On the equation: $\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)=2$

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abstract

We give the real roots of the equation:

$$\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)=2\,,\;\; x>0$$

where $\Gamma(x)$ is the Gamma function.

keywords: Gamma function, integrals, number Pi.

I. Introduction

Recall that:

1.1. The Gamma function is defined by

$$\Gamma(z)=\int_0^\infty e^{-t}\,t^{z-1}\,dt\;\;,\;\;\mathrm{Re}(z)>0 \tag{1}$$

When $\mathrm{Re}(z)\leq 0$, $\Gamma(z)$ is defined by analytic continuation.

1.2. Some functional relations.

$$\Gamma(z+1)=z\,\Gamma(z) \tag{2}$$

$$\sqrt{\pi}\;\Gamma(2\,z)=2^{2\,z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) \tag{3}$$

1.3. The number Pi is defined by

$$\pi=4\sum_{n=0}^\infty\frac{(-1)^n}{2\,n+1}=3.141592\ldots \tag{4}$$

a related integral is

$$\pi=2\int_0^\infty\left|\Gamma\left(\frac{1}{2}+i\,x\right)\right|^2 dx\;\;,\;\;i=\sqrt{-1} \tag{5}$$

II. The real roots of $\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)=2$

2.1. The real roots of $\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)=2\ ,\ \ x>0$:

$$\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)-2=0 \Longrightarrow \begin{cases} x=\alpha=0.4551832402231255\ ... \\ x=\beta=2.3153691196079344\ ... \end{cases} \tag{6}$$

2.2. Graphics

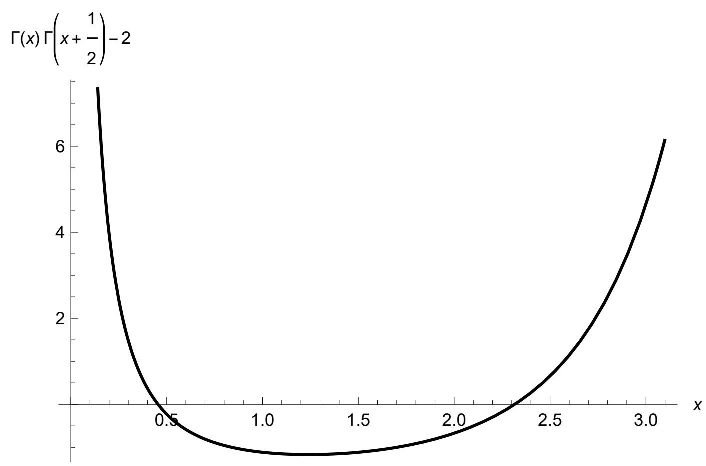

Figure 1.

2.3. Iteration 1.

$$x_{n+1}=\frac{1}{2}\,\Gamma(1+x_n)\,\Gamma\left(\frac{1}{2}+x_n\right),\ \ x_1=0\ \Longrightarrow x_n \longrightarrow \alpha=0.4551\ ... \tag{7}$$

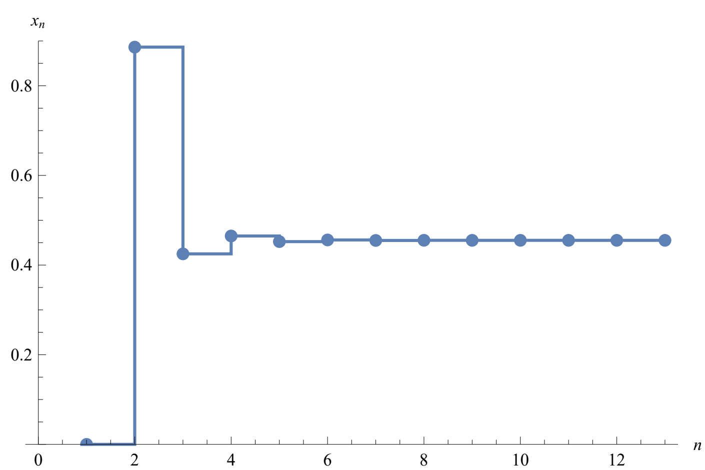

Figure 2.

2.3. Iteration 2.

$$x_{n+1}=x_n-\frac{1}{3}\left(\Gamma(x_n)\,\Gamma\left(\frac{1}{2}+x_n\right)-2\right)\ ,\ \ x_1=2.3 \Longrightarrow x_n \longrightarrow \beta=2.3153\ ... \tag{8}$$

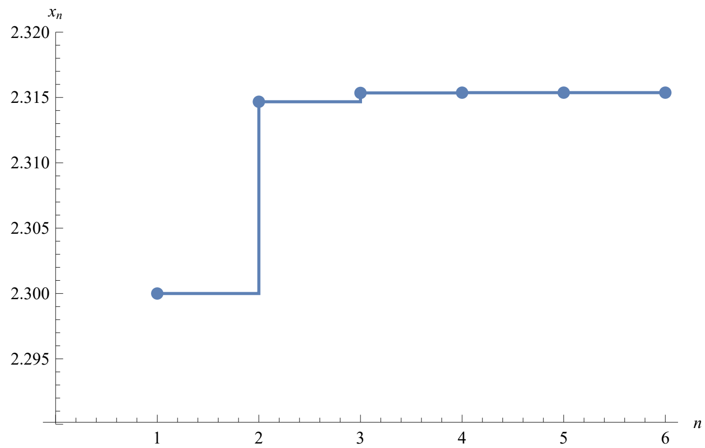

Figure 3.

III. Formulas

$$\sqrt{\pi} = \int_0^\infty |\Gamma(\alpha + i x)|^2 \, dx = \int_0^\infty |\Gamma(\beta + i x)|^2 \, dx \tag{9}$$

$$\pi = \frac{2^{4\alpha}}{(\Gamma(2\alpha))^2} = \frac{2^{4\beta}}{(\Gamma(2\beta))^2} \tag{10}$$

$$\sqrt{\pi} = \frac{2^{2\alpha}}{\Gamma(2\alpha)} = \frac{2^{2\beta}}{\Gamma(2\beta)} \tag{11}$$

$$2^{2(\beta-\alpha)} = \frac{\Gamma(2\beta)}{\Gamma(2\alpha)} \tag{12}$$

$$\pi = \frac{2^{2(\alpha+\beta)}}{\Gamma(2\alpha)\,\Gamma(2\beta)} \tag{13}$$

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