],\ \left|\hat{j}^{I}\right\rangle=\left|j^{I}\right\rangle+za^Ib_J\left|i^J\right\rangle,\ \left|\hat{j}_J\right]=\left|j_J\right] \tag{25}$$

We see that momentum conservation is satisfied for the shifted momenta i.e. $\hat{p}_i+\hat{p}_j=p_i+p_j$:
$\left|\hat{p}_i\right\rangle=\left|\hat{i}^{I}\right\rangle\left[\hat{i}_I\right|=\left|i^{I}\right\rangle\left[i_I\right|-zb_Ia^J\left|i^{I}\right\rangle\left[j_J\right|,\ \left|\hat{p}_j\right\rangle=\left|\hat{j}^{I}\right\rangle\left[\hat{j}_I\right|=\left|j^{I}\right\rangle\left[j_I\right|+za^Ib_J\left|i^J\right\rangle\left[j_I\right|$
The vector q is given by

$$q = b_I a^J \left| i^I \right\rangle \left[j_J \right| = -a^I b^J \left| i_J \right\rangle \left[j_I \right| \tag{26}$$

At first one has to check the asymptotics. For $m_i = 0, m_j \neq 0$ one gets for $\left| i_I \right] = \left(\left| i \right] \quad 0 \right)$ and $\left| n_i \right] = 0$ from (25) by comparing with (19) $\left| \hat{i}_I \right] = \left(\left| \hat{i} \right] \quad 0 \right) = \left(\left| i \right] \quad 0 \right) - z \left(b_1 \quad b_2 \right) a^J \left| j_J \right] = ! \left(\left| i \right] - z a^J \left| j_J \right] \quad 0 \right) \Rightarrow b_1 = 1, b_2 = 0$. The other shifts then automatically coincide with (19). The equation for a^I is for compatibility with (18).

$$m_i = 0, m_j \neq 0 \Rightarrow b^I = (0 \quad 1), b_I = (1 \quad 0), q = \left| i \right\rangle a^J \left[j_J \right|, a^I = (1 \quad a) \tag{27}$$

Similarly for $m_i \neq 0, m_j = 0$ one gets for $\left| j^I \right\rangle = \left(\left| j \right\rangle \quad 0 \right)$ and $\left| n_J \right\rangle = 0$ from (25) by comparing with (22) $\left| \hat{j}^I \right\rangle = \left(\left| \hat{j} \right\rangle \quad 0 \right) = \left(\left| j \right\rangle \quad 0 \right) - z \left(a^1 \quad a^2 \right) b_J \left| i^J \right\rangle = ! \left(\left| j \right\rangle - z b_J \left| i^J \right\rangle \quad 0 \right) \Rightarrow a^1 = 1, a^2 = 0$. Again the other shifts are identical with (22). The equation for b^I makes the limit compatible with BCFW in (18).

$$m_i \neq 0, m_j = 0 \Rightarrow a^I = (1 \quad 0), a_I = (0 \quad -1),\ q = b_I \left| i^I \right\rangle \left[j \right|, b^I = (b \quad 1) \tag{28}$$

Now we have to check the onshell conditions in (17). The first one $2q^2 = \mathrm{Tr}\{q \cdot \overline{q}\} = 0$ with $\overline{q} = -a^K b^L \left| j_K \right] \left\langle i_L \right|$ gives $2q^2 = a^I b^J a^K b^L \left\langle i_L\ i_J \right\rangle \left[j_I\ j_K \right] = a^I b^J a^K b^L \cdot m_i \varepsilon_{LJ} \cdot -m_j \varepsilon_{IK} = -m_i m_j a_K a^K b^J b_J = 0$ due to (8). The next one yields: $2 q \cdot p_i = b_I a^J \left\langle i^K\ i^I \right\rangle \left[j_J\ i_K \right] = b_I a^J \cdot -m_i \varepsilon^{KI} \left[j_J\ i_K \right] = m_i b^K a^J \left[j_J\ i_K \right] = 0$ and for the third one we get: $2 q \cdot p_j = b_I a^J \left\langle j^K\ i^I \right\rangle \left[j_J\ j_K \right] = b_I a^J \cdot -m_j \varepsilon_{JK} \left\langle j^K\ i^I \right\rangle = -m_j a^I b^J \left\langle j_I\ i_J \right\rangle = 0$. So in summary we get two onshell conditions, which should determine the 2-vectors a^I, b^I

$$a^I b^J \left[j_I\ i_J \right] = 0 \tag{29}$$

$$a^I b^J \left\langle j_I\ i_J \right\rangle = 0 \tag{30}$$

Inspecting equations (21) and (24) we make the ansatz

$$a^I = (1 \quad a), b^I = (b \quad 1) \tag{31}$$

Inserting this in (29), (30) using (3) and evaluating the dot products between a^I, b^J and the spinors j_I, i_J one obtains two equations from which one can obtain :

$$b \left[j\ i \right] + \left[j\ n_i \right] + a\, b \left[n_j\ i \right] + a \left[n_j\ n_i \right] = 0 \tag{32}$$

$$b \left\langle n_j\ n_i \right\rangle - \left\langle n_j\ i \right\rangle - a\, b \left\langle j\ n_i \right\rangle + a \left\langle j\ i \right\rangle = 0 \tag{33}$$

We again check the two limiting cases. For the case described in (27) one obtains from (32) $0 = 0$ and from (33) $a = \left\langle n_j\ i \right\rangle / \left\langle j\ i \right\rangle$. For the case in (28) one gets from (32) $b = -\left[j\ n_i \right] / \left[j\ i \right]$ and from (33) $0 = 0$, therefore these equations contain the limiting cases, if one particle is massless and one massive. One can solve (32),(33) by solving both for a and equating them and similarly for b, resulting in quadratic equations for a and b:

$$a^2 \left\langle j \middle| p_i \middle| n_j \right] + a \left(\left\langle j \middle| p_i \middle| j \right] - \left\langle n_j \middle| p_i \middle| n_j \right] \right) - \left\langle n_j \middle| p_i \middle| j \right] = 0 \tag{34}$$

$$b^2 \left\langle n_i \middle| p_i \middle| i \right] - b \left(\left\langle i \middle| p_j \middle| i \right] - \left\langle n_i \middle| p_j \middle| n_i \right] \right) - \left\langle i \middle| p_j \middle| n_i \right] = 0 \tag{35}$$

The solution of (34) under the condition $n_j \neq 0$ is:

$$a = \frac{-\langle j|p_i|j] + \langle n_j|p_i|n_j] + \sqrt{\left(\langle j|p_i|j] - \langle n_j|p_i|n_j]\right)^2 + 4\langle j|p_i|n_j]\langle n_j|p_i|j]}}{2\langle j|p_i|nj]} \tag{36}$$

In the case (27) $m_i = 0, m_j \neq 0, n_i = 0, b = 0, p_i = |\,i\rangle[\,i\,|$ only the plus sign gives the correct value $a = \langle n_j\ i\rangle / \langle j\ i\rangle$.
The solution of (35) requiring $n_i \neq 0$ is

$$b = \frac{\langle i|p_j|i] - \langle n_i|p_j|n_i] - \sqrt{\left(\langle i|p_j|i] - \langle n_i|p_j|n_i]\right)^2 + 4\langle i|p_j|n_i]\langle n_i|p_j|i]}}{2\langle n_i|p_j|i]} \tag{37}$$

In the case (28) $m_i \neq 0, m_j = 0, n_j = 0, a = 0, p_j = |\,j\rangle[\,j\,|$ only the minus sign gives the value $b = -[j\ n_i]/[j\ i]$.

With the shifts in (26), the vector q in (27) and the solutions (36) , (37) momentum conservation and onshell conditions are satisfied for particles with mass. A severe problem is of course to apply this to real amplitudes with massive particles and we will not discuss it here.

4. Summary

In summary we have considered massive spinors and formulated them as 2-vectors, which makes it easy to obtain a couple of relations between them. We avoid entirely the display of $SL(2,\mathbb{C})$ indices, which simplifies many formulas considerably. An example for expanding the three particle amplitude in the high energy limit is shown. Finally we comment on recursion relations for massive spinors and show that it is possible to maintain momentum conservation and the onshell conditions. The application of these recursion relations is however left as an open problem.

Appendix A

Here we provide another explicit representation of massive spinors based on the standard momentum

$$p^\mu = \begin{pmatrix} E & P\sin(\theta)\cos(\phi) & P\sin(\theta)\sin(\phi) & P\cos(\theta) \end{pmatrix} \tag{A1}$$

Using the Pauli matrices, the momentum can be written in spinor notation $p = p_{\alpha\dot\alpha} = p_\mu \sigma^\mu$ and $\bar p = p^{\dot\alpha\alpha} = p_\mu \bar\sigma^\mu$,

$$p = p_{\alpha\dot\alpha} = \begin{pmatrix} E - P(cc - ss^*) & -2Pcs^* \\ -2Pcs & E + P(cc - ss^*) \end{pmatrix},\ \bar p = p^{\dot\alpha\alpha} = \begin{pmatrix} E + P(cc - ss^*) & 2Pcs^* \\ 2Pcs & E - P(cc - ss^*) \end{pmatrix} \tag{A2}$$

In analogy to (3) we now write the massive spinors in the 2-vector notation,

$$|\,i^I\rangle = \left(|\,i\rangle\ \ |\,n_i\rangle\right),\ \langle i^I| = \left(\langle i|\ \ \langle n_i|\right),\ |\,i^I] = \left(-|\,n_i]\ \ |\,i]\right),\ [\,i^I| = \left(-[\,n_i|\ \ [\,i|\right)$$
$$|\,i_I\rangle = \left(|\,n_i\rangle\ \ -|\,i\rangle\right),\ \langle i_I| = \left(\langle n_i|\ \ -\langle i|\right),\ |\,i_I] = \left(|\,i]\ \ |\,n_i]\right),\ [\,i_I| = \left([\,i|\ \ [\,n_i|\right) \tag{A3}$$

The spinors i and n_i are now explicitly given as

$$|\,i\rangle = \sqrt{E_i + P_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix},\ |\,n_i\rangle = \sqrt{E_i - P_i}\begin{pmatrix} -c_i \\ -s_i \end{pmatrix},\ \langle i| = \sqrt{E_i + P_i}\begin{pmatrix} c_i \\ s_i^* \end{pmatrix},\ \langle n_i| = \sqrt{E_i - P_i}\begin{pmatrix} -s_i \\ c_i \end{pmatrix}$$
$$|\,i] = \sqrt{E_i + P_i}\begin{pmatrix} c_i \\ s_i \end{pmatrix},\ |\,n_i] = \sqrt{E_i - P_i}\begin{pmatrix} -s_i^* \\ c_i \end{pmatrix},\ [\,i| = \sqrt{E_i + P_i}\begin{pmatrix} -s_i \\ c_i \end{pmatrix},\ [\,n_i| = \sqrt{E_i - P_i}\begin{pmatrix} -c_i \\ -s_i^* \end{pmatrix} \tag{A4}$$

The momentum is still $p_i = |\,i^I\rangle[\,i_I\,| = |\,i\rangle[\,i\,| + |n_i\rangle[n_i|$ and the relations in (5) remain

$$\langle p_i\ n_i\rangle = -m_i\ ,\ [p_i\ n_i] = +m_i \tag{A5}$$

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