What Was Division by Zero?; Division by Zero Calculus and New World (Compact Version)

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Abstract Based on the preprint survey paper ([23]), we will introduce the importance of the division by zero and its great impact to elementary mathematics and mathematical sciences for some general people. For this purpose, we will give its global viewpoint in a self-contained manner by using the related references.

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Keywords Division by zero·Division by zero calculus·Differential equation·Analysis·Infinity·Discontinuous·Point at infinity·Laurent expansion·Conformal mapping·Stereographic projection·Riemann sphere·Horn torus·Elementary geometry·Zero and infinity· $1/0 = 0/0 = z/0 = \tan(\pi/2) = 0$

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1 Introduction

For the long history of division by zero, see [1, 17]. S. K. Sen and R. P. Agarwal [24] quite recentry referred to our paper [4] in connection with division

by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book in the following way:

"Thou shalt not divide by zero" remains valid eternally.

However, in [20] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

For the long tradition on the division by zero, people may not be accepted our new results against many clear evidences on our division by zero, however, a physicist stated as follows:

Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out (2017.10.14.08:55).

The common sense on the division by zero with the long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on differential coefficients we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. In this paper, in a new and definite sense, we will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$. In particular, we will introduce several fundamental concepts in calculus, Euclidean geometry, analytic geometry, complex analysis and differential equations. We will see new properties on the Laurent expansion, singularity, derivative, extension of solutions of differential equations beyond analytical and isolated singularities, and reduction problems of differential equations. On Euclidean geometry and analytic geometry, we will find new fields by the concept of the division by zero. We will give many concrete properties in mathematical sciences from the viewpoint of the division by zero. We will know that the division by zero is our elementary and fundamental mathematics.

The contents in ([23]) are as follows.

4. We can divide the numbers and analytic functions by zero.
5. General division and usual division.
6. Division by zero calculus.
7. Derivatives of functions.
8. Differential equations.
9. Euclidean spaces and division by zero calculus.
10. Analytic functions and division by zero calculus.
11. The Descartes circle theorem.
12. Horn torus models and division by zero calculus – a new world.

2 Division by Zero

The division by zero with the mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [4] for example.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [4, 27]:

Proposition 2.1 *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ such that*

$$F(a,b)F(c,d) = F(ac,bd)$$

for all

$$a, b, c, d \in \mathbf{C}$$

and

$$F(a,b) = \frac{a}{b}, \quad a, b \in \mathbf{C}, b \neq 0.$$

Then, we obtain, for any $a \in \mathbf{C}$

$$F(a,0) = 0.$$

In the long mysterious history of the division by zero, this proposition seems to be decisive.

Following Proposition 2.1, we should **define**

$$F(b,0) = \frac{b}{0} = 0,$$

and we should consider that for the mapping

$$W = f(z) = \frac{1}{z}, \tag{2.1}$$

the image $f(0)$ of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere. As the representation of the point at infinity on the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value at the origin by limiting like $+\infty$ and $-\infty$ or by the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotele. – However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([8, 9]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidean geometry, analytic geometry, complex analysis and the theory of differential equations at an undergraduate level and furthermore to our basic idea for the space and universe.

The simple field structure containing division by zero was established by M. Yamada ([7]) in a natural way. For a simple introduction, H. Okumura [15] discovered the very simple essence that:

To divide by zero is to multiply by zero.

For the operator properties of the generalized fractions, see [27].

3 Division by Zero Calculus

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to

functions, we need the concept of the division by zero calculus for the sake of unique determination of the results and for other reasons.

We will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{3.1}$$

we **define** the identity

$$f(a) = C_0. \tag{3.2}$$

Apart from the motivation, we define the division by zero calculus by (3.2). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

In addition, we will refer to an interesting viewpoint of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-a} dz.$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus.

We **define** the value of the function $f(z)$ at the singular point $z = a$ with the above Cauchy integral.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples to the author by the system that

$$\tan \frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp \frac{1}{x}(x = 0) = 1,$$

and others following the questions of the author.

4 We Can Divide the Numbers and Analytic Functions by Zero

In the division by zero like $1/0, 0/0$ the important problem was on their definitions. We will give our interpretation.

Based on the division by zero calculus, the meaning (definition) of

$$\frac{1}{0} = 0$$

is given by $f(0) = 0$ for the function $f(z) = 1/z$. Similarly, the definition

$$\frac{0}{0} = 0$$

is given by $f(0) = 0$ for the function $f(z) = 0/z$.

In the division by zero, the essential problem was in the sense of the division by zero (**definition**) $z/0$. Many confusions and simple history of division by zero may be looked in [14].

In order to give the precise meaning of division by zero, we will give a simple and affirmative answer, for a famous rule that we are not permitted to divide the numbers and functions by zero. In our mathematics, **prohibition** is a famous word for the division by zero.

For any analytic function $f(z)$ around the origin $z = 0$ that is permitted to have any singularity at $z = 0$ (of course, any constant function is permitted), we can consider the value, by the division by zero calculus

$$\frac{f(z)}{z^n} \tag{4.1}$$

at the point $z = 0$, for any positive integer n. This will mean that from **the form** we can consider it as follows:

$$\frac{f(z)}{z^n} \mid_{z=0} . \tag{4.2}$$

For example,

$$\frac{e^x}{x^n} \mid_{x=0} = \frac{1}{n!}.$$

This is the definition of our division by zero (general fraction). In this sense, we can divide the numbers and analytic functions by zero. For $z \neq 0$, $\frac{f(z)}{z^n}$ means the usual division of the function $f(z)$ by z^n.

The content of this subsection was presented by [20].

For many applications, see the original survey paper ([23]).

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Greeting at ICRAMA2019 (16-18 July, 2019) :

Thank you very much for the very honorable invitation.

India is a very specially great country for Japan, because our basic ideas for human life, human beings and our basic culture are based on Buddhism; religion and philosophy.

India is great for mathematical sciences and philosophy, because basic arithmetic operations were discovered by Brahmāgupta in 628 with zero, negative numbers and so on. Of course, his basic ideas were derived on the long history of India for void, nothing, infinity, non-existence and existence and so on. For example, in Vedas, we can find the decimal number system in very old days.

I am extremely happy to introduce our recent results on the division by zero. Surprisingly enough, Brahmāgupta said 0/0=0 in 628, thirteen hundred years ago. However, our world history shows that his result is wrong and we have still in confusions on the division by zero. However, his result and idea are right.

Indeed, I would like to introduce this fact clearly with its great impact and many applications.

Thank you. 2019.7.16.10:20-25.

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