Perihelion Advance formula inference from Newton gravity law *Relative-Velocity Dependence* completed

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Abstract

While the original Newton's law of gravitation does not lead to the formula in question, the same law *relative-velocity dependence* completed does, briefly, with no hypothesis. **Keywords**: perihelion advance; interactions relative-velocity dependence.

1 Introduction

1.1 Perihelion advance formula

As known, *Perihelion/Periastron advance/rotation/ precession/shift* are names of the small remainder of the angular perihelion advance, δ, per revolution, of a planet orbiting the sun—or in general of a body orbiting an astron—not accounted for by Newton's law of gravitation. This slight deviation from Kepler 's first law was discovered by Urbain Le Verrier (1859) [1] for Mercury by calculations, and the well known formula of the effect,

$$\delta = \frac{6\pi G M_{\odot}}{c^2 a(1-\epsilon^2)}\ , \qquad (1)$$

was found by Paul Gerber [2] (1898) from some premises later regarded as inconsistent. A consistent inference was put forward by Albert Einstein (1916), via GTR [3]. Now we deduce it from *Newton's gravity law RVD*[1] *completed.*

[1]RVD stands for Relative-Velocity Dependence/Dependent (according to context)

1.2 RVD[1] completion of Newton's gravitation law

Let M and m be two point masses, and $\vec{r}$ the position vector of m with respect to M, i.e., $\vec{r}$ has its initial point at M and the terminal point at m or, in other words, m lies in the gravitational field of M; denote as usually $\vec{v}=\dot{\vec{r}}$ the *relative* velocity of m with respect to M. In usual notations, Newton's law of gravitation writes $\vec{F}_N = -GMm\vec{r}/r^3 = m\vec{g}_N$. *Newton's law of gravitation* (empirically) *RVD completed* is

$$\vec{F} = \vec{F}_N \left[1 + 3\frac{v^2}{c^2} + 4\left(\frac{v}{c}\right)^{\gamma} \frac{v_{\shortparallel}}{c}\right], \qquad (2)$$

where $v_{\shortparallel}$ is the component of $\vec{v}$ along the field, $v_{\shortparallel}=\dot{r}$, and $\gamma = 1.8$ (or $\gamma = 9/5$); using $\vec{g} = \vec{F}/m$ (*force per unit mass*, or *gravitational field strength*, or *gravitational acceleration*), Eq. (2) writes

$$\vec{g} = \vec{g}_N \left[1 + 3\frac{v^2}{c^2} + 4\left(\frac{v}{c}\right)^{\gamma} \frac{v_{\shortparallel}}{c}\right]. \qquad (2')$$

Of course, this is not a theory of gravitation, but simply a completion of Newton's law.

2 Perihelion advance formula inference

Unlike Gerber, whose reasoning has ultimately been considered both inconsistent and unclear, we either perform or mention all likely useful steps. We do this rather by metamorphic successive equalities than by words.

Newton's law of motion, $M_\oslash \vec{a} = \vec{F}$, of a mass $M_\oslash$ (as a planet) in the gravitational field of a mass $M_\odot$ (as the sun) taken as origin, $M_\oslash \ll M_\odot$ so that the center of the masses $M_\odot$ and $M_\oslash$ be approximately at $M_\odot$ (not the case of binary pulsars, for instance), writes

$$\ddot{\vec{r}} = -\frac{GM_\odot \vec{r}}{r^3}\left[1 + 3\frac{v^2}{c^2} + 4\left(\frac{v}{c}\right)^\gamma \frac{\dot{r}}{c}\right], \qquad (3)$$

where $\gamma = 1.8 = 9/5$ as mentioned in subsection 1.2.

Apply $\vec{r}\times$ to both sides of Eq. (3) and note that $\vec{r}\times\ddot{\vec{r}} = d(\vec{r}\times\dot{\vec{r}})/dt = \dot{\vec{L}}/M_\oslash$, where $\vec{L}$ is the angular moment of $M_\oslash$, obtaining $\dot{\vec{L}} = \vec{0}$, i.e., $\vec{L}$ is constant (Kepler's second law). As $\vec{L} = M_\oslash \vec{r}\times\vec{v}$, we have $\vec{r}\vec{L} = 0$, hence $\vec{r}$ keeps lying in a plain perpendicular to a constant vector $\vec{L}$, i.e., the motion is planar, because of which a plane polar coordinates system (ρ,φ) is convenient, in fact its three-dimensional extension (ρ,φ,z)—cylindrical coordinate system—with the same origin (at $\rho=0$) and the z-axis along $\vec{L}$; however, we continue using the notation $\vec{r}$ instead of shifting to $\vec{\rho}$. Also use the notation $\vec{1}_d$ for the unit vector of a given direction d, for instance $\vec{1}_v \equiv \vec{v}/|\vec{v}| = \vec{v}/v$, and $(\vec{1}_x, \vec{1}_y, \vec{1}_z)$ instead of $(\vec{\imath}, \vec{\jmath}, \vec{k})$ as the basis of unit vectors of a coordinate system (x, y, z). Thus one can write

$$\left.\begin{array}{l} \vec{r} = r\vec{1}_r\,, \quad \dot{\vec{r}} \equiv \vec{v} = \dot{r}\vec{1}_r + r\dot{\varphi}\vec{1}_\varphi\,, \\ \ddot{\vec{r}} \equiv \vec{a} = (\ddot{r} - r\,\dot{\varphi}^2)\vec{1}_r + (2\dot{r}\,\dot{\varphi} + r\,\ddot{\varphi})\vec{1}_\varphi\,, \end{array}\right\} \qquad (4)$$

$$\vec{L}/M_\oslash = \vec{r}\times\vec{v} = r^2\dot{\varphi}\vec{1}_z = 2\Omega\vec{1}_z = 2\vec{\Omega}\,, \qquad (5)$$

where $\vec{\Omega}$ is the *areolar velocity* (and Ω the *areolar speed*). Inserting expressions (4) in Eq. (3) and equating the components for each $\vec{1}_r$ and $\vec{1}_\varphi$, yield two equations:

$$\ddot{r} - r\,\dot{\varphi}^2 = -\frac{GM_\odot}{r^2}\left[1 + 3\frac{v^2}{c^2} + 4\left(\frac{v}{c}\right)^\gamma \frac{\dot{r}}{c}\right], \qquad (6)$$

$$2\dot{r}\,\dot{\varphi} + r\,\ddot{\varphi} = 0\,. \qquad (7)$$

Eq. (7) writes $(1/r)d(r^2\dot{\varphi})/dt = 0$, hence $r^2\dot{\varphi} = 2\Omega =$ `constant`, finding again Kepler's second law.

Change variable $t \to \varphi$, so having

$$\frac{d}{dt} = \dot{\varphi}\,\frac{d}{d\varphi} = \frac{2\Omega}{r^2}\frac{d}{d\varphi}\,,$$
$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{2\Omega}{r^2}\frac{d}{d\varphi}\left(\frac{2\Omega}{r^2}\frac{d}{d\varphi}\right) =$$
$$= \frac{2\Omega}{r^2}\frac{d}{d\varphi}\left[\frac{2\Omega}{r^2}\frac{d^2}{d\varphi^2} - \frac{4\Omega}{r^3}\left(\frac{d}{d\varphi}\right)^2\right] = \left(\frac{2\Omega}{r^2}\right)^2\left[\frac{d^2}{d\varphi^2} - \frac{2}{r}\left(\frac{d}{d\varphi}\right)^2\right],$$

thus, using primes for derivatives with respect to φ,

$$\dot{r} = \frac{2\Omega}{r^2}r'\,, \qquad \ddot{r} = \left(\frac{2\Omega}{r^2}\right)^2\left(r'' - \frac{2}{r}r'^2\right).$$

Insert this expression of $\ddot{r}$ in Eq. (6) and divide both sides by $(2\Omega/r^2)^2$, obtaining

$$r'' - 2\frac{r'^2}{r} - r = -\frac{GM_\odot r^2}{(2\Omega)^2}\left[1 + 3\frac{v^2}{c^2} + 4\left(\frac{v}{c}\right)^\gamma \frac{\dot{r}}{c}\right]. \qquad (8)$$

By function change $r \to u$, as $r = \ell/u$, where ℓ is an arbitrary constant, we have $r' = -\ell u'/u^2$, and $r'' = -\ell u''/u^2 + 2\ell u'^2/u^3$, so the left side of Eq. (8) becomes $-\ell u''/u^2 - \ell/u = (-\ell/u^2)(u'' + u)$; also $v^2 = \dot{r}^2 + r^2\dot{\varphi}^2 = (2\Omega/r^2)^2(r'^2 + r^2) = (2\Omega/\ell)^2(u'^2 + u^2)$; with these preparations Eq. (8) writes

$$u'' + u = \frac{GM_\odot \ell}{(2\Omega)^2}\Big[1 + 3\left(\frac{2\Omega}{c\,\ell}\right)^2(u'^2 + u^2) - 4\left(\frac{2\Omega}{c\,\ell}\right)^{\gamma+1}(u'^2 + u^2)^{\gamma/2}u'\Big]\,,$$

which, after setting the arbitrary constant ℓ, and defining a non-dimensional constant κ as

$$\ell = \frac{(2\Omega)^2}{GM_\odot}\,, \quad \kappa \equiv \left(\frac{2\Omega}{\ell\,c}\right)^2 = \left(\frac{GM_\odot}{2\Omega\,c}\right)^2 = \frac{GM_\odot}{\ell\,c^2}\,, \qquad (9)$$

finally writes

$$u'' + u = 1 + 3\kappa\,(u'^2 + u^2) - 4\kappa^{(\gamma+1)/2}(u'^2 + u^2)^{\gamma/2}u'. \qquad (10)$$

The next step is to solve Eq. (10) whose non-linear terms contain in factor the powers 1 and $(\gamma+1)/2$ of κ that carries the RVD effect. As κ is small (2.663×10^{-8} for Mercury, decreasing to 2.666×10^{-10} for Pluto), we treat the non-linear terms as a small perturbation, solving the equation approximately, by successive approximations, $u_0, u_1, u_2, \ldots$, replacing the non-linear terms in equation with their precedent approximation, and neglecting all terms having in factor κ^ν with $\nu > (\gamma+1)/2$.

If κ were zero, then Eq. (10) would be just that in the Newton case, $u_0'' + u_0 = 1$, whose solution is $u_0 = 1 + \epsilon\cos\varphi$, meeting the condition of passing through periastron at $\varphi=0$, ϵ being the eccentricity. Taking as the zeroth approximation just the Newton solution u_0 is convenient for a fast convergence. Corresponding to the sequence of approximations $\{u_n\}$ we have a sequence of *linear* equations,

$$\left.\begin{array}{l} u_1''+u_1=1+3\kappa(u_0'^2+u_0^2)-4\kappa^{(\gamma+1)/2}(u_0'^2+u_0^2)^{\gamma/2}u_0',\\ u_2''+u_2=1+3\kappa(u_1'^2+u_1^2)-4\kappa^{(\gamma+1)/2}(u_1'^2+u_1^2)^{\gamma/2}u_1',\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{array}\right\} \tag{11}$$

So, using the expression $u_0 = 1+\epsilon\cos\varphi$, the first of these equations becomes

$$\begin{array}{l} u_1'' + u_1 = 1 + 3\kappa(1+\epsilon^2+2\epsilon\cos\varphi)\\ \quad + 4\epsilon\kappa^{(\gamma+1)/2}(1+\epsilon^2+2\epsilon\cos\varphi)^{\gamma/2}\sin\varphi\,. \end{array} \tag{12}$$

The general solution of this linear non homogeneous equation is the sum of a particular solution u_{1p} and the general solution, $c_1\sin\varphi + c_2\cos\varphi$, of the homogeneous equation $u_1'' + u_1 = 0$. Denoting by $h(\varphi)$ the whole second side (the non-homogeneity term) of Eq. (12),

$$\begin{array}{l} h(\varphi) = 1 + 3\kappa(1+\epsilon^2+2\epsilon\cos\varphi)\\ \quad + 4\epsilon\kappa^{(\gamma+1)/2}(1+\epsilon^2+2\epsilon\cos\varphi)^{\gamma/2}\sin\varphi\,, \end{array} \tag{13}$$

the general solution of Eq. (12), directly verifiable by differentiation, is

$$\begin{array}{l} u_1(\varphi) = c_1\sin\varphi + c_2\cos\varphi\\ + \sin\varphi\int_0^\varphi h(\tau)\cos\tau\,d\tau - \cos\varphi\int_0^\varphi h(\tau)\sin\tau\,d\tau\,, \end{array} \tag{14}$$

and its derivative

$$\begin{array}{l} u_1'(\varphi) = c_1\cos\varphi - c_2\sin\varphi\\ + \cos\varphi\int_0^\varphi h(\tau)\cos\tau\,d\varphi \;+\; \sin\varphi\int_0^\varphi h(\tau)\sin\tau\,d\tau\,. \end{array} \tag{15}$$

Now determine constants c_1 and c_2 using the initial conditions (the same for all approximations u_n), $u_1(0) = 1+\epsilon$, and $u_1'(0) = 0$, directly, without expliciting the integrals. Obviously, from (14), $u_1(0) = c_2$, hence $c_2 = 1 + \epsilon$, and from (15), $u_1'(0) = c_1$, hence $c_1 = 0$; insert these values in (15),

$$\begin{array}{l} u_1'(\varphi) = -(1+\epsilon)\sin\varphi + \cos\varphi\int_0^\varphi h(\tau)\cos\tau\,d\tau\\ \quad + \sin\varphi\int_0^\varphi h(\tau)\sin\tau\,d\tau\,. \end{array} \tag{16}$$

Note that our sequence of successive approximations $\{u_n\}_{n\in\mathcal{N}}$ —neglecting the terms having in factor κ^ν for $\nu > 3/2$—stops at $n = 1$, since the second of Eqs. (11) (for u_2) coincides with the first (for u_1). In other words, u_1 contains the whole RVD effect of periastron shift in our pre-established approximation, $\kappa^\nu \approx 0$ for $\nu > (\gamma+1)/2$.

By its definition, perihelion (or periastron) is a point of extreme (minimum distance), hence $u_1' = 0$ at that point. Expecting a periastron shift δ after a revolution means that $u_1' = 0$ at $\varphi = 2\pi+\delta$ (instead of $\varphi = 2\pi$ in the Newton case). Because of the smallness of κ, a small δ is to be expected, so that we approximate $\sin\delta \approx \delta$, $\cos\delta \approx 1$, $\delta^2 \approx 0$, and $\kappa\delta \approx 0$, i.e., neglect δ^ν for $\nu \geq (\gamma+1)/2$. From Eq. (16), using these approximations, as well as the general fact that $f(x+\delta) \approx f(x) + \delta f'(x)$, and (13), we have successively

$$\begin{array}{l} u_1'(2\pi+\delta) \approx -(1+\epsilon)\delta + \int\limits_0^{2\pi+\delta} h(\varphi)\cos\varphi d\varphi\\ +\delta \int\limits_0^{2\pi+\delta} h(\varphi)\sin\varphi\,d\varphi \approx -(1+\epsilon)\delta + \int\limits_0^{2\pi} h(\varphi)\cos\varphi d\varphi\\ +\delta\,h(2\pi) + \delta\int\limits_0^{2\pi} h(\varphi)\sin\varphi d\varphi + 0\\ \approx -\epsilon\delta + \int\limits_0^{2\pi} h(\varphi)\cos\varphi d\varphi + 0\\ \approx -\epsilon\delta + 6\kappa\epsilon\left(\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right)\Big|_{\varphi=0}^{2\pi}\\ = -\,\epsilon\delta + 6\pi\kappa\epsilon\,, \end{array}$$

whence, as $\epsilon \neq 0$,

$$\delta = 6\pi\kappa\,. \tag{17}$$

Hence the perihelion shift δ is positive, i.e., an advance, indeed.Eq. (17) coincides with the well-known formula (1), via the third form of κ in (9), and $\ell = a\,(1-\epsilon^2)$, ℓ being the *semilatus rectum* of an ellipse in polar coordinates, $r = \ell/(1+\epsilon\cos\varphi)$. Q.E.D.

References

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