The Proof of Goldbach's Conjecture

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October. 21, 2019

Abstract

Since the set of $A_{S(+)}$ and $A_{S(\times)}$ [1] is a bijective function, we use the improved the theorem of asymptotic density to prove that there exist prodcut of two odd primes in any $A_{S(\times)}$. At the same time, in any $A_{S(+)}$, the sum of two odd primes can be obtained.

Keywords: Goldbach's conjecture, bijective function, asymptotic density, Inequality transform
Mathematics Subject Classification (2010): Primary 11A41; Secondary 11A05,11A25

1. Introduction

The Goldbach's conjecture, every even integer greater than 4 can be expressed as the sum of two primes, until now it has not been proven. In this paper we will prove that it is true.

1.1 Concepts and Propositions

Let M_1 be the midpoint of any sequence of odd numbers and M_1 is an odd number greater than or equal to 3, where M_1 assumptions are known. We have the following sequence of S_1:

$$1,3,\cdots,M_1-2,M_1,M_1+2,\cdots,2M_1-3,2M_1-1; \tag{1}$$

Let M_2 be the midpoint of any sequence of odd numbers and M_2 is an even number greater than or equal to 4 and does not exist in the the sequence, where M_2 assumptions are known. We have the following sequence of S_2:

$$1,3,\cdots,M_2-1,(M_2),M_2+1,\cdots,2M_2-3,2M_2-1; \tag{2}$$

By (1), multiply each number in the sequence that is symmetric to the midpoint to get a new sequence of $S_1(\times)$, the first item is the product of 1 and $2M_1-1$:

$$1\times(2M_1-1),3\times(2M_1-3),\cdots,M_1\times M_1; \tag{3}$$

[1] The definitions of $A_{S(+)}$ and $A_{S(\times)}$ can be found in introduction 1.1

By (2), multiply each number in the sequence that is symmetric to the midpoint to get a new sequence of $S_2(\times)$, the first item is the product of 1 and $2M_2-1$:

$$1\times(2M_2-1),3\times(M_2-3),\cdots,(M_2-1)\times(M_2+1); \tag{4}$$

For (3), change sign of "$\times$" into sign of " + "and get sequence of $S_1(+)$

$$1+(2M_1-1),3+(2M_1-3),\cdots,M_1+M_1; \tag{5}$$

Meanwhile, for (4), change sign of "$\times$" into sign of "+" and get sequence of $S_2(+)$

$$1+(2M_2-1),3+(2M_2-3),\cdots,(M_2-1)+(M_2+1); \tag{6}$$

Let the $A_{S_1(+)}$ and $A_{S_1(\times)}$ are the set of $S_1(+)$ and $S_1(\times)$, then $A_{S_1(+)}$ and $A_{S_1(\times)}$ is bijective. So if we prove that there have product of two primes in any $A_{S_1(\times)}$ or $A_{S_2(\times)}$, Goldbach's conjecture will be proved.

1.2. Properties of $S_1(\times)$, $S_2(\times)$ Sequence

In view of the particularity of $S_1(\times)$, $S_2(\times)$ Sequence, we can get the some properties of following:

(I) In $S_1(\times)$ and $S_2(\times)$, all numbers are two or more odd prime factors.

(II) For $S_1(\times)$, M_1 must be at least two prime factors. if M_1 is a prime number, the proposition hold.

(III) The frequency of all the prime factors appear in the $S_1(\times)$ and $S_2(\times)$ sequence is twice of the prime factor cycle, and if M_1 or M_2 contains the prime factor, the frequency is once of this prime factor.

(IV) For $S_1(\times)$, M_1^2 is the largest odd number. In $S_2(\times)$, $(M_2-1)\times(M_2+1)$ is the largest odd number.

(V) For $S_1(\times)$, real rows are $(M_1-1)\div 2$, and for $S_2(\times)$, real rows are $(M_2-2)\div 2$.

1.3. An Example of $S_1(\times)$ Sequence

In $S_1(\times)$, take $M_1=25$ as an example.(see the table below). Let P_i denote the i-th prime, and the numbers greater than or equal to three prime factors is Multi-Factor Number(MFN). For each MFN, rank its prime factors from the smallest to the largest and name it with first smallest prime factor, even if there are two prime factors, in order to prove simplification, we also treat it as the MFN. For example, in $S_1(\times)$, when $H_1=\{2,3,5,6,8,9,11,12\}$, all numbers are MFN of p_1. At the same time, let MFN of p_x denote the biggest of first digit of MFN in $S_1(\times)$, when $H_1=1$, MFN of p_x is $5\times5\times5\times5$.

Let α_i denote the frequency of occupied by the i-th prime, that there always have $\alpha_i = 1$ (see 1.2(III)) or $\alpha_i = 2$.

Let H_1 is real row, then

$$H_1 = \frac{M_1 - 1}{2} = 12 \tag{7}$$

$S_1(+)$	$S_1(\times)$	H_1	$MFNofp_1$	$MFNofp_x$	α_1	α_x
25+25	$25 \times 25 = 5 \times 5 \times 5 \times 5$	1		✓		1
23+27	$23 \times 27 = 3 \times 3 \times 3 \times 23$	2	✓		1	
21+29	$21 \times 29 = 3 \times 7 \times 29$	3	✓		1	
19+31	19×31	4				
17+33	$17 \times 33 = 3 \times 11 \times 27$	5	✓		1	
15+35	$15 \times 35 = 3 \times 5 \times 5 \times 27$	6	✓		1	1
13+37	13×37	7				
11+39	$11 \times 39 = 3 \times 11 \times 13$	8	✓		1	
9+41	$9 \times 41 = 3 \times 3 \times 41$	9	✓		1	
7+43	7×43	10				
5+45	$5 \times 45 = 3 \times 3 \times 5 \times 5$	11	✓		1	1
3+47	3×47	12	✓		1	
1+49	1×49	13	×		×	×

1.4.Theorem

The following are two important theorems involved in the proof.

Theorem 1. *For asymptotic density of sequence, the set A_s of integers not divisible by any of the prime number $p_1, p_2, \cdots, p_X$, has density*

$$\delta(A_s) = 1 - \sum_i \frac{1}{p_i} + \sum_{i<j} \frac{1}{p_i} \times \frac{1}{p_j} - \sum_{i<j<k} \frac{1}{p_i} \times \frac{1}{p_j} \times \frac{1}{p_k} + \cdots + (-1)^x \frac{1}{p_1} \times \frac{1}{p_2} \times \cdots \times \frac{1}{p_x}, \tag{8}$$

Heilbronn[2] and Rohrbach [3] proved that

$$\delta(A_s) \geqq \prod_{i=1}^{x} (1 - \frac{1}{p_i}). \tag{9}$$

Theorem 2. *Let $\delta(A_{S_1(\times)})$ or $\delta(A_{S_2(\times)})$ is asymptotic density of the product of two primes in the $S_1(\times)$ or $S_2(\times)$, and let $\delta(A_{S_1(+)})$ or $\delta(A_{S_2(+)})$ is asymptotic density of of the sum of two primes in the $S_1(+)$ or $S_2(+)$. According to bijective principle, then, must have*

$$\delta(A_{S_1(\times)}) = \delta(A_{S_1(+)}) \tag{10}$$

and

$$\delta(A_{S_2(\times)}) = \delta(A_{S_2(+)}) \tag{11}$$

[2]H.Heilbronn, "On an inequality in the elementary theory of number", Proc.Cambridge Philos. Soc.vol.55(1937) pp.641-665.

[3]H.Rohrbach, "Beweis einer zahlentheoretische Ungleichung",J.Reine Angew.Math.vol.187(1950)pp.193-196.

2. Lemmas

In this section, through the proofs of several Lemmas, we can obtain all conditions for proof of Goldbach's conjecture.

2.1. lemma 1.

For $S_1(\times)$ or $S_2(\times)$, let α_i is occurrence frequency of p_i in the sequence, the set $A_{S_1(\times)}$ or $A_{S_2(\times)}$ of integers not divisible by any of the prime number $p_1, p_2, \cdots, p_X$, has density

$$\delta(A_{S_1(\times)}) = \prod_{i=1}^{x}(1 - \frac{\alpha_i}{p_i}) \tag{12}$$

$$\delta(A_{S_2(\times)}) = \prod_{i=1}^{x}(1 - \frac{\alpha_i}{p_i}) \tag{13}$$

Proof. By **Theorem 1**, for $S_1(\times)$ or $S_2(\times)$, and α_i is occurrence frequency of p_i, we can get

$$\delta(A_{S_1(\times)}) = 1 - \sum_i \frac{\alpha_i}{p_i} + \sum_{i<j} \frac{\alpha_i}{p_i} \times \frac{\alpha_j}{p_j} - \sum_{i<j<k} \frac{\alpha_i}{p_i} \times \frac{\alpha_j}{p_j} \times \frac{\alpha_k}{p_k} + \cdots + (-1)^x \frac{\alpha_1}{p_1} \times \frac{\alpha_2}{p_2} \times \cdots \times \frac{\alpha_x}{p_x}, \tag{14}$$

$$= \prod_{i=1}^{x}(1 - \frac{\alpha_i}{p_i}) \tag{15}$$

the same reason

$$\delta(A_{S_2(\times)}) = 1 - \sum_i \frac{\alpha_i}{p_i} + \sum_{i<j} \frac{\alpha_i}{p_i} \times \frac{\alpha_j}{p_j} - \sum_{i<j<k} \frac{\alpha_i}{p_i} \times \frac{\alpha_j}{p_j} \times \frac{\alpha_k}{p_k} + \cdots + (-1)^x \frac{\alpha_1}{p_1} \times \frac{\alpha_2}{p_2} \times \cdots \times \frac{\alpha_x}{p_x}, \tag{16}$$

$$= \prod_{i=1}^{x}(1 - \frac{\alpha_i}{p_i}) \tag{17}$$

lemma 1 is proved.

2.2. lemma 2.

By (15) (17), we can obtain a simplified inequality

$$\delta(A_{S_1(\times)}) \geqslant \frac{p_1 - \alpha_1}{p_x} \tag{18}$$

$$\delta(A_{S_2(\times)}) \geqslant \frac{p_1 - \alpha_1}{p_x} \tag{19}$$

Proof. From any two continuous prime numbers p_i and p_{i+1} must have the relationship $p_{i+1} \geqslant p_i + 2$. when $\alpha_i = 1$, or $\alpha_i = 2$, therefore, we could perform reduction using $p_{i+1} \geqslant p_i + 2$ on the formula (15) (17).

Change

$$\prod_{i=1}^{x}\frac{(p_i-\alpha_i)}{p_i}=(\frac{p_1-\alpha_1}{p_1})\times(\frac{p_2-\alpha_2}{p_2})\times\cdots\times(\frac{p_x-\alpha_x}{p_x}) \tag{20}$$

Into

$$\prod_{i=1}^{x}\frac{(p_i-\alpha_i)}{p_i}\geqslant\frac{p_1-\alpha_1}{p_x} \tag{21}$$

Get

$$\delta(A_{S_1(\times)})\geqslant\frac{p_1-\alpha_1}{p_x}$$
$$\delta(A_{S_2(\times)})\geqslant\frac{p_1-\alpha_1}{p_x}$$

lemma 2 is proved.

2.3.Lemma 3.

For any $S_1(\times)$ or $S_2(\times)$, when $p_x\geqslant p_2$, $p_2=5$, M_1 or M_2 with p_x have the following relationship

$$\frac{M_1-1}{2}\geqslant p_x \tag{22}$$

$$\frac{M_2-2}{2}\geqslant p_x \tag{23}$$

Proof.

(I) According to the defintion of $S_1(\times)$, $S_1(+)$ at the same time, if $S_1(\times)$ exist MFN of p_x, then $p_x\times p_x\times p_x$ must be minimal form and that can be written as only one form in $S_1(+)$

$$2M_1=p_x+p_x\times p_x \tag{24}$$

Converting the formula (24) to

$$M_1-1=\frac{p_x+p_x\times p_x-2}{2} \tag{25}$$

For (25), when $p_x\geqslant p_2$, $p_2=5$, obviously there always have

$$M_1-1>2p_x$$

.

(II) Similarly, since $S_2(\times)$ and $S_2(+)$ at the same time, we can get

$$M_2-2=\frac{p_x+p_x\times p_x-4}{2} \tag{26}$$

For (26), when $p_x\geqslant p_2$, $p_2=5$, obviously there always have

$$M_2-2>2p_x$$

For $S_1(\times)$ and $S_2(\times)$, if there appears any MFN of p_x larger than $p_x\times p_x\times p_x$, by the reason of (25) and (26), when $p_x\geqslant p_2$, always able to suitable (22) (23).
Lemma 2 is proved.

2.4.Lemma 4.

For any $S_1(\times)$ or $S_2(\times)$, when $p_x = p_1$, $p_1 = 3$, there always exists the product of two prime numbers.

Proof. According to the defintion of $a_i = 1$ or $a_i = 2$, if the real row of $S_1(\times)$ or $S_2(\times)$ greater than or equal to 3, that is $H_1 \geqslant 3$ or $H_2 \geqslant 3$, by (18) (19), there always have

$$\delta(A_{S_1(\times)}) \geqslant \frac{p_1 - \alpha_1}{p_x} \geqslant \frac{1}{3} \tag{27}$$

$$\delta(A_{S_2(\times)}) \geqslant \frac{p_1 - \alpha_1}{p_x} \geqslant \frac{1}{3} \tag{28}$$

If the real row of $S_1(\times)$ and $S_2(\times)$ equal to 2, that is $H_1 = 2$ or $H_2 = 2$, at this time, only have following form

when $M_1 = 5$, have $3 \times 7, 5 \times 5$, and when $M_2 = 6$, have $3 \times 9, 5 \times 7$, always have the product of two prime numbers.

when the real row of $S_1(\times)$ or $S_2(\times)$ is one, then only exist following form, $M_1 = 3$, has 3×3, and $M_2 = 4$, has 3×5, also has the product of two prime numbers.

Lemma 4 is proved.

3. Conclusion

Through the above proof, if we get any

$$\delta(A_{S_1(\times)}) \geqslant \frac{1}{H_1} \tag{29}$$

and

$$\delta(A_{S_2(\times)}) \geqslant \frac{1}{H_2} \tag{30}$$

By the reason of **Theorem 2**, Goldbach's conjecture must be true.

Proof. Since (18) (19), can get

$$\delta(A_{S_1(\times)}) \geqslant \frac{H_1 \times \frac{p_1 - \alpha_1}{p_x}}{H_1} \tag{31}$$

$$\delta(A_{S_2(\times)}) \geqslant \frac{H_2 \times \frac{p_1 - \alpha_1}{p_x}}{H_2} \tag{32}$$

By the introduction 1.2 (V)

$$H_1 = \frac{M_1 - 1}{2} \tag{33}$$

$$H_2 = \frac{M_2 - 2}{2} \tag{34}$$

Put (33) into (31), and (34) into (32), due to (22) (23) of lemma 3, and $p_1 - \alpha_i \geqslant 1$, when $p_x \geqslant p_2$, $p_2 = 5$, always have

$$\delta(A_{S_1(+)}) \geqslant \frac{1}{H_1} \tag{35}$$

and

$$\delta(A_{S_2(+)}) \geqslant \frac{1}{H_2} \tag{36}$$

When $p_x = p_1$, $p_1 = 3$, the reason for the establishment of (29) (30), see lemma 4.

Through the above proof, we get the conclusion that Goldbach's conjecture was established.

References

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