# Periodic sequences of a certain kind of progressions

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Abstract. A few progressions and their periodic sequences.

Keywords. periodic sequence, progression, prime nubmer, Fermat's little theorem

## 0. Introduction.

We define some progressions , and study their periodic sequences to find the rule related to them.

## 1. Periodicity of a progression(1).

Now we define a progression as follows.

Let k and n be also a positive integer more than 1, then

$$\begin{aligned} a_{\mathrm{n,k}} &= 1 && \text{(when n = 1)} \\ &= (a_{\mathrm{n-1,k}}{+}\mathrm{n})^{\mathrm{k}-1} \pmod{\mathrm{k}} && \text{(when n > 1)} \end{aligned}$$

One by one we survey the shortest periods of the progressions of this kind, for some cases of k.

(e.q.) When k=2, then $\{a_{\mathrm{n,2}}\}=\{1, 1, 0, 0, 1, 1, 0, 0, 1, 1, \ldots\}$.
This progression seems periodic and its shortest period is assumed 4.
When k=3, then $\{a_{\mathrm{n,3}}\}=\{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots\}$.
This progression seems periodic and its shortest period is assumed 3.
When k=4, then $\{a_{\mathrm{n,4}}\}=\{1, 3, 0, 0, 1, 3, 0, 0, 1, 3, 0, \ldots\}$.
This progression seems periodic and its shortest period is assumed 4.
Periodicity of progressions is easily found for now (See Table 1).

Table 1: ( A.S.P. means the assumed shortest period.)

| k / n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | A.S.P. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 4 |
| 3 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 3 |
| 4 | 1 | 3 | 0 | 0 | 1 | 3 | 0 | 0 | 1 | 3 | 0 | 0 | 1 | 3 | 4 |
| 5 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 5 |
| 6 | 1 | 3 | 0 | 4 | 3 | 3 | 4 | 0 | 3 | 1 | 0 | 0 | 6 | 1 | 12 |
| 7 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 7 |
| 8 | 1 | 3 | 0 | 0 | 5 | 3 | 0 | 0 | 1 | 3 | 0 | 0 | 5 | 3 | 8 |
| 9 | 1 | 0 | 0 | 7 | 0 | 0 | 4 | 0 | 0 | 1 | 0 | 0 | 7 | 0 | 9 |

Theorem 1

Let l be a positive integer. If $a_{\mathrm{n,k}}=a_{\mathrm{n+l,k}}$ and k|l (i.e. l is divisible by k.) for the above-mentioned progression $\{a_{\mathrm{n,k}}\}$, then $\{a_{\mathrm{n,k}}\}$ has a period equal to l.

*Proof.*

We will prove deductively, that if $a_{\mathrm{n+m,k}}=a_{\mathrm{n+m+l,k}}$ then $a_{\mathrm{n+m+1,k}}=a_{\mathrm{n+m+l+1,k}}$ where m is a non-negative integer.

When m=0 evidently $a_{\mathrm{n,k}}=a_{\mathrm{n+l,k}}$.

Furthermore if $a_{\mathrm{n+m,k}}=a_{\mathrm{n+m+l,k}}$ then $a_{\mathrm{n+m+1,k}}\equiv(a_{\mathrm{n+m,k}}+\mathrm{n+m+1})^{\mathrm{k}-1}$ (mod k)$\equiv(a_{\mathrm{n+m+l,k}}+\mathrm{n+m+l+1})^{\mathrm{k}-1}$(mod k)$=$ $a_{\mathrm{n+m+l+1,k}}$, for l$\equiv$0(mod k).

This completes Theorem 1.

□

Theorem 2

Suppose k is a prime number larger than 2.

If n$\equiv$0 or n$\equiv$k$-$1 (mod k) then $a_{\mathrm{n,k}}=0$, otherwise $a_{\mathrm{n,k}}=1$.

*Proof.*

When k=3 then $a_{1,3}=1$, $a_{2,3}=(a_{1,3}+2)^2$(mod 3)=0, $a_{3,3}=(a_{2,3}+3)^2$(mod 3)=9(mod 3)=0, $a_{4,3}=(a_{3,3}+4)^2$(mod 3)=1(mod 3)=1.

Therefore $a_{1,3}=1=a_{4,3}$, so 3 is a period of this progression.

This completes Theorem 2 for k=3.

When k is larger than 3 then, applying Fermat's little theorem[1], $a_{1,\mathrm{k}}=1$, $a_{2,\mathrm{k}}=(a_{1,\mathrm{k}}+2)^{\mathrm{k}-1}$(mod k)$=3^{\mathrm{k}-1}$(mod k)=1, $a_{3,\mathrm{k}}=(a_{2,\mathrm{k}}+3)^{\mathrm{k}-1}$(mod k)$=4^{\mathrm{k}-1}$ (mod k) =1,. . .,$a_{\mathrm{k}-1,\mathrm{k}}=(a_{\mathrm{k}-2,\mathrm{k}}+\mathrm{k}-1)^2$(mod k)=0(mod k)=0, . . ., $a_{\mathrm{k,k}}=(a_{\mathrm{k}-1,\mathrm{k}}+\mathrm{k})^2$(mod k)=0(mod k)=0.

Also $a_{\mathrm{k+1,k}}$=($a_{\mathrm{k,k}}$+k+1)$^2$(mod k)=1(mod k)=1, so k is a period of this progression.

This completes Theorem 2 for k is larger than 3.

□

2. Periodicity of a progression(2).

Now we define another progression as follows.

Let k and n be also a positive integer more than 1, then

$$b_{\mathrm{n,k}} = 1 \qquad \text{(when n = 1)}$$
$$= (b_{\mathrm{n-1,k}} - \mathrm{n})^{\mathrm{k}-1} \text{ (mod k)} \qquad \text{(when n > 1)}$$

Periodicity of progressions is easily found for now (See Table 2).

Table 2: ( A.S.P. means the assumed shortest period.)

| k / n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | A.S.P. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 4 |
| 3 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 3(*) |
| 4 | 1 | 3 | 0 | 0 | 3 | 1 | 0 | 0 | 3 | 1 | 0 | 0 | 3 | 1 | 0 | 4(*) |
| 5 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 5(*) |
| 6 | 1 | 5 | 2 | 4 | 5 | 5 | 4 | 2 | 5 | 1 | 2 | 2 | 1 | 5 | 2 | 12 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 7(*) |

Theorem 3

Let l be a positive integer. If $b_{\mathrm{n,k}}$=$b_{\mathrm{n+l,k}}$ and k|l (i.e. l is divisible by k.) for the above-mentioned progression $\{b_{\mathrm{n,k}}\}$, then $\{b_{\mathrm{n,k}}\}$ has a period equal to l.

*Proof.*

We will prove deductively, that if $b_{\mathrm{n+m,k}}$=$b_{\mathrm{n+m+l,k}}$ then $b_{\mathrm{n+m+1,k}}$=$b_{\mathrm{n+m+l+1,k}}$ where m is a non-negative integer.

When m=0 evidently $b_{\mathrm{n,k}}$=$b_{\mathrm{n+l,k}}$.

Furthermore if $b_{\mathrm{n+m,k}}$=$b_{\mathrm{n+m+l,k}}$ then $b_{\mathrm{n+m+1,k}}$≡($b_{\mathrm{n+m,k}}$−n−m−1)$^{\mathrm{k}-1}$ (mod k)≡($b_{\mathrm{n+m+l,k}}$−n−m−l+1)$^{\mathrm{k}-1}$(mod k)= $b_{\mathrm{n+m+l+1,k}}$, for l≡0(mod k).

This completes Theorem 3 similarly as Theorem 1.

□

references
[1] Patrick St-Amant, International Journal of Algebra, Vol. 4, 2010, no. 17-20, 959-994