A Possible Theory of Partial Differential Equations

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Abstract

The current gold standard for solving nonlinear partial differential equations, or PDEs, is the simplest equation method, or SEM. As a matter of fact, another prior technique for solving such equations, the G'/G-expansion method, appears to branch from the simplest equation method (SEM). This study will discuss a new method for solving PDEs called the generating function technique (GFT), may establish a new precedence with respect to SEM. First, this study shows how GFT relates to SEM and the G'/G-expansion method. Next, the paper describes a new theorem that incorporates GFT, Ring and Knot theory in the finding of solutions to PDEs. Then the novel technique is applied in the derivation of new solutions to the Benjamin-Ono, QFT and Good Boussinesq equations. Finally, the study concludes via a discourse on the reasons why the technique is likely better than SEM and G'/G-expansion method, the scope and range of what GFT could ultimately accomplish, and the elucidation of a putative new branch of calculus, called "diversification".

1. Introduction

Many notable mathematicians, like Lawrence Evans, suggest a general theory of [nonlinear] partial differential equations cannot exist. He claims there can never be a pithy theory to describe partial differential equations due to its vast number of [diverse] sources [1]. However, there are semi-analytical methods, like Adomian decomposition and homotopy analysis, which have been shown to solve a large variety of NPDEs [2,3]. Unfortunately, these techniques are not purely analytical and come with extremely high computational costs and are very time-consuming. Therefore, one must truly ask can one find or erect a purely analytical method for solving partial differential equations, especially NPDEs?

Stone-Weierstrass theorem states that a continuous function can be closely approximated to a polynomial [4]. Assuming the polynomial is a formal power series of at least an exponential function, it should converge to the exact solution of partial differential equations with the right coefficients [5]. Ultimately, if one wishes to devise a method that can solve a wide variety of partial differential equations, (s)he may have to heed this theorem and utilize a formal power series of an exponential function with the appropriate coefficients [6].

In the paper, a technique, called the Generating Function[s] Technique (GFT), for solving at least homogeneous partial differential equations will be discussed. First, the paper will show how the method incorporates a set of Laurent series of formal power series with a solution, derived from an auxiliary/characteristic equation, and trigonometric-based coefficients; thus, the paper will compare GFT to other methods (i.e. the simplest equation (SEM), G'/G-expansion methods). Next, the study will show how the set of formal power series, hence general and exact solution to the partial differential equations are connected to polynomial rings and knot polynomials via theorem. Then the paper will apply the theory in several examples. Finally, the study will conclude with a more exquisite explanation on why the method is more highly effective in comparison to other techniques, what other functions GFT can perform, and provide evidence of the possible existence of a branch of calculus, called "diversification".

2. Methodology

The relationship between generating functions and the solution to the Riccati equation.

The Riccati equation, a first-order ordinary differential equation (ODE), is the following expression:

$$\phi'(\xi) + \phi(\xi)^2 + \phi(\xi) = 0,$$

where ϕ is the solution to the equation and ξ is the [transformed] variable [7]. Solution ϕ is defined as:

$$\phi(\xi) = \frac{1}{e^{\xi - c_1} - 1}.$$

Now consider a generating function γ, or:

$$\gamma(\xi) = \sum_{i=0}^{\infty} p_i \, f(\xi)^i ,$$

where f is some function in terms of ξ and p_i is the i-th coefficient or parameter in the formal power series [8]. If one lets function f equal and parameter a_i equal the Lucas L_i combinatorial number about zero, or $2\cos^2(i\pi/2)$, the generating function γ becomes:

$$\gamma(\xi) = \sum_{i=0}^{\infty} \frac{1}{2} \, L_i(0) \, e^{\frac{i\xi}{2}}$$

$$\gamma(\xi) = \frac{1}{1 - e^{\xi - c_1}}.$$

It is noteworthy to state γ is equal to negative ϕ. In other words, the solution to the Riccati equation can be redefined as a generating function.

The relationship of other quintessential expressions and generating functions.

There are other important functions used to solve [nonlinear] PDEs that can be defined as generating functions. The table below provides a list of relationships between generating functions and quintessential expressions utilized in solving [nonlinear] PDEs.

$\gamma(\xi)$	$f(\xi)$	p_i
$\frac{1}{1 - e^{\xi - c_1}}$	$e^{\xi - c_1}$	$2\,B_i(0)$ $\mathrm{Le}_i(0)$
$\frac{1}{1 - e^{\xi - c_1}}$	$e^{\frac{1}{2}(\xi - c_1)}$	$L_i(0)$
$\frac{1}{e^{\xi - c_1} + 1}$	$e^{\frac{1}{2}(\xi - c_1)}$	$2\,U_i(0)$
$\frac{1}{\cosh(\xi - c_1) - 1}$	$e^{\xi - c_1}$	$2\,H_i(0)$
$-\mathrm{csch}\left(\frac{1}{2}(\xi - c_1)\right)$	$e^{\frac{1}{2}(\xi - c_1)}$	$2\,F_i(0)$
$\mathrm{sech}\left(\frac{1}{2}(\xi - c_1)\right)$	$e^{\frac{1}{2}(\xi - c_1)}$	$2\sqrt{F_i(0)}$

The general solution associated with GFT

Consider the following expression:

$$p_s \geq \left| \frac{d_l - d_n}{p_n - p_l} - n_n + 1 \right|,$$

where p_s is the power of the solution u in a putative series or the level of Laurent serial truncation for solution u, d_l is the highest degree of the linear terms, d_n is the total degree of a nonlinear term, p_n is the total power of the same nonlinear term, p_l is the highest power of the linear term which is one, and n_n is the number of basic nonlinear terms (including the source type).

SEM defines the general solution of a [nonlinear] PDE as a rudimentary linear combination or simple sum of the solution to the Riccati equation, or:

$$U(\xi) = \sum_{i=0}^{p_s} \eta_i \, \phi(\xi)^i,$$

where η_i is the i-th coefficient or parameter [9,10]. The Riccati equation serves as an auxiliary equation to SEM and more specifically the G'/G-expansion method [11,12].

Now considered the [transformed] general solution for GFT which involves a [truncated] Laurent series [13]. If one lets p_s equal γ, then the putative [transformed] general solutions u (or U) to many PDEs is defined as:

$$U(\xi) = \sum_{i=-p_s}^{p_s} \sum_{j=1}^{2} \left(a_{ij} \left(\sum_{k=0}^{\infty} 2\, f(\xi)^k \, F_k(0)^j \right)^i + b_{ij} \left(\sum_{k=0}^{\infty} 2\, f(\xi)^k \, U_{C_k}(0)^j \right)^i \right).$$

, where the Fibonacci k-th number/parameter given/for zero is the following expression:

$$F_k(0) = \sin\left(\frac{\pi k}{2}\right)$$

and the Chebyshev U k-th number/parameter given/for zero is expressed as:

$$C_k(0) = \cos\left(\frac{\pi k}{2}\right).$$

Note: the ansatz transformed variable ξ is a linear array of intermediates/variables, or the following expression:

$$\xi = \alpha t + \beta x,$$

where α and β are coefficients to the variables or intermediates t and x, respectively. This expression is only for 1 + 1 dimensional system.

If one wishes not to work with coefficients with negative indices, then shift the [truncated] Laurent series via p_s, like:

$$U(\xi) = \sum_{i=0}^{2\,p_s} \sum_{j=1}^{2} \left(a_{\mathrm{ij}} \left(\sum_{k=0}^{\infty} 2\, f(\xi)^k\, F_k(0)^j \right)^{i-p_s} + b_{\mathrm{ij}} \left(\sum_{k=0}^{\infty} 2\, f(\xi)^k\, U_{C_k}(0)^j \right)^{i-p_s} \right).$$

The latter expression/general solution involves an offset. Through GFT, the auxiliary/characteristic equation used for the facilitation of SEM and the G'/G-expansion method is a basic first-order ODE, or:

$$f'(\xi) + f(\xi) = 0.$$

Its solution is simply defined as:

$$f(\xi) = c_1\, e^{-\xi}.$$

Using the solution to the above basic auxiliary equation in the general solution to some principal partial differential equation will give rise to hyperbolic secant, hyperbolic cosecant, hyperbolic sine, hyperbolic cosine via Fibonacci or sine-based parameters/generating functions and expressions involving one plus hyperbolic tangent and cotangent via Chebyshev U or cosine-based parameters/generating functions raised by various powers.

The degree of "diversity" of solutions u of [nonlinear] PDEs established by GFT will be dependent upon the complexity of the auxiliary equation used. The auxiliary equation of GFT, which is used to derive f, hence generating function γ, can be any order linear ODE just as long as it does not surpass the order of the differential equation being solved. This will be further discussed in the conclusion section of this paper.

3. Theorem

Let u_g be the general solution while u_e be the exact solution to the differential equation F, defined as:

$$F(u,u_t,u_x,u_{tt},u_{xx},u_{xt},..) = 0.$$

Definition 3.1*: the general solution u_g, which is a set of formal power series and their multiplicative inverses, is a ring formed from the set of polynomials in one or more indeterminates with coefficients in another ring/field, or $u_g \in {}_R[[x]]^{\{E\}}$. The general solution u_g may also include hyperbolic trigonometric functions (i.e. hyperbolic secant, hyperbolic cosecant, etc.) raised by various powers which are generally polynomial ring analogs.*

Definition 3.2*: transformed general solution U_g, which is a set of formal power series and their multiplicative inverses, is a ring formed from the set of polynomials in one indeterminate with coefficients in another ring/field, or $U_g \in {}_R[[\xi]]^{\{E\}}$. The general solution U_g may also include hyperbolic functions raised by various powers which are polynomial ring analogs.*

A formal power series (of an exponential function) establishes a polynomial ring $R[[x]]^{\{E\}}$ [14,15,16]. (Note: {E} designates a exponentiated entity.) On the other hand, Maclaurin/Taylor series establishes a polynomial ring analog [17]. The multiplicative inverse of some formal power series will produce either another formal power series or the analog of a polynomial ring $R[[x]]^{\{E\}}$. Since polynomial rings can be commutative and associative, or undergo both addition and multiplication, another polynomial ring [analog] is generated by raising the power of a formal power and Maclaurin/Taylor series. In essence, the general solution u_g and its transformed general solution U_g is a set of Laurent series of polynomial rings and their analogs. Since polynomial rings can be commutative and associative, their net sum is a larger polynomial ring [analog].

Lemma 3.3: *if U_g is a polynomial ring, then the transformed differential equation F will be a polynomial ring also. In other words, $F \in R[[\xi]]^{\{E\}}$.*

Lemma 4.4: *if the common denominator C of transformed differential equation F is a polynomial ring, or $C \in R[[\xi]]^{\{E\}}$, then the product of the common denominator C and transformed differential equation F is another polynomial ring P, or $C \times F = P \in R[[\xi]]^{\{E\}}$.*

Plugging in U_g into a differential equation F will form a sum of differential polynomial rings which will be designate F. Since differential polynomial rings can be commutative and associative like other polynomial rings, their sum is an even larger polynomial ring F.

Lemma 3.5: *since P is a polynomial ring, then ideal I is a subset of the polynomial ring P, or: $I \subseteq P$. Also, ideal I possesses at least one set of generators (i.e. $\langle \xi \rangle$, $\langle e^{\xi} \rangle$, etc.).*

The common denominator of the differential equation F is the product of knots. The product of knot polynomials involves the connected sum of knots [18]. The connected sum of knots will form a commutative ring C (Schubert's theorem) [19]. The product of polynomials is another polynomial. In other words, the product of knots of knot polynomials/invariants of polynomials and polynomial rings will [probably] establish another polynomial ring P. The subset of the latter polynomial ring will form a new ideal I which possesses a set of generators (i.e. $< \xi > = \{\xi, \xi^2, ..., \xi^n\}$, $< e^{\xi} > = \{e^{\xi}, e^{2\xi}, ..., e^{n\xi}\}$) [20].

Lemma 3.6: *if the coefficients of the polynomial ideal I are made to equal to zero, then the exact solution u_e may exist.*

Definition 3.7: *the exact solution u_e, is a polynomial subring of the transformed general solution U_g.*

The coefficients associated with the generators, linked to the ideal I, form algebraic equations that should equal zero. Thus, an individual would consider the ideal I to be trivial. With the trivial ideal I, one is able to determine the values of the constants (i.e. a_{ij}, b_{ij}, α, β, etc.) of the set of [truncated] Laurent series and its formal power series; (s)he is able to derive at least one exact solution u_e for the differential equation *F*.

***Lemma 3.8**: if the exact solution u_e exists for the differential equation F, then the differential equation F will vanish when the exact solution u_e is plugged into the equation.*

The examples shown below will provide proof. Once an exact solution u_e is placed into the differential equation *F*, an individual will obtain zero. In other words, the differential equation *F* will become a "zero" polynomial ring after introduction of the general solutions u_g with solved constants (i.e. coefficients/parameters, etc.).

***Theorem 3.9**: if one is dealing with a [homogeneous] partial differential equation F, which occurs in the physical universe, then (s)he can utilize a set of Laurent series of formal power series, comprised of combinatorial numbers (specifically Fibonacci and Chebyshev U numbers about zero)/trigonometric-based parameters and some function f (which is the solution to a [linear] ordinary differential equation), to find exact solutions u_e to the equation F.*

This theorem is analogous to, but not the same as the Cauchy-Kovalevskaya theorem. Both theorems suggest that if a (system of) equation[s] is analytical, then the solution[s] will be analytical. However, this new theorem does not require Cauchy initial or other conditions (i.e. Neumann, Dirichlet) for the derivation of exact solutions.

4. Examples

All calculations were performed with Mathematica®. The supplemental to this paper contains Mathematica® spreadsheets for each example. Finally, all general transformed general solutions U will be based upon polynomial exponential knots and/or rings.

4.1 *A 2nd order linear parabolic equation*

A 2nd order linear parabolic equation is defined as follows:

$$u_t + u_{xx} = 0.$$

The transformed LPDE F in terms of the transformed solution U(ξ) is:

$$\alpha U_{\xi} + \beta^2 U_{\xi\xi} = 0.$$

First, one needs to determine which auxiliary equation (s)he wishes to use. In this example, the paper will utilize a basic first-order linear ODE given below:

$$f'(\xi) + f(\xi) = 0.$$

Then, (s)he must calculate the possible maximal/minimal power of the solution p_s. An individual would obtain a p_s equal to -1. Then (s)he must plug in the p_s value to the transformed general solution U(ξ). Next, the individual must apply the transformed general solution U into the transformed LPDE. By multiplying this transformed LPDE with its common denominator, (s)he will produce a large expression that can produce up to fourteen algebraic equations associated with the set of generators $\langle e^{\xi} \rangle$. The fourteen algebraic equations are used to solve for constants a_{ij}, b_{ij}, α and β whenever possible. Substituting in the previously described constants into the transformed general solution will give rise to the final exact solution[s] u like:

$$u(x, t) = \frac{a_{10}\, e^{\beta(x-\beta t)}}{c_1} + a_{11} + a_{21} + b_{10} + b_{11} + b_{21}.$$

4.2. *The Benjamin-Ono equation.*

The nonlinear Benjamin-Ono equation is defined as follows:

$$u_t + u_{xx} + uu_x = 0.$$

The transformed NPDE F in terms of the transformed solution U(ξ) is:

$$\alpha U_{\xi} + \beta^2 U_{\xi\xi} + \beta U U_{\xi} = 0.$$

First, considers and solves the following first-order linear ODE to get the solutions that would be derived via SEM:

$$f'(\xi) + f(\xi) = 0.$$

Then one must calculate the possible maximal/minimal power of the solution p_s. An individual would obtain a p_s equal to 1. Then (s)he must plug in the p_s value to the transformed general solution U(ξ). Next, the individual must apply the transformed general solution U into the transformed NPDE. By multiplying this transformed NPDE with its common denominator, (s)he will produce a large expression that can produce at most eighteen algebraic equations linked to the generator set $\langle e^{\xi} \rangle$. The eighteen algebraic equations are used to solve for constants a_{ij}, b_{ij}, α and β whenever possible. Substituting in the previously described constants into the transformed general solution will give rise to the final exact solution[s] u like:

$$u(x, t) = \frac{2\beta e^{\beta x}}{e^{\beta x} - c_1\, e^{\beta t(a_{11}+a_{21}+\beta+b_{10}+b_{11}+b_{21})}} + a_{11} + a_{21} + b_{10} + b_{11} + b_{21}.$$

Now an individual can derive new solutions if they change the characteristic/auxiliary equation to a second-order linear ODE given below:

$$f''(\xi) + f'(\xi) + f(\xi) = 0.$$

The solution to this equation is as follows:

$$f(\xi) = c_1\, e^{-\xi/2} \sin\left(\frac{\sqrt{3}\ \xi}{2}\right) + c_2\, e^{-\xi/2} \cos\left(\frac{\sqrt{3}\ \xi}{2}\right).$$

Then (s)he must plug in the same p_s value and the to the transformed general solution $U(\xi)$. Next, the individual must apply the transformed solution U into the transformed NPDE. By multiplying this transformed NPDE with its common denominator, (s)he will produce a large expression that can produce at most fourty-five algebraic equations linked to generator sets $\left\langle e^{\xi/2} \right\rangle$ and $\left\langle e^{\frac{1}{2} i \sqrt{3}\ \xi} \right\rangle$. The fourty-five algebraic equations are used to solve for constants a_{ij}, b_{ij}, α and β whenever possible. Substituting in the previously described constants into the transformed general solution will give rise to the final exact solution[s] u like:

$$u(x, t) = \frac{\sqrt[6]{-1}\ c_1 \left(-\left(\sqrt{3} + i\right) \beta\, c_1\, e^{\beta^2 t} - (1 - i)\sqrt{2}\ \sqrt{(-1)^{5/6}\ \beta^2}\ e^{\frac{1}{2}\left(1 + i\sqrt{3}\right)\beta(\beta t + x)}\right)}{c_2^{1}\, e^{\beta^2 t} - e^{\beta\left(i\sqrt{3}\ \beta t + i\sqrt{3}\ x + x\right)}}$$

and

$$u(x, t) = 2\beta\left(\frac{2\left(\sqrt{3} - i\right) e^{2\xi}}{\left(-c_2^{1} \sin^2\left(\frac{\sqrt{3}\ \xi}{2}\right) - i\, c_2^{1} \sin\left(\sqrt{3}\ \xi\right) + c_2^{1} \cos^2\left(\frac{\sqrt{3}\ \xi}{2}\right) + e^{\xi}\right)\left(c_2^{1} \sin\left(\sqrt{3}\ \xi\right) + i\, c_2^{1} \cos\left(\sqrt{3}\ \xi\right) - i\, e^{\xi}\right)} - 2i\sqrt{3} - 2\right),$$

where $\xi = \beta x + 2\left(\beta^2 + i\sqrt{3}\ \beta^2\right) t$.

The latter solutions to the equation are new or exotic.

4.3. *The nonlinear QFT[-like] equation.*

The QFT[-like] equation is defined as the following expression:

$$u_{tt} + u_{xx} + u + u^3 = 0.$$

The transformed NPDE F in terms of the transformed solution $U(\xi)$ is:

$$\alpha^2 U_{\xi\xi} + \beta^2 U_{\xi\xi} + U + U^3 = 0.$$

First, one must calculate the possible maximal/minimal power of the solution p_s. An individual would obtain a p_s equal to 1. Then (s)he must determine the solution to the following auxiliary equation:

$$f''(\xi) + f(\xi) = 0,$$

which is:

$$f(\xi) = c_2 \sin(\xi) + c_1 \cos(\xi).$$

Next, (s)he plugs in the p_s value and the solution to the to the characteristic/auxiliary equation into the transformed general solution U(ξ). Next, the individual must apply the transformed solution into the transformed NPDE. By multiplying this transformed NPDE with its common denominator, (s)he will produce a large expression that can produce at most thirteen algebraic equations linked to the set of generators $\langle e^{i\xi} \rangle$. The thirteen algebraic equations are used to solve for constants a_{ij}, b_{ij}, α and β whenever possible. Substituting in the previously described constants into the transformed general solution will give rise to the final exact solution[s] u like:

$$u(x, t) = \frac{i\sqrt{2}}{\sqrt{-c_1^2 - 2\, c_1 \sin\left(2\alpha t - \sqrt{1-4\alpha^2}\, x\right) + \left(c_1^2 + 1\right)\cos\left(2\alpha t - \sqrt{1-4\alpha^2}\, x\right)}}$$

and

$$u(x, t) = \frac{2i}{-c_2^2 \sin^2\left(\alpha t + \sqrt{-\alpha^2 - \frac{1}{2}}\, x\right) - i\, c_2^2 \sin\left(2\alpha t + \sqrt{-4\alpha^2 - 2}\, x\right) + c_2^2 \cos^2\left(\alpha t + \sqrt{-\alpha^2 - \frac{1}{2}}\, x\right) - 1} + i.$$

The above solutions are considered new or exotic.

4.4. *The Good Boussinesq[-like] equation.*

The Good Boussinesq[-like] equation is defined as the following expression:

$$u_{tt} + u_{xx} + u_{xxxx} + (u^2/2)_{xx} = 0.$$

The transformed NPDE F in terms of the transformed solution U(ξ) is:

$$\alpha^2 U_{\xi\xi} + \beta^2 U_{\xi\xi} + \beta^4 U_{\xi\xi\xi\xi} + \beta^2 (U^2/2)_{\xi\xi} = 0.$$

First, one must calculate the possible maximal/minimal power of the solution p_s. An individual would obtain a p_s equal to 2. Next (s)he must find a solution to the auxiliary equation which is a linear ODE, like:

$$f^{(3)}(\xi) + f(\xi) = 0.$$

The above equation solution is:

$$f(\xi)=c_1\,e^{-\xi}+c_2\,e^{\xi/2}\sin\left(\frac{\sqrt{3}\ \xi}{2}\right)+c_3\,e^{\xi/2}\cos\left(\frac{\sqrt{3}\ \xi}{2}\right).$$

Then the individual must plug in the p_s value and the solution of the characteristic/auxiliary equation into the transformed general solution U(ξ). Next, the individual must apply the transformed solution U into the transformed NPDE. By multiplying this transformed NPDE with its common denominator, (s)he will produce a large expression which can produce at most four-hundred and sixty-five algebraic equations linked to generator sets $\left\langle e^{\xi/2}\right\rangle$ and $\left\langle e^{\frac{1}{2}i\sqrt{3}\ \xi}\right\rangle$. These algebraic equations are used to solve for constants a_{ij}, b_{ij}, α and β whenever possible. Substituting in the previously described constants into the transformed general solution will give rise to the final exact solution[s] u like:

$$u(x,\,t)=\frac{96\left(1-i\sqrt{3}\right)\beta^2\,c_2^4\,e^{2\xi}}{\left(\left(c_2^4\,e^{2\xi}-1\right)\cos\left(\sqrt{3}\ \xi\right)+i\left(c_2^4\,e^{2\xi}+1\right)\sin\left(\sqrt{3}\ \xi\right)\right)^2},$$

where $$\xi=\beta\,x+\frac{\sqrt{16\sqrt{3}\ \beta^4-16\,i\,\beta^4-\sqrt{3}\ \beta^2-i\,\beta^2}\ t}{\sqrt{\sqrt{3}+i}}.$$

The above solution is considered new or exotic.

5. Conclusions

From one dimension to beyond.

GFT can be used to solve a large range of PDEs including problems that have more than one spatial dimension. This paper primarily focused on the generation of soliton-based solutions for (1 + 1) PDEs; thus, the "bilinear" form of GFT is only utilized in this study. If one needs to solve (N + 1) PDEs, where $N \geq 2$, then the individual just adds more coefficients and variables or intermediates to ξ, like the following for N = 3:

$$\xi = \alpha t + \beta_1 x + \beta_2 y + \beta_3 z,$$

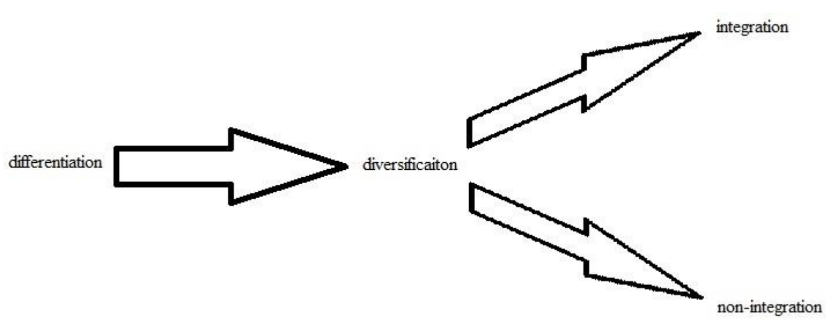
then make the appropriate transformations to the PDE. Next, (s)he can find the exact solution by committing the same steps used to solve 1 + 1 equations, but one must also solve for additional coefficients of the added variables or intermediates if deemed necessary. (Generally, (s)he just needs to solve for α concerning the other coefficients.) Therefore, "multilinear" GFT would be needed to solve N + 1 PDEs. An individual can also apply "unilinear" GFT to solve ordinary differential equations, by restricting ξ to one specific coefficient and variable/intermediate product, like αt, then committing the same steps that are described above.

The distinction between SEM, its specific extension G'/G-expansion and GFT.

The major difference in SEM and GFT is how their auxiliary equations are used. The auxiliary equation utilized in SEM creates the template by which solutions are established while the generating function performs that same task for GFT. The auxiliary equation usage in GFT is for adding complexity or greater diversification of the template through which solutions are established. As discussed before, an auxiliary equation that is a first-order linear ODE will let GFT create solutions similar to SEM. It is important to note these solutions tend to be primarily comprised of a [hyperbolic] secant, cosecant, tangent or cotangent function in the numerator position. If an individual uses a higher-order auxiliary equation, (s)he will produce a greater variety of solutions where "differing (and large) combinations" of [hyperbolic] sine, cosine and exponential functions appear in both the numerator and denominator positions.

A new branch of calculus.

In traditional calculus, there are two well-known branches of mathematics called "differentiation" and "integration". Due to the invention of various techniques to derive solutions to nonlinear PDEs, another branch, "non-integration", came about: the techniques represent a process through which an individual can by-pass the process of integration to derive a solution to a differential equation. This paper may have identified another subfield of calculus, "diversification" which exists between the three fields. It allows one to "proactively" generate a plethora of distinct exact solutions for both integrable and non-integrable equations via changing the order and/or presence of terms in the auxiliary equation.



Several principles can be elucidated in the field. For instance, the more terms the auxiliary equation has, the greater the number of possible exact solutions one can generate. Another principle would state that one can use inverse Z-transforms to define the coefficients of the formal power series solutions to a (particular auxiliary and) principal equation. Then (s)he can utilize permutations of the set of exact solutions generated from a (particular auxiliary and) principal equation to establish different exact solutions via taking either the arithmetic or geometric mean of the defined coefficients of the formal power series solutions present in the set. In other words, the above states that a set of exact solutions is the summation of the product between a set of coefficients (dependent upon the i-th iteration) and the [exponentiated] linear array of intermediates, or:

$$\mathbf{S_u} = \sum_{i=0}^{\infty} \mathbf{S_P(i) X^{\{E\}i}} ,$$

where the set of exact solutions S_u is defined as:

$$\mathrm{S_u} = \{\mathrm{u_1, u_2, \ldots, u_n}\} ,$$

and where the set of parameter/coefficients S_P which is:

$$\mathrm{S_P(i)} = \{\mathrm{p_1(i), p_2(i), \ldots, p_n(i)}\} .$$

The arithmetic mean μ_a and geometric mean μ_g for the entire set of exact solutions with proportional arguments would be described as:

$$\mu_a(S_u) = \sum_{i=0}^{\infty} \mu_a(S_P(i)) \left(X^{\{E\}}\right)^i ,$$

where $\mu_a(S_P(i)) = \dfrac{(\mathrm{p_1(i) + p_2(i) + \cdots + p_n(i)})}{\mathrm{n}}$

and

$$\mu_g(S_u) = \sum_{i=0}^{\infty} \mu_g(S_P(i)) \left(X^{\{E\}}\right)^i ,$$

where $\mu_g(S_P(i)) = (\mathrm{p_1(i) p_2(i) \cdots p_n(i)})^{1/n}$.

Both the arithmetic and geometric mean operations exemplify the fact that exact solutions, derived from at least GFT, are polynomial rings, which again are associative and commutative; these operations exploit the ability of the exact solutions to undergo some variation of addition and multiplication. Note: the arithmetic mean μ_a represents the "centroid" to the entire set of exact solutions S_u.

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