Coordinate Transformation and Static Charged Sphere in General Relativity

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Abstract

We consider a static charged sphere in general relativity. We make a coordinate transformation of a specific form. The electromagnetic energy-momentum tensor in the transformed coordinates is shown to be zero contrary to what is expected.

1 Electromagnetic potential and field

Let $A_\mu(t,x,y,z)$ and $g_{\mu\nu}(t,x,y,z)$ be the electromagnetic potential and metric tensor respectively. The electromagnetic field is

$$F_{\mu\nu}(t,x,y,z) = A_{\nu,\mu}(t,x,y,z) - A_{\mu,\nu}(t,x,y,z) \tag{1}$$

For a scalar function $\phi(t,x,y,z)$ define

$$\hat{A}^\mu(t,x,y,z) = A^\mu(t,x,y,z) + (g^{\mu\alpha}\phi_{,\alpha})(t,x,y,z) \tag{2}$$

We have by (1) and (2)

$$\begin{aligned} F_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu,\mu} - A_{\mu,\nu} + \phi_{,\nu\mu} - \phi_{,\mu\nu} = (A_\nu + \phi_{,\nu})_{,\mu} - (A_\mu + \phi_{,\mu})_{,\nu} \\ &= (g_{\nu\alpha}[A^\alpha + g^{\alpha\beta}\phi_{,\beta}])_{,\mu} - (g_{\mu\alpha}[A^\alpha + g^{\alpha\beta}\phi_{,\beta}])_{,\nu} = (g_{\nu\alpha}\hat{A}^\alpha)_{,\mu} - (g_{\mu\alpha}\hat{A}^\alpha)_{,\nu} \end{aligned} \tag{3}$$

2 Static charged sphere and Einstein field equations

Let there be a static charged sphere of total charge Q and mass M centred at the origin. Let the charge and mass densities be spherically symmetric. For this charged sphere let the metric $g_{\mu\nu}(r)$ of isotropic coordinate form

$$-a(r)dt^2 + b(r)(dx^2 + dy^2 + dz^2) \tag{4}$$

satisfy the Einstein field equations

$$G_{\mu\nu} = 8\pi\left[g^{\sigma\tau}F_{\mu\sigma}F_{\nu\tau} - \frac{1}{4}g_{\mu\nu}g^{\sigma\alpha}g^{\tau\beta}F_{\sigma\tau}F_{\alpha\beta}\right] + 8\pi T_{\mu\nu} \tag{5}$$

where $T^{\mu\nu}(r)$ is the energy-momentum tensor of matter. Require the electromagnetic energy-momentum tensor is not zero and

$$A_0(r) \qquad A_1(r) = A_2(r) = A_3(r) = 0 \tag{6}$$

Define $h_{\mu\nu}(r) = g_{\mu\nu}(r) - \eta_{\mu\nu}$. Require $rA_\mu(r)$ and $rh_{\mu\nu}(r)$ have finite limits as r goes to infinity. Consequently $r[a^{-1}(r)-1]$ and $r[b^{-1}(r)-1]$ have finite limits as r goes to infinity. Require also for small Q and M that

$$|A_0(r)| << 1 \qquad |h_{\mu\nu}(r)| << 1 \tag{7}$$

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3 Coordinate transformation

Let

$$\phi(t,x,y,z) = x \tag{8}$$

hence by (2), (6), and (8)

$$\hat{A}^0(r) = -(a^{-1}A_0)(r) \qquad \hat{A}^1(r) = b^{-1}(r) \qquad \hat{A}^2(r) = \hat{A}^3(r) = 0 \tag{9}$$

Let Q and M be small so that $b(r)$ is approximately one. Consider the tranformation from x, y, z coordinates to x', y', z' coordinates given by

$$x' = \int_0^x b(\sqrt{u^2+y^2+z^2})du \qquad y' = y \qquad z' = z \tag{10}$$

From the inverse of this transformation define the function φ by $x = \varphi(x', y', z')$. Define the coordinate transformation from t', x', y', z' coordinates to t, x, y, z coordinates by

$$t = t' - \int_0^{x'} (a^{-1}A_0)\Big(\sqrt{\varphi^2(u',y',z') + y'^2 + z'^2}\Big)du' \qquad x = \varphi(x',y',z') \qquad y = y' \qquad z = z' \tag{11}$$

The inverse of this transformation tranforms $\hat{A}^\mu(r)$ of (9) to $\hat{A}'^\mu(x', y', z')$ so that

$$\hat{A}'^0(x',y',z') = 0 \qquad \hat{A}'^1(x',y',z') = 1 \qquad \hat{A}'^2(x',y',z') = 0 \qquad \hat{A}'^3(x',y',z') = 0 \tag{12}$$

4 Size of metric perturbation

We have by (10) that

$$\frac{\partial x}{\partial y'} = -y' b^{-1}(\sqrt{\varphi^2(x',y',z') + y'^2 + z'^2}) \int_0^{\varphi(x',y',z')} \frac{\frac{db}{dr}(\sqrt{u'^2 + y'^2 + z'^2})}{\sqrt{u'^2+y'^2+z'^2}} du' \tag{13}$$

Now $r[b(r) - 1]$ and $r[b^{-1}(r) - 1]$ have finite limits as r goes to infinity hence $r^2(db/dr)(r)$ has finite limit as r goes to infinity. Consequently the integral is finite as x' goes to infinity and goes to zero as $\sqrt{y'^2 + z'^2}$ goes to infinity. For small Q and M since $b(r) - 1$ is small we then have $\partial x/\partial y'$ is small. We have by (11) that

$$\frac{\partial t}{\partial y'} = -\int_0^{x'} \frac{\frac{d(a^{-1}A_0)}{dr}(\sqrt{\varphi^2(u',y',z') + y'^2 + z'^2})}{\sqrt{\varphi^2(u',y',z') + y'^2 + z'^2}} [\varphi(u',y',z')\partial_{y'}\varphi(u',y',z') + y']du' \tag{14}$$

Now $r(a^{-1}A_0)(r)$ has finite limit as r goes to infinity. Consequently $r^2(d(a^{-1}A_0)/dr)(r)$ has a finite limit as r goes to infinity. Also we just showed $\partial\varphi/\partial y' = \partial x/\partial y'$ is small for small Q and M. Consequently the integral is finite as x' goes to infinity. Also $\partial t/\partial y'$ will go to zero as $\sqrt{y'^2 + z'^2}$ goes to infinity. For small Q and M we then have $\partial t/\partial y'$ is small. Also we have

$$\begin{aligned} \frac{\partial t}{\partial t'} = 1 \qquad \frac{\partial t}{\partial x'} &= -(a^{-1}A_0)(\sqrt{\varphi^2(x',y',z') + y'^2 + z'^2}) \\ \frac{\partial x}{\partial t'} = 0 \qquad \frac{\partial x}{\partial x'} &= b^{-1}(\sqrt{\varphi^2(x',y',z') + y'^2 + z'^2}) \qquad \frac{\partial y}{\partial y'} = 1 \end{aligned} \tag{15}$$

We can then conclude for small Q and M that

$$\left| \frac{\partial x^\mu}{\partial x'^\nu} - \delta^\mu_\nu \right| << 1 \tag{16}$$

Now

$$g'_{\mu\nu}(x',y',z') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(\sqrt{\varphi^2(x',y',z') + y'^2 + z'^2}) \tag{17}$$

and define $h'_{\mu\nu}(x',y',z') = g'_{\mu\nu}(x',y',z') - \eta_{\mu\nu}$. By (7) and (16) we have for small Q and M that

$$|h'_{\mu\nu}(x',y',z')| << 1 \tag{18}$$

5 Contradiction

We have by (3) transformed to t', x', y', z' coordinates and (12) that

$$F'_{\mu\nu} = A'_{\nu,\mu} - A'_{\mu,\nu} = (g'_{\nu\alpha}\hat{A}'^{\alpha})_{,\mu} - (g'_{\mu\alpha}\hat{A}'^{\alpha})_{,\nu} = g'_{\nu 1,\mu} - g'_{\mu 1,\nu} = h'_{\nu 1,\mu} - h'_{\mu 1,\nu} \tag{19}$$

Assuming the Principal of General Covariance and transforming (5) to t', x', y', z' coordinates and using (19) we have $h'_{\mu\nu}(x',y',z')$ satisfies

$$G'_{\mu\nu} = 8\pi g'^{\sigma\tau}[h'_{\sigma 1,\mu} - h'_{\mu 1,\sigma}][h'_{\tau 1,\nu} - h'_{\nu 1,\tau}] - 2\pi g'_{\mu\nu} g'^{\alpha\sigma} g'^{\beta\tau}[h'_{\tau 1,\sigma} - h'_{\sigma 1,\tau}][h'_{\beta 1,\alpha} - h'_{\alpha 1,\beta}] + 8\pi T'_{\mu\nu} \tag{20}$$

By (18) and (20) we have $h'_{\mu\nu}(x',y',z')$ approximately satisfies

$$G'_{\mu\nu}(x',y',z') = 8\pi T_{\mu\nu}(r') \tag{21}$$

From (21) we can conclude that the electromagnetic energy-momentum tensor in t', x', y', z' coordinates is zero. This is a contradiction since we started with a charged sphere with nonzero electromagnetic energy-momentum tensor.

References

[1] K. De Paepe, *Physics Essays*, September 2007