Periodic sequences of a certain kind of progressions

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Abstract. A progression and the periodic sequences of the progressions of this kind.

Keywords. periodic sequence, progression, Fermat's little theorem

0. Introduction.

We define a progression, and study the periodic sequences of the progressions of this kind.

1. Definition of a progression.

Now we define a progression as follows.

Let k be a positive integer and n be also a positive integer more than 1, then

$$\begin{aligned} a_{\mathrm{n,k}} &= 1 && \text{(when n = 1)} \\ &= (a_{\mathrm{n-1,k}}+\mathrm{n})^{\mathrm{k}-1} \pmod{\mathrm{k}} && \text{(when n > 1)} \end{aligned}$$

2. Periodicity of progressions.

One by one we survey the shortest periods of the progressions of this kind, for some cases of k.

When k=2, then $\{a_{\mathrm{n},2}\}=\{1, 1, 0, 0, 1, 1, 0, 0, 1, 1, \ldots\}$.

This progression seems periodic and we easily assume its shortest period is 4.

When k=3, then $\{a_{\mathrm{n,3}}\}=\{1,0,0,1,0,0,1,0,0,1,\ldots\}$.

This progression seems periodic and we easily assume its shortest period is 3.

When k=4, then $\{a_{\mathrm{n,4}}\}=\{1,3,0,0,1,3,0,0,1,3,0,\ldots\}$.

This progression seems periodic and we easily assume its shortest period is 4.

When k=5, then $\{a_{\mathrm{n,5}}\}=\{1,1,1,0,0,1,1,1,0,0,1,1,1,0,\ldots\}$.

This progression seems periodic and we easily assume its shortest period is 5.

Periodicity of progressions is easily found for now.

Theorem 1

Let l be a positive integer. If $a_{\mathrm{n,k}}=a_{\mathrm{n+l,k}}$ and k|l (i.e. l is divisible by k.) for the above-mentioned progression $\{a_{\mathrm{n,k}}\}$,
then $\{a_{\mathrm{n,k}}\}$ has a period equal to l.

Proof.

We will prove deductively, that if $a_{\mathrm{n+m,k}}=a_{\mathrm{n+m+l,k}}$ then $a_{\mathrm{n+m+1,k}}=a_{\mathrm{n+m+l+1,k}}$.

When m=0 evidently $a_{\mathrm{n,k}}=a_{\mathrm{n+l,k}}$.

Furthermore if $a_{\mathrm{n+m,k}}=a_{\mathrm{n+m+l,k}}$ then $a_{\mathrm{n+m+1,k}}\equiv(a_{\mathrm{n+m,k}}+\mathrm{n+m+1})^{\mathrm{k-1}}$ (mod k)$\equiv(a_{\mathrm{n+m+l,k}}+\mathrm{n+m+l+1})^{\mathrm{k-1}}$(mod k)$=a_{\mathrm{n+m+l+1,k}}$.

This completes Theorem 1.

□

Theorem 2

Suppose k a prime number larger than 2.

If n≡0 or n≡k−1 (mod k) then $a_{\mathrm{n,k}}=0$, otherwise $a_{\mathrm{n,k}}=1$.

Proof.

When k=3 then $a_{1,3}=1$, $a_{2,3}=(a_{1,3}+2)^2$(mod 3)=0, $a_{3,3}=(a_{2,3}+3)^2$(mod 3)=9(mod 3)=0, $a_{4,3}=(a_{3,3}+4)^2$(mod 3)=1(mod 3)=1.

Therefore $a_{1,3}=1=a_{4,3}$, so 3 is a period of this progression.

This completes Theorem 2 for k=3.

When k is larger than 3 then, by applying Fermat's little theorem, $a_{1,\mathrm{k}}=1$, $a_{2,\mathrm{k}}=(a_{1,\mathrm{k}}+2)^{\mathrm{k-1}}$(mod k)$=3^{\mathrm{k-1}}$(mod k)=1, $a_{3,\mathrm{k}}=(a_{2,\mathrm{k}}+3)^{\mathrm{k-1}}$(mod k)$=4^{\mathrm{k-1}}$

(mod k) $=1,\ldots,a_{k-1,k}=(a_{k-2,k}+k-1)^2$(mod k)=0(mod k)=0, $\ldots,a_{k,k}$ $=(a_{k-1,k}+k)^2$(mod k)=0(mod k)=0.

Also $a_{k+1,k}=(a_{k,k}+k+1)^2$(mod k)=1(mod k)=1, so k is a period of this progression.

This completes Theorem 2 for k is larger than 3.

□