On the integer solutions to the equation $x! + x = x^n$

Mathematics Thesis

Miika Rankaviita

Vaasan Lyseo Upper Secondary School

April 6, 2019

Abstract

This thesis explains the solution to the problem of finding all of the integer pair solutions to the equation $x! + x = x^n$. A detailed explanation is given so that anyone with high school mathematics background can follow the solution. This paper is a translation of my diplom work in Vaasa Lyseo Upper Secondary School.

Contents

1 Introduction

"Mathematics is the queen of science,

and number theory is the queen of mathematics."

-Carl Friedrich Gauss

In this thesis, I investigate the integer solutions to the equation $x! + x = x^n$.

$$x! + x = x^n \tag{†}$$

Both sides of the equation can be divided by x ($x \neq 0$), which gives the following.

$$(x-1)! + 1 = x^m \tag{‡}$$

I set $m := n-1$. These notations stay the same through out the whole thesis.

The reader is not assumed to have knowledge beyond high school mathematics curriculum, but we are going to deal with number theory, which may not have been covered in any obligatory course.

2 Reducing the set of possible solutions

We will start searching for the answer to the problem by reducing the number of possible solutions to the equation. It is easy to see that x and n cannot be negative. If x is negative then $x!$ is not defined ($\Gamma(x+1) = \pm\infty$). On the other hand, if n is negative, the right side of the equation $x! + x = x^n$ is rational while the left side is integer. Thus we arrive at a contradiction, so n cannot be negative.

Also, x and n cannot be zero. If $x = 0$ then the equation becomes $0! + 0 = 0^n \iff 1 = 0$, which is false. If $n = 0$ then the left side is $x^n = x^0 = 1$. But the right side is divisible by x so x must divide 1 also or $x = 0$. We saw that $x = 0$ is cannot be a solution, so that leaves us with $x = 1$. Let us substitute $(x, n) = (1, 0)$ to the equation (†):

$$\begin{aligned} 1! + 1 &= 1^0 \\ 2 &= 1, \end{aligned}$$

which is false.

We conclude that $x, n \in \mathbb{Z}_+$.

2.1 Modular arithmetic

Next we will inspect the equation using modular arithmetic, which is a natural approach when finding integer solutions to equations. If some x satisfies the equation $(x-1)! + 1 = x^m$, it must also satisfy the following equation.

$$(x-1)! + 1 \equiv x^m \pmod{x} \tag{1}$$

Since $x \equiv 0 \pmod{x}$, the right side of the congruence is zero.

$$(x-1)! + 1 \equiv 0 \pmod{x} \tag{2}$$

One must note that we are dealing with an implication, not an equevalence. If x is a solution to the equation (1), it is not necessarily a solution to the equation (‡). But using this approach, we can at least reduce the number of possible solutions.

Lemma 1. *Integer k divides $(k-1!)$, if and only if k is composite and $k \neq 4$.*

Proof. Let us assume that its possible to factor k so that $k = ab$. Let us also assume that $a \neq b$, in other words k is not a square of a prime number. Integers a and b are smaller than $k - 1$ because they are factors of k, and neither is exatly k. Since the factors of $(k - 1)!$, given by the definition of factorial, are all smaller than or equal to $k-1$, a and b are divisors of $(k-1)!$.

Let us assume that k is a square of a prime number. In other words $k = q^2$, where $q \in \mathbb{P}$. Let us also assume that $k - 1 = q^2 - 1 \geq 2q$. It follows that also $q^2 - 1 \geq q$. Because q and $2q$ are smaller than or eaqual to $q^2 - 1$, they are divisors of $(q^2 - 1)! = (k - 1)!$.

If we instead assume that $q^2 - 1 < 2q$, the inequality can be solved.

$$\begin{aligned} & q^2 - 2q - 1 < 0 \\ \iff & 1 - \sqrt{2} < q < 1 + \sqrt{2} \end{aligned}$$

Since q is a positive integer, it can only be 1 or 2. $1^2 = 1$ is not a composite, and $2^2 = 4$ is the only exception.

If k is prime, it does not divide any number that is smaller than it $(\forall l < k : k \nmid l)$. Thus k does not divide any one of the factors of $(k - 1)!$.

In conclusion, we have proved that the statements is satisfied by composites that are not squares of prime numbers. Then we proved it for squares of prime numbers (not including the number 4). Finally we proved the statement for prime numbers. Thus the statement is proved for all positive integers. □

Theorem 1. *The integer x must be a prime number.*

Proof. Let us assume against the satetment that x is composite.

$$\begin{aligned}(2) \implies 1 \cdot (x-1)! + 1 &\equiv 0 \pmod{x}\\ x\frac{(x-1)!}{x} + 1 &\equiv 0 \pmod{x}\end{aligned}$$

Based on *Lemma 1*, $\frac{(x-1)!}{x}$ is an integer, so $x\frac{(x-1)!}{x} \equiv 0 \pmod{x}$.

$$\begin{aligned}x\frac{(x-1)!}{x} + 1 &\equiv 0 \pmod{x}\\ 0 + 1 &\equiv 0 \pmod{x}\end{aligned}$$

This is a contradiction. Therefore x cannot be a composite number. □

3 An upper bound for x

It's possible to find an upper bound for x by first inspecting m.

3.1 An upper bound for m

Let us find the upper bound for m with respect to x.

Theorem 2. *The equation $(x-1)! + 1 = x^m$ has the upper bound $m < x$.*

Proof. When $x > 1$,

$$\frac{(x-1)!}{x^{x-1}} = \frac{\overbrace{(x-1)(x-2)(x-3)\cdots 3 \cdot 2 \cdot 1}^{(x-1)}}{\underbrace{x \cdot x \cdot x \cdots x \cdot x \cdot x}_{(x-1)}} < 1,$$

because each x in the nominator is greater than the corresponding factor of $(x-1)!$. The inequality can be manipulated in the following way.

$$\begin{aligned}
\frac{(x-1)!}{x^{x-1}} &< 1 \\
(x-1)! &< x^{x-1} \\
x \cdot (x-1)! &< x \cdot x^{x-1} \\
1 + (x-1)! \leq x \cdot (x-1)! &< x \cdot x^{x-1},
\end{aligned}$$

when $x \neq 1$.

Thus we get $(x-1)! + 1 < x^x$. From the equation (‡) we know that $(x-1)! + 1 = x^m$. Thus we get the following.

$$x^m < x^x$$

The function $\log_x(n)$ is strictly increasing when $x > 1$, so

$$\begin{aligned}
\log_x(x^m) &< \log_x(x^x) \\
\iff m &< x.
\end{aligned}$$

$\square$

3.2 Carmichael's theorem

The following theorem is an important one in modular arithmetic:

$$x^{\varphi(n)} \equiv 1 \pmod{n},$$

where x and n do not have common factors, and where $\varphi(n)$ is Euler's totient function. This theorem is called *Euler's theorem.*

$\varphi(n)$ denotes the number of positive integers, which are smaller than n and which do not share any factors with the number n. For example, $\varphi(8) = 4$, because 8 does not share any factors with the numbers $1, 3, 5$ and 7.

Using Euler's theorem, we get the following.

$$3^{\varphi(8)} = 3^4 \equiv 1 \pmod 8$$

The totient function has an important property: its value is easy to calculate when $n = p^r$, where p is prime:

$$\varphi(n) = p^{r-1}(p-1)$$

Euler's theorem and the totient function are not perfect. An improved version of the theorem is called Carmichael's theorem. The theorem is the following:

$$x^{\lambda(n)} \equiv 1 \pmod n$$

The theorem reminds of Euler's theorem, but φ is replaced by Carmichael λ-function. The λ-function is a better version of Euler's totient function, because it gives the *smallest* possible positive integer m that satisfies the equation

$$x^m \equiv 1 \pmod n.$$

For example, $\varphi(8) = 4$, but $\lambda(8) = 2$. The reader can check that the following statement holds.

$$3^{\lambda(8)} = 3^2 \equiv 1 \pmod 8$$

The value of Carmichael λ can be calculated using the following equation.

$$\lambda(n) = \mathrm{lcm}[\lambda(p_1^{r_1}), \lambda(p_2^{r_2}), \ldots, \lambda(p_{k-1}^{r_{k-1}}), \lambda(p_k^{r_k})],$$

where $n = p_1^{r_1} \cdot p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} \cdot p_k^{r_k}$ is the prime factorization of n, and lcm is a function that gives the lowest common multiple of the values.

The value of the λ-function for powers of prime numbers can be calculated in the following way.

$$\lambda(p^n) = \begin{cases} \varphi(p^n) & \text{, when } p \text{ is odd or } p^n = 2, 4 \\ \frac{1}{2}\varphi(p^n) & \text{, when } p \text{ is even, but } n \text{ is not 1 or 2} \end{cases}$$

The value of the Euler totient function is easy to calculate for powers of the primes, so we can simplify the value of the λ-function further.

$$\lambda(p^n) = \begin{cases} p^{n-1}(p-1) & \text{, when } p \text{ is odd or } p^n = 2, 4 \\ 2^{n-2}(2-1) & \text{, when } p = 2\text{, but } n \text{ is not 1 or 2} \end{cases}$$

3.3 A lower bound for m

From the equation (‡) we get the following equation.

$$(x-1)! = x^m - 1 \tag{3}$$

This equation can be inspected through modular arithmetic the same way as in the last chapter, but this time the modulus is $(x-1)!$.

$$\begin{aligned} (x-1)! &= x^m - 1 \\ \implies (x-1)! &\equiv x^m - 1 \pmod{(x-1)!} \\ x^m &\equiv 1 \pmod{(x-1)!} \end{aligned}$$

We learnt that m can be solved using the Carmichael's theorem, if x and $(x-1)!$ do not share any divisors. Because $x < (x-1)!$ when $x > 1$, using *Lemma 1* we can note that they do not have any common divisors (except when $x = 1$). Thus

$$m = \lambda((x-1)!).$$

But we arrived at the result in the context of modular arithmetic, which must be taken into consideration:

$$1 = 1^k \equiv (x^m)^k = x^{mk} \pmod{(x-1)!}$$

It is important to note that m needs a coefficient outside the context of modular arithmetic. So we get the following.

$$m = k\lambda((x-1)!),$$

where k is a natural number. If (x, m) is a solution to the equation (‡), m must be of the form $k\lambda((x-1)!)$.

Let us consider the λ-function next.

Lemma 2. *The following inequality holds for the λ-function, when $x \geq 4$.*

$$\lambda(x!) \geq \frac{x!}{2 \cdot x\#}$$

Proof. Let us recall the definition of the λ-function.

$$\lambda(n) = \operatorname{lcm}[\lambda(p_1^{r_1}), \lambda(p_2^{r_2}), \ldots, \lambda(p_{k-1}^{r_{k-1}}), \lambda(p_k^{r_k})]$$

Thus you need to know the factorization of a number to calculate the value of the totient function. If we want to figure out the value of $\lambda(x!)$, we must find the factorization of $x!$. It's not necessary, in this proof, to find the exact factorization. But one must note that every prime that is smaller than x appears in the factorization of $x!$, because every integer less than or equal to x is a divisor of $x!$. Hence we can note the value of $\lambda(x!)$ in the following way.

$$\lambda(x!) = \operatorname{lcm}[\lambda(2^{r_1}), \lambda(3^{r_2}), \ldots, \lambda(p_{k-1}^{r_{k-1}}), \lambda(p_k^{r_k})],$$

where p_k is the gratest prime number less than or equal to x.

The value of the totient function for a power of prime can be calculated in the following way.

$$\lambda(p^n) = \begin{cases} p^{n-1}(p-1) & \text{, when } p \text{ is odd or } p^n = 2, 4 \\ 2^{n-2}(2-1) & \text{, when } p = 2\text{, but } n \text{ is not 1 or 2} \end{cases}$$

If we assume that $x \geq 4$, then r_1 (the exponent of 2 in the prime factorization) is always greater than 2, because $x! = 2 \cdot 3 \cdot 4 \cdots x$ and $2 \cdot 4 = 2^3$ always divides $x!$. Thus $\lambda(2^{r_1}) = 2^{n-2}(2-1) = 2^{n-2}$ and we get the following equation.

$$\lambda(x!) = \mathrm{lcm}[2^{r_1-2}, 2 \cdot 3^{r_2-1}, \ldots, (p_{k-1}-1)p_{k-1}^{r_{k-1}-1}, (p_k-1)p_k^{r_k-1}]$$

Each power of an odd prime is multiplied by some cofficient $(p-1)$. If we ignore the coefficient, we get an expression that is smaller than the value of the totient function.

$$\mathrm{lcm}[2^{r_1-2}, 2 \cdot 3^{r_2-1}, \ldots, (p_k-1)p_k^{r_k-1}] \geq \mathrm{lcm}[2^{r_1-2}, 3^{r_2-1}, \ldots, p_k^{r_k-1}]$$

Based on the definition of the least common multiple, it's clear that the following equality holds.

$$\begin{aligned} \mathrm{lcm}[2^{r_1-2}, 3^{r_2-1}, \ldots, p_k^{r_k-1}] &= 2^{r_1-2} \cdot 3^{r_2-1} \cdots p_k^{r_k-1} \\ &= \frac{1}{2} \cdot \frac{2^{r_1} \cdot 3^{r_2} \cdots p_k^{r_k}}{2 \cdot 3 \cdots p_k} \\ &= \frac{1}{2} \cdot \frac{x!}{2 \cdot 3 \cdots p_k} \end{aligned}$$

Each prime less than or equal to x are multiplied together in the denominator of the fraction. This can be expressed using the primorial function: $n\#$ denotes the product of all primes less than or equal to n.

When all of the observations are combined, we get a lower bound for $\lambda(x!)$.

$$\lambda(x!) \geq \frac{x!}{2 \cdot x\#}$$

$\square$

Using the lemma, it is easy to figure out a lower bound for m.

Theorem 3. *If (x, m) is a solution to $(\ddagger)$, then m has the following lower bound, when $x \geq 5$.*

$$\frac{(x-1)!}{2(x-1)\#} < m$$

Proof. We showed previously that $m = k\lambda((x-1)!)$, where k is a natural number. If $k = 0$, then $m = 0$. Let us sumbstitute $m = 0$ to the equation $(\ddagger)$:

$$(x-1)! + 1 = x^0$$
$$(x-1)! + 1 = 1$$
$$(x-1)! = 0$$

There is no integer, for which $x! = 0$ holds, so let us concentrate on the case when $k > 0$.

It is clear that $\lambda((x-1)!) < k \cdot \lambda((x-1)!) = m$, for all $k > 0$. The following inequality follows from the *Lemma 2*.

$$\frac{(x-1)!}{2 \cdot (x-1)\#} \leq \lambda((x-1)!) < m,$$

when $x - 1 \geq 4 \iff x \geq 5$. Thus the lower bound of m is

$$\frac{(x-1)!}{2 \cdot (x-1)\#} < m.$$

$\square$

3.4 The upper bound

Theorem 2 gives an upper bound of $m < x$. **Theorem 3** gives a lower bound $\frac{(x-1)!}{2(x-1)\#} < m$, when $x \geq 5$. If (x, m) is a solution to the equation $(\ddagger)$, then the following inequality holds.

$$\frac{(x-1)!}{2(x-1)\#} < m < x$$

We can note that x is a solution only when the following inequality holds (and $x \geq 5$).

$$\frac{(x-1)!}{2(x-1)\#} < x$$

It is possible to solve the inequality, whereupon we get an upper bound for the possible values x can have.

Theorem 4. *The variable x has the following upper bound.*

$$x \leq 6$$

Proof. Let us expand the factorial function in the inequality, which we got above.

$$\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdots (x-2) \cdot (x-1)}{2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdots p_k} < x,$$

where p_k is the greatest prime, for which $p_k \leq x-1$. The primes in the numerator and denominator can be cancelled. In addition, the coefficiont 4 of the numerator can be cancelled with the extra 2 in the denominator. The number 4 is in the numerator, because $x \geq 5$.

$$\frac{\cancel{2} \cdot \cancel{3} \cdot \overset{2}{\cancel{4}} \cdot \cancel{5} \cdots (x-2) \cdot (x-1)}{\cancel{2} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{7} \cdots \cancel{p_k}} < x \tag{4}$$

$$2 \cdot 6 \cdot 8 \cdots (x-2) \cdot (x-1) < x$$

The inequality is most of the time false, because $2 \cdot (x-1) < x$ is intuitively false for positive integers $x > 1$.

We will build a more precise argument and find the solution to the inequality. If $x-1$ is prime, it is also in the denominator of the fraction in (4). Thus, $x-1$ will get cancelled, and the inequality becomes the following.

$$2 \cdot 6 \cdot 8 \cdots (x-3) \cdot (x-2) < x$$

If $x-1$ is not prime, then $x-1$ doesn't get cancelled. If $x-1$ is prime and the inequality is true, then $2(x-2) < x \iff 2x-4-x < 0 \iff x < 4$. If it is not prime and the inequality holds, then $2(x-1) < x \iff 2x-2-x < 0 \iff x < 2$. But we assume that $x \geq 5$, so the inequality does not hold for any integer *in the domain.*

But we must take into consideration the cases, when $x = 5$ and $x = 6$. When $x = 5$, the inequality is the following.

$$\frac{\cancel{2} \cdot \cancel{3} \cdot \cancelto{2}{4}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{3}} < 5$$
$$2 < 5$$

Thus the inequality is true when $x = 5$.

When $x = 6$, the inequality is the following, and it too holds.

$$\frac{\cancel{2} \cdot \cancel{3} \cdot \cancelto{2}{4} \cdot \cancel{5}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{5}} < 6$$
$$2 < 6$$

Thus the upper bound of x is 6. $\square$

When the upper bound for x has been found, the solution is near. The factorial is not defined for negative integers, so the lower bound of x is 1. Based on **Theorem 1**, it is enough to check primes between one and six. The solutions can be checked in the following way. We get $n = \log_x(x! + x)$ from the equation (†). If $\log_x(x! + x)$ is an integer for some integer x, then x is a solution to the equation.

Let us check all the primes $1 \leq p \leq 6$.

$$\log_2(2! + 2) = 2 \in \mathbb{Z}$$
$$\log_3(3! + 3) = 2 \in \mathbb{Z}$$
$$\log_5(5! + 5) = 3 \in \mathbb{Z}$$

Thus the only integer pair solutions (x, n) to the equation $x! + x = x^n$ are $(2, 2)$, $(3, 2)$ and $(5, 3)$.

A Generalizations

After solving the problem, I wanted to investigate its generalizations. However for lack of time, I haven't found any satisfactory results but I want to share my discoveries more freely in the appendix.

I have studied the general form of the equation (†):

$$ax! + bx = cx^n,$$

where a, b and c are non-zero integer coefficients.

A.1 $ax! + x = x^n$

I have approached the equation by simplifying it. I began by changing the equation to the form $ax! + x = x^n$.

The equation is equivalent with the following equation.

$$a(x-1)! + 1 = x^m \tag{5}$$

There are two possibilities: either $\gcd(a, x) \neq 1$, or $\gcd(a, x) = 1$. If $\gcd(a, x) \neq 1$, then the following equation holds.

$$a(x-1)! + 1 \equiv x^m \pmod{\gcd(a, x)}$$
$$0 + 1 \equiv 0 \pmod{\gcd(a, x)}$$

This equation is true only if $\gcd(a, x) = 1$, which contradicts the assumption. Therefore $\gcd(a, x) = 1$.

The equation (5) can be inspected from the perspective of modular arithmetic with modulus x:

$$a(x-1)! + 1 = x^m$$
$$a(x-1)! \equiv -1 \pmod{x}$$
$$(x-1)! \equiv -a^{-1} \pmod{x}$$

In this case, we can use Wilson's theorem, which states that x is a composite number if and only if $(x-1)! \not\equiv -1 \pmod{x}$.

Thus, if $a^{-1} \not\equiv 1$, then x is composite. On the basis of *Lemma 1*, either $x = 4$ or $(x-1)! \equiv 0 \pmod{x}$. $x = 4$ can be substituted in the equation (5):

$$a(4-1)! + 1 = 4^m$$
$$2(3a) + 1 = 2(2 \cdot 4^{m-1})$$

The LHS of the equation is odd and the RHS is even $(m \neq 1)$. Thus the only solutions would be the trivial family of solutions: $(a, m) = (0, 0)$.

Let's, therefore, assume that $(x-1)! \equiv 0 \pmod{x}$.

$$(x-1)! \equiv -a^{-1} \pmod{x}$$
$$0 \equiv -a^{-1} \pmod{x}$$
$$0 \cdot a \equiv -a^{-1} \cdot a \pmod{x}$$
$$0 \equiv -1 \pmod{x}$$

The equation is true only when $x = 1$. Let us substitute it to the equation (5).

$$a(1-1)! + 1 = 1^m$$
$$a + 1 = 1^m$$
$$a + 1 = 1$$
$$\implies a = 0$$

Thus the solution is $(a, x, m) = (0, 1, m_0)$, where m_0 is any non-negative integer.

Lastly, one must inspect the case, where $a^{-1} \equiv 1$, when x is a prime by Wilson's theorem. If $a \not\equiv 1$, then 1 cannot be the inverse of a. Therefore $a \equiv 1$

(mod x) $\iff x \mid a-1$. When searching the solutions, one must check the prime factors of $a-1$.

For example, let's solve the equation $7x! + x = x^n$. The primefactors of $7 - 1 = 6$ are 2 and 3. Let's check the possible solutions in the same way as when solving the equation (†).

$$\log_2(7 \cdot 2! + 2) = 4 \in \mathbb{Z}$$
$$\log_3(7 \cdot 3! + 3) = 3,464\ldots \notin \mathbb{Z}$$

Thus the equation has exactly one solution: $(x, n) = (2, 4)$.

A.2 $x! + bx = x^n$

The variable b can be isolated in the same way, and we get the following equation.

$$(x-1)! + b = x^m \tag{6}$$

Let us assume that x is a prime. The (6) can be inspected through modular arithmetic, once again.

$$(x-1)! + b \equiv x^m \pmod{x}$$
$$(x-1)! \equiv -b \pmod{x}$$

Because x is prime, the equation $(x-1)! \equiv -1 \pmod{x}$ holds by Wilson's theorem. Therefore $b \equiv 1 \pmod{x}$. So, outside of modular arithmetic, the following holds.

$$b = kx + 1$$

for some integer k.

Therefore we assume again that x is not prime. Based on *Lemma 1*, $(x-1)! \equiv 0 \pmod{x}$, if $x \neq 4$.

$$(x-1)! + b \equiv x^m \pmod{x}$$
$$0 + b \equiv 0 \pmod{x}$$

Thus x divides b.

For example, if we want to find the integer solutions to the equation $x! + 15x = x^n$, we must check the divisors of 15, the number 4 and the prime factors of 14. The divisors of 15 are 1, 3, 5 and 15. The prime factors of 14 are 2 and 7. We can check the possible solutions:

$$\log_2(2! + 15 \cdot 2) = 5 \in \mathbb{Z}$$
$$\log_3(3! + 15 \cdot 3) = 3,578 \ldots \notin \mathbb{Z}$$
$$\log_4(4! + 15 \cdot 4) = 3,196 \ldots \notin \mathbb{Z}$$
$$\log_5(5! + 15 \cdot 5) = 3,276 \ldots \notin \mathbb{Z}$$
$$\log_7(7! + 15 \cdot 7) = 4,391 \ldots \notin \mathbb{Z}$$
$$\log_{15}(15! + 15 \cdot 15) = 10,302 \ldots \notin \mathbb{Z}$$

Thus our equation has exactly one solution: $(x, n) = (2, 5)$.

A.3 $x! + x = cx^n$

We can simplify this equation too.

$$(x-1)! + 1 = cx^m \tag{7}$$

Let's inspect all the possible values of c with respect to x. Thus, we'll temporarily think of x as a constant, and c as a variable.

1. Let's assume that $0 < c < x$. Then it follows that $c \mid (x-1)!$.

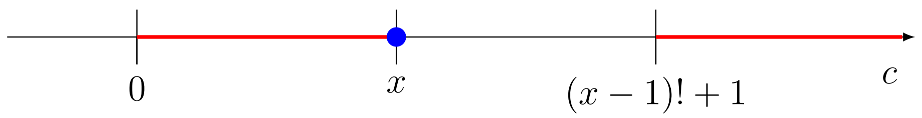
$$(x-1)! + 1 \equiv cx^m \pmod{c}$$
$$0 + 1 \equiv 0 \pmod{c}$$

This is a contradiction, so $c \neq 1$.

2. If $c = x$, the equation becomes $x! + x = x^{n+1}$. Its solutions correspond to the solutions of the equation (†).

3. If $(x-1)! + 1 < c$, then $c \nmid (x-1)! + 1$. The larger number cannot dived the smaller. But the RHS of the equation $(x-1)! + 1 = cx^n$ is divisible by c. This is a contradiction.

The following figure illustrates what is stated above.



There are no solutions on the red area. The solutions on the blue dot are known. But I have not yet studied closely the area that is not colored.

The blue point gives three solutions to the problem. If c corresponds to an x of some solution to the equation (†), then the solution of the equation $x! + x = cx^n$ is $(x, n-1)$, where (x, n) is the corresponding solution to (†). For example, one solution to the equation (†) is $(x, n) = (2, 2)$. Thus, if $c = 2$, then the solution to the equation $x! + x = 2x^n$ is $(2, 2-1) = (2, 1)$.

Because I haven't found more precise results about the interval $x < c \leq (x-1)! + 1$, we must check all of the possible values of x that satisfy the inequality.

For example, let's solve the equation $x! + x = 6x^n$. The values of x, for which $x < 6 \leq (x-1)! + 1$ hold, are 4 and 5. Let us therefore check the solutions.

$$\log_4(\frac{4!+4}{6}) = 1,111\ldots \notin \mathbb{Z}$$
$$\log_5(\frac{5!+5}{6}) = 1,886\ldots \notin \mathbb{Z}$$

Thus the equation $x! + x = 6x^n$ has no integer solutions.

I encourage the reader to investigate the solutions of the general form, and to come up with new general forms. For example, I have myself considered the equation $x! + p(x) = x^n$, where $p(x)$ is some polynomial with integer coefficients. The search for solutions of the general forms continues.