The Non-forced Spherical Pendulum: Semi-numerical Solutions

Richard J. Mathar*

Max-Planck Institute of Astronomy, Königstuhl 17, 69117 Heidelberg, Germany

(Dated: September 9, 2019)

Classical mechanics models the *plane* pendulum as a point mass fastened to a pole by a cord of fixed length. The mass is released at some distance from the pole. It moves along a section of a circle; the circle lies in a plane defined by the pole, the initial place, and the direction of the gravitational force. This manuscript deals with semi-numerical solutions of the equations of motion of the *spherical* pendulum. This pendulum has some azimuthal velocity and non-vanishing angular momentum. The cord restricts the motion to the surface of a sphere. The instantaneous plane of motion of the mass is no longer constant.

PACS numbers: 45.20.Jj

I. COORDINATE SYSTEM

A. Cartesian Coordinates and Angles

An ideal point mass is suspended from a pendulum of cord length l. The mass has Cartesian coordinates x, y, z, where the point $x = y = z = 0$ is the lowest point reachable by the mass and where it will rest. We assume the pendulum restricts the mass position to the sphere at distance l from $(x, y, z) = (0, 0, l)$:

$$x^2 + y^2 + (z - l)^2 = l^2. \tag{1}$$

The constraint introduces two angles: The first, φ (measured in radians), is the angle between the cord and the vertical of the pole. The second is an angle λ of the azimuth,

$$\tan\lambda = \frac{y}{x}. \tag{2}$$

λ defines the instantaneous plane of the motion. There is a rectangular triangle with vertices at $(0, 0, z)$, (x, y, z) and $(0, 0, l)$ which has side lengths l, $\sqrt{x^2 + y^2}$ and $l - z$ from which φ can be obtained:

$$\sin\varphi = \frac{\sqrt{x^2 + y^2}}{l}. \tag{3}$$

$$\cos\varphi = \frac{l - z}{l} = 1 - z/l; \quad z = l(1 - \cos\varphi). \tag{4}$$

$$\sin\lambda = \frac{y}{\sqrt{x^2 + y^2}}; \cos\lambda = \frac{x}{\sqrt{x^2 + y^2}}; \tag{5}$$

$$x = \cos\lambda\sqrt{x^2 + y^2} = l\sin\varphi\cos\lambda; \tag{6}$$

$$y = \sin\lambda\sqrt{x^2 + y^2} = l\sin\varphi\sin\lambda; \tag{7}$$

---

* https://www.mpia-hd.mpg.de/~mathar

B. Time Derivatives

The derivatives of the Cartesian coordinates with respect to time — indicated by a dot on top of the variables — are

$$\dot{x} = l\cos\lambda\frac{d}{dt}\sin\varphi + l\sin\varphi\frac{d}{dt}\cos\lambda = l\dot{\varphi}\cos\lambda\cos\varphi - l\dot{\lambda}\sin\varphi\sin\lambda. \tag{8}$$

$$\dot{y} = l\sin\lambda\frac{d}{dt}\sin\varphi + l\sin\varphi\frac{d}{dt}\sin\lambda = l\dot{\varphi}\sin\lambda\cos\varphi + l\dot{\lambda}\sin\varphi\cos\lambda. \tag{9}$$

$$\dot{z} = -l\frac{d}{dt}\cos\varphi = l\dot{\varphi}\sin\varphi. \tag{10}$$

The squared velocity is

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2(\dot{\lambda}^2\sin^2\varphi + \dot{\varphi}^2). \tag{11}$$

The projection of the acceleration into the direction of the suspension is

$$\frac{1}{l}[x\ddot{x} + y\ddot{y} + (z-l)\ddot{z}] = -l(\dot{\varphi}^2 + \dot{\lambda}^2\sin^2\varphi), \tag{12}$$

which is basically the negated squared velocity — equivalent to the standard relation between centrifugal force and squared angular velocity.

II. ENERGIES

We set the zero of the potential energy V at the origin of the Cartesian Coordinates — the lowest point accessible to the pendulum — and assume that the gravitational potential is homogeneous:

$$V = mgz. \tag{13}$$

m is the mass at the end of the pendulum and g the acceleration. The Kinetic Energy K is proportional to the square of the velocity v:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{14}$$

III. LAGRANGIAN

A. Euler-Lagrange equations

The Lagrangian of the system is [1–3]

$$\mathcal{L} = K - V = \frac{1}{2}ml^2(\dot{\lambda}^2\sin^2\varphi + \dot{\varphi}^2) - mgl(1-\cos\varphi). \tag{15}$$

The Euler-Lagrange equations of the two generalized coordinates are

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}; \tag{16}$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\lambda}}, \tag{17}$$

explicitly

$$\frac{1}{2}ml^2\dot{\lambda}^2 2\sin\varphi\cos\varphi - mgl\sin\varphi = \frac{d}{dt}[\frac{1}{2}ml^2 2\dot{\varphi}]; \tag{18}$$

$$0 = \frac{d}{dt}[\frac{1}{2}ml^2\sin^2\varphi 2\dot{\lambda}]. \tag{19}$$

$$\dot{\lambda}^2\sin\varphi\cos\varphi - \frac{g}{l}\sin\varphi = \frac{d}{dt}\dot{\varphi}. \tag{20}$$

B. Conical Pendulum

There is one particular solution where $\ddot{\lambda} = \dot{\varphi} = 0$ such that

$$\dot{\lambda}^2 \cos\varphi = g/l. \tag{21}$$

This system is known as the conical pendulum because the mass swings on a circle of constant distance to the pole such that the the forces exerted by the cord and by the gravitation are keeping φ and $\dot{\lambda}$ constant in time.

The centrifuge limit is $\varphi \to \pi/2$, $\cos\varphi \to 0$ and $\dot{\lambda} \to \infty$

IV. DIFFERENTIAL EQUATIONS OF MOTION

A. Separation of canonical variables

A first integral of the second order differential equation for $\ddot{\lambda}$ from (19) is obvious:

$$\sin^2\varphi\dot{\lambda} = L_0. \tag{22}$$

It represents the conservation of angular momentum. ($l^2\dot{\lambda}^2\sin^2\varphi$ is the squared tangential velocity $\dot{x}^2 + \dot{y}^2$, and $l^2\sin^2\varphi = x^2 + y^2$ is the squared distance to the pole. So $m^2l^4\sin^4\varphi\dot{\lambda}^2$ is the squared angular momentum, and the equation above is up to constants the angular momentum.) It also implicitly says that the general solutions avoids $\varphi = 0$, the point of lowest potential, because that would force L_0 to fall to an abrupt and intermediate non-continuous zero at that point.

The second order differential equation (20) is

$$\ddot{\varphi} - \dot{\lambda}^2 \sin\varphi\cos\varphi + \frac{g}{l}\sin\varphi = 0. \tag{23}$$

The usual approach for differential equations that do not contain the first derivative [4, §1.12.4.11]: multiply by $2\dot{\varphi}$ to obtain a first order differential equation:

$$2\dot{\varphi}\ddot{\varphi} - 2\dot{\lambda}^2\dot{\varphi}\sin\varphi\cos\varphi + \frac{2g}{l}\dot{\varphi}\sin\varphi = 0. \tag{24}$$

$$\frac{d}{dt}[\dot{\varphi}^2] - 2\dot{\lambda}^2\dot{\varphi}\sin\varphi\cos\varphi + \frac{2g}{l}\dot{\varphi}\sin\varphi = 0. \tag{25}$$

Elimination of $\dot{\lambda}^2$ via (22) decouples the two differential equations:

$$\frac{d}{dt}[\dot{\varphi}^2] - 2L_0^2\dot{\varphi}\frac{\cos\varphi}{\sin^3\varphi} + \frac{2g}{l}\dot{\varphi}\sin\varphi = 0. \tag{26}$$

$$\frac{d}{dt}[\dot{\varphi}^2] - 2L_0^2\frac{d}{dt}[-\frac{1}{2\sin^2\varphi}] - \frac{2g}{l}\frac{d}{dt}[\cos\varphi] = 0. \tag{27}$$

The format now is similar to the radial equations in a gravitational potential where the squared angular momentum modifies the effective radial potential. The integration constant is written as a function of φ_0 at some reference point in time:

$$\dot{\varphi}^2 + L_0^2\frac{1}{\sin^2\varphi} - \frac{2g}{l}\cos\varphi = \dot{\varphi}_0^2 + L_0^2\frac{1}{\sin^2\varphi_0} - \frac{2g}{l}\cos\varphi_0. \tag{28}$$

Starting at the upper angular limit φ_u, the angle drops, $\dot{\varphi} < 0$, and its cosine increases, $(d/dt)\cos\varphi > 0$. Starting from the φ_l, the signs are the opposite.

$$\dot{\varphi} = \mp\sqrt{\dot{\varphi}_0^2 + L_0^2\frac{1}{\sin^2\varphi_0} - \frac{2g}{l}\cos\varphi_0 - L_0^2\frac{1}{\sin^2\varphi} + \frac{2g}{l}\cos\varphi}. \tag{29}$$

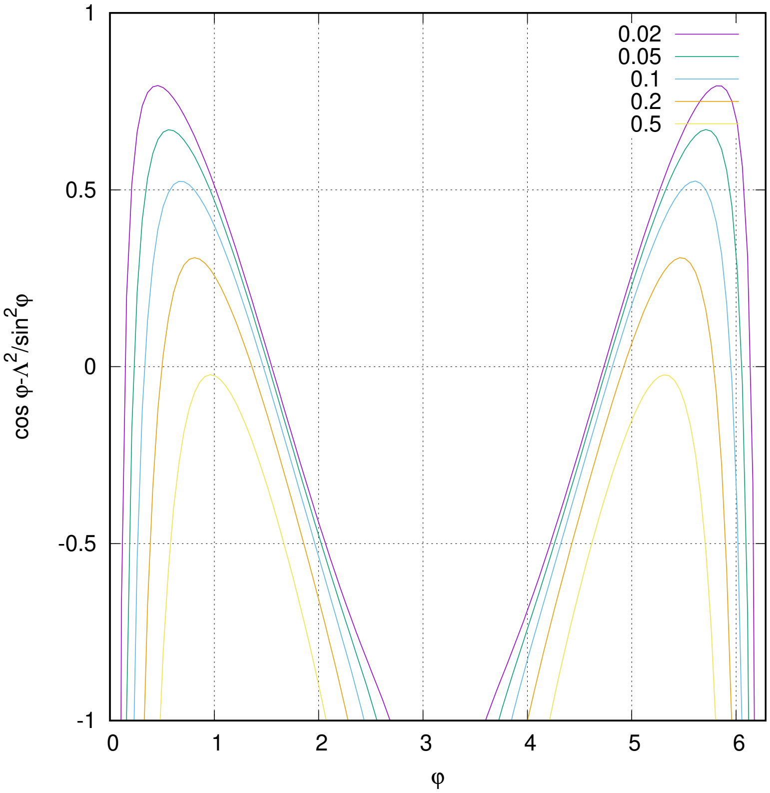

FIG. 1. The discriminant $D(\varphi)$ as a function of φ for five values of Λ^2.

$$\mp \int_{\varphi_0} \frac{d\varphi}{\sqrt{\dot\varphi_0^2 + L_0^2 \frac{1}{\sin^2\varphi_0} - \frac{2g}{l}\cos\varphi_0 - L_0^2\frac{1}{\sin^2\varphi} + \frac{2g}{l}\cos\varphi}} = \int_0 dt = t. \tag{30}$$

We solve the homogeneous equation, i.e., we shift the time variable such that $\dot\varphi_0 = 0$, measuring time from one of the extreme positions of the motions at maximum or minimum amplitude φ:

$$\mp \int_{\varphi_0} \frac{d\varphi}{\sqrt{\cos\varphi - \frac{lL_0^2}{2g}\frac{1}{\sin^2\varphi} - \cos\varphi_0 + \frac{lL_0^2}{2g}\frac{1}{\sin^2\varphi_0}}} = \sqrt{\frac{2g}{l}}t. \tag{31}$$

Define a unitless time $T \equiv \sqrt{2g/l}t$ and the unitless angular momentum

$$\Lambda^2 \equiv lL_0^2/(2g) \tag{32}$$

— which means, define the time derivative in (22) also on the new scale — to reduce the radial integral to

$$\mp \int_{\varphi_0} \frac{d\varphi}{\sqrt{\cos\varphi - \Lambda^2\frac{1}{\sin^2\varphi} - \cos\varphi_0 + \Lambda^2\frac{1}{\sin^2\varphi_0}}} = T. \tag{33}$$

The discriminant

$$D(\varphi) \equiv \cos\varphi - \Lambda^2/\sin^2\varphi \tag{34}$$

is periodic with period 2π, illustrated in Figure 1. The maximum is where $(d/d\varphi)D = 0$, the conical pendulum:

$$\sin^4\hat\varphi = 2\Lambda^2\cos\hat\varphi \tag{35}$$

The is a quartic equation in $\cos\hat\varphi$ and could be solved directly. A suitable series expansion is

$$\hat\varphi = 2^{1/4}\sqrt{\Lambda}[1 + \frac{\sqrt{2}}{24}\Lambda - \frac{7}{320}\Lambda^2 - \frac{65\sqrt{2}}{3584}\Lambda^3 - \frac{1045}{73728}\Lambda^4 - \frac{1785\sqrt{2}}{720896}\Lambda^5 + \frac{14973}{6815744}\Lambda^6 + \frac{153439\sqrt{2}}{62914560}\Lambda^7 + \cdots], \tag{36}$$

$$\sin\hat{\varphi} = 2^{1/4}\sqrt{\Lambda}\left[1 - \frac{\sqrt{2}}{8}\Lambda - \frac{3}{64}\Lambda^2 - \frac{3\sqrt{2}}{512}\Lambda^3 + \frac{35}{8192}\Lambda^4 + \cdots\right]. \tag{37}$$

The value $D(\hat{\varphi})$ at that maximum is

$$D(\hat{\varphi}) = 1 - \sqrt{2}\Lambda - \frac{1}{4}\Lambda^2 - \frac{\sqrt{2}}{16}\Lambda^3 - \frac{1}{32}\Lambda^4 - \frac{3\sqrt{2}}{512}\Lambda^5 + \frac{11\sqrt{2}}{8192}\Lambda^7 + \frac{3}{2048}\Lambda^8 + \cdots \tag{38}$$

It is positive if Λ is in the range

$$0 \le \Lambda^2 < \frac{2}{3^{3/2}} \approx 0.384900179. \tag{39}$$

B. Equilibrium Points

If the value of $D(\varphi) - D(\varphi_0)$ becomes zero, the pendulum has reached a point of zero radial velocity, either a point at maximum amplitude or a point of closest swing by the pole. The initial conditions of the individual trajectory are set by shifting the plots of Figure 1 up or down until the curve becomes zero at the value of φ_0 that is selected to be an equilibrium point. $D(\varphi) - D(\varphi_0)$ is essentially the negative value of the energy E at the equilibrium points.

Thus having fixed φ_0 and $D(\varphi_0) \equiv D_0$, one task is to find the other root(s) of the equation

$$\cos\varphi - \Lambda^2/\sin^2\varphi - D_0 = 0, \tag{40}$$

the second equilibrium point. Multiplied by $\sin^2\varphi$, this yields a cubic equation for the cosine,

$$\cos^3\varphi - \cos^2\varphi D_0 - \cos\varphi + \Lambda^2 + D_0 = 0. \tag{41}$$

Some comments of solving this without recourse to complex arithmetic are added in Appendix B. An overview of the solutions is given in Figure 2. If $D_0 < 0$, some parts of the trajectory may push the mass transiently through negative values of the cosine, i.e., above the horizon of the suspension.

The substitution $\cos\varphi = \theta$ rewrites the integral (33) as an elliptic integral [5, 6][7, (17.4.68)][8, 3.131.5][9, (235.00)].

$$\mp\int_{\varphi_0} \frac{d\varphi}{\sqrt{\cos\varphi - \Lambda^2/\sin^2\varphi - D_0}} = T \tag{42}$$

$$\pm\int_{\theta_0} \frac{d\theta}{\sqrt{-\theta^3 + D_0\theta^2 + \theta - \Lambda^2 - D_0}} = T. \tag{43}$$

Supposed the three roots $\cos\varphi_u = \theta_u$, $\cos\varphi_l = \theta_l$, and $\cos\varphi_s = \theta_s$ of the cubic equation of the cosine are known, time T is an Elliptic Integral of the First Kind:

$$\pm\int_{\theta_0} \frac{d\theta}{\sqrt{-(\theta-\theta_l)(\theta-\theta_u)(\theta-\theta_s)}} = T. \tag{44}$$

$$\pm F(\phi\backslash\pi/2 - \tau) = \frac{1}{2}\sqrt{\theta_l - \theta_s}T, \tag{45}$$

where $\sin^2\phi = (\theta_l - \theta_s)(\theta - \theta_u)/[(\theta_l - \theta_u)(\theta - \theta_s)]$, $\sin^2\tau = (\theta_u - \theta_s)/(\theta_l - \theta_s)$. Integration between the two equilibrium points θ_u and θ_l is the quarter period $P/4$

$$F(\pi/2\backslash\pi/2 - \tau) = \frac{1}{2}\sqrt{\theta_l - \theta_s}P/4. \tag{46}$$

(The nomenclature is here adapted to the plane pendulum, where the mass returns to the same farthest position after *two* passages through the lowest point.) Numerical evaluation of the Elliptic Integrals gives the curves of Figure 3.

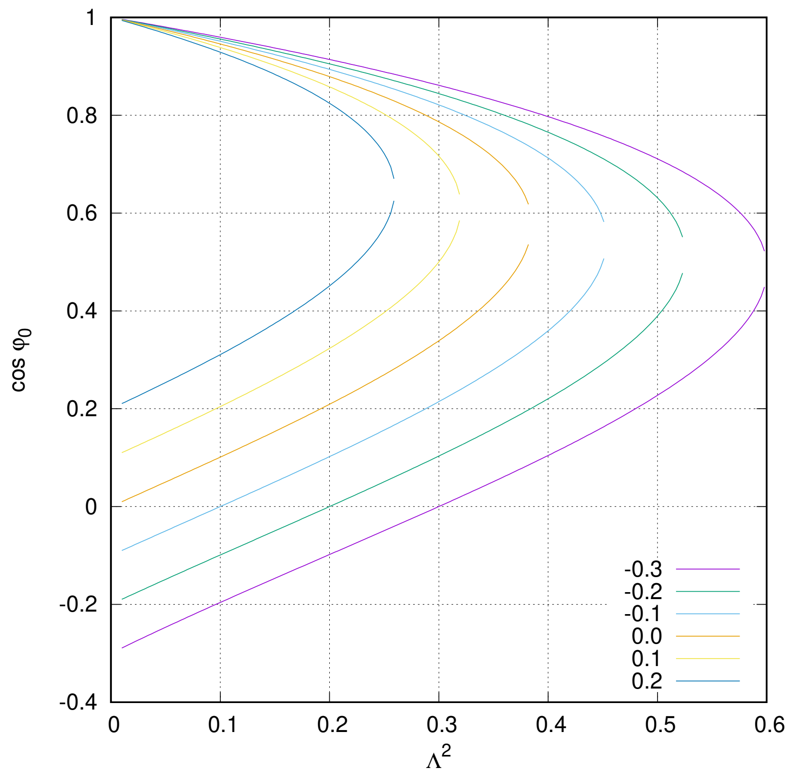

FIG. 2. The cosines of the two equilibrium positions, $\cos\varphi_u$ and $\cos\varphi_l$, as a function of Λ^2 for 6 different values of D_0 in the range of -0.3 to 0.2. The maximum D_0 for which solutions exist depends on Λ via Eq. (38).

V. TRAJECTORY

A. Of the Radial Angle

Equation (33) describes the evolution of the angle φ as a function of dimensionless time T,

$$\frac{d}{dT}\varphi = \mp\sqrt{\cos\varphi - \Lambda^2/\sin^2\varphi - D_0}. \tag{47}$$

The upper and lower signs apply starting at the farther or closer equilibrium point, respectively The associated second order differential equation (23) of the dimensionless variables is

$$\frac{d^2}{dT^2}\varphi = \Lambda^2\frac{\cos\varphi}{\sin^3\varphi} - \frac{1}{2}\sin\varphi. \tag{48}$$

At the start position we might substitute Λ^2 via (40),

$$\frac{d^2}{dT^2}\varphi_0 = \Lambda^2\frac{\cos\varphi_0}{\sin^3\varphi_0} - \frac{1}{2}\sin\varphi_0 = \frac{3\cos^2\varphi_0 - 1 - 2D_0\cos\varphi_0}{2\sin\varphi_0} \tag{49}$$

so the lowest non-trivial order of the solution near a turning point starts

$$\varphi = \varphi_0 + \frac{1}{2}\frac{3\cos^2\varphi_0 - 1 - 2D_0\cos\varphi_0}{2\sin\varphi_0}T^2 + O(T^3). \tag{50}$$

Iterative insertion into (48) gives a power series of φ in terms of T. This format is complicated and not converging well if the entire amplitude from φ_l to φ_u needs to be covered.

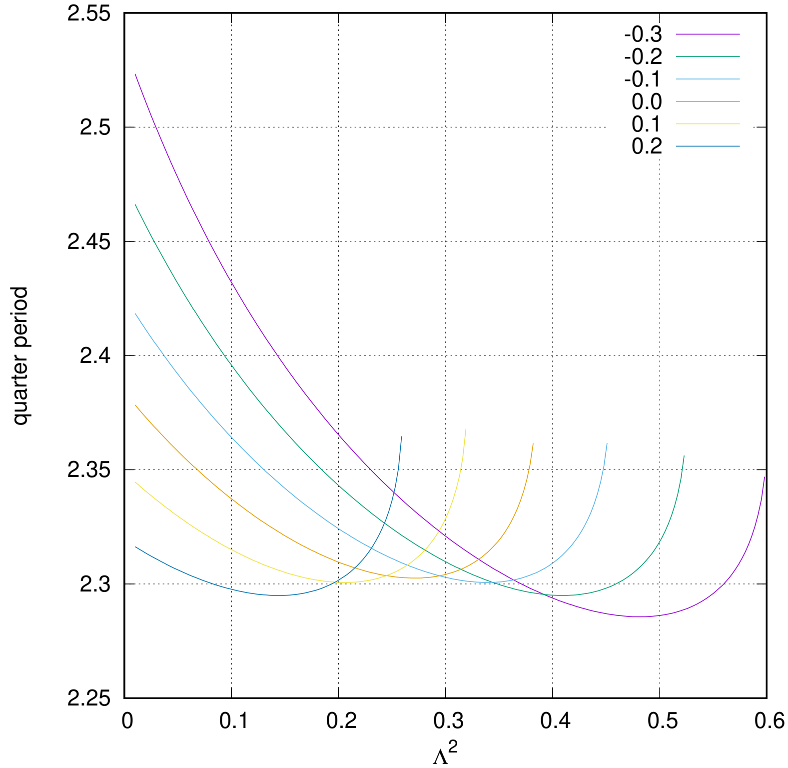

FIG. 3. The quarter period $P/4$ as a function of Λ^2 for the same 6 different signed values of D_0 as in Figure 2.

B. Of the Altitude

The differential equation of the cosine of the angle (basically proportional to the altitude z or the gravitational energy V) turns out to be simpler than Equation (48) for the angle. Let the tick mark denote derivatives with respect to T. Then

$$\frac{d}{dT}\cos\varphi = -\sin\varphi\varphi'. \tag{51}$$

$$\frac{d^2}{dT^2}\cos\varphi = -\cos\varphi(\varphi')^2 - \sin\varphi\varphi'' \tag{52}$$

Insertion of (47) and (48) eliminates Λ:

$$\frac{d^2}{dT^2}\cos\varphi = \frac{1}{2} - \frac{3}{2}\cos^2\varphi + D_0\cos\varphi. \tag{53}$$

We bootstrap the series expansion of $\cos\varphi$ as a function of T by repeated integration of (53) from $\varphi_0 = \varphi_u$ or φ_l. The iteration starts with the observation that the linear term vanishes, $(d/dT)\cos\varphi_0 = -\sin\varphi_0 \cdot 0 = 0$. The quadratic order is copied from (53):

$$\cos\varphi = \cos\varphi_0 + \frac{1}{2}(\frac{1}{2} - \frac{3}{2}\cos^2\varphi_0 + D_0\cos\varphi_0)T^2 + O(T^3). \tag{54}$$

The iteration of such polynomial expressions of T inserts the square into the left hand side of (53), integrates twice over T, re-installs the constant term $\cos\varphi_0$, and arrives at a polynomial that is correct to the previous order plus 2.

$$\cos\varphi = \cos\varphi_0 + \frac{1-3\cos^2\varphi_0+2D_0\cos\varphi_0}{4}T^2 - \frac{(3\cos\varphi_0-D_0)(1-3\cos^2\varphi_0+2D_0\cos\varphi_0)}{48}T^4 \tag{55}$$

$$-\frac{(45\cos^2\varphi_0-30D_0\cos\varphi_0-9+2D_0^2)(1-3\cos^2\varphi_0+2D_0\cos\varphi_0)}{2880}T^6$$
$$-\frac{(3\cos\varphi_0-D_0)(90\cos^2\varphi_0-60D_0\cos\varphi_0-27+D_0^2)(1-3\cos^2\varphi_0+2D_0\cos\varphi_0)}{80640}T^8+\cdots \tag{56}$$

By defining power series coefficients c_i,

$$\cos\varphi = \cos\varphi_0 + \frac{1-3\cos^2\varphi_0+2D_0\cos\varphi_0}{4}T^2\sum_{i\ge 0}c_i(T/2)^{2i}, \tag{57}$$

the c_i are recursively computable, rooted at $c_0=1$, as

$$c_i = -(3\cos\varphi_0-D_0)\frac{c_{i-1}}{(i+1/2)(i+1)} - \frac{1-3\cos^2\varphi_0+2D_0\cos\varphi_0}{2}\frac{3}{(i+1/2)(i+1)}\sum_{j=0}^{i-2}c_jc_{i-2-j}. \tag{58}$$

In numerical practice this power series is computed separately for $\varphi_0=\varphi_u$ and $\varphi_0=\varphi_l$, and both curves are stitched at some intermediate point (for example at the inflection point where $(d^2/dT^2)\cos\varphi$ is zero).

Rephrasing the power series as a function of time measured in units of the quarter period — which is a matter of multiplying the coefficients with powers of $P/4$ — this can be written as a sum of Chebyshev Polynomials Of The First Kind of $4T/P$, and is equivalent to a Fourier time series of $\cos\varphi$.

C. Evolution of the Azimuth

Equation (22) for the speed of the azimuth plane reads

$$\sin^2\varphi\lambda' = \Lambda \tag{59}$$

in the unitless time scale T. The sine term of the left hand side can be rephrased by the coefficients c_i which were computed in Section V B with the aid of Eq. (53):

$$\sin^2\varphi = \frac{2}{3} + \frac{2}{3}\frac{d^2}{dT^2}\cos\varphi - \frac{2}{3}D_0\cos\varphi$$
$$= \frac{2}{3}(1-D_0\cos\varphi_0) + \frac{8}{3}\frac{1-3\cos^2\varphi_0+2D_0\cos\varphi_0}{4}\sum_{i\ge 0}c_i(i+1)(i+1/2)(T/2)^{2i}$$
$$-\frac{2}{3}D_0\frac{1-3\cos^2\varphi_0+2D_0\cos\varphi_0}{4}T^2\sum_{i\ge 0}c_i(T/2)^{2i}. \tag{60}$$

In principle we need a series reversion to get λ' as a function of T from here. In concrete, the power series of λ' as a function of T can be derived with Leibniz' rule [7, (3.3.8)][8, 0.42]

$$\frac{d^j}{dT^j}(\sin^2\varphi\lambda') = 0 = \sum_{k=0}^{j}\binom{j}{k}\frac{d^k}{dT^k}\sin^2\varphi\frac{d^{j-k}}{dT^{j-k}}\lambda', \quad (j\ge 1). \tag{61}$$

The derivatives $(d^k/dT^k)\sin^2\varphi$ are essentially the coefficients of the power series of the previous equation.

VI. SUMMARY

We have established Eq. (58) to compute the power series of the cosine of the rope angle of the spherical pendulum versus the vertical as a function of the time, and use the same power series in Eq. (61) to compute the power series of the azimuth angle.

Appendix A: Initial Conditions

The initial conditions of the trajectory may be specified by initial position vector (x, y, z) and initial velocity vector $(\dot{x}, \dot{y}, \dot{z})$. The transition to the dimensionless control parameters of our description may be installed as follows: The initial angles φ and λ are obtained from (4) and (5). $\dot{\lambda}$ is computed by the time derivative of (2),

$$\frac{1}{\cos^2 \lambda}\dot{\lambda} = \frac{\dot{y}x - y\dot{x}}{x^2}. \tag{A1}$$

$\dot{\varphi}$ is computed by the time derivative of (4),

$$\dot{\varphi} \sin\varphi = \dot{z}/l. \tag{A2}$$

L_0 follows from (22), then Λ^2 from (32), then D_0 from (40).

The initial position has 3 coordinates, but the constraint to a sphere of radius l reduces these to 2 parameters, and since the system is invariant with respect to a rotation around the pole axis (in the spirit of the Noether Theorem the 'cause' of the conservation of angular momentum), only z is relevant. The initial velocity has 3 coordinates, but since the motion is bound to the sphere surface, it must be orthogonal to the initial position: $x\dot{x} + y\dot{y} + (z - l)\dot{z} = 0$, so 2 velocity parameters are independent. These $1 + 2$ independent coordinates have been transfused to the 2 parameters Λ^2 and D_0 above. The remaining third piece of information hides as the time T since the transit through one of the equilibrium points — or equivalently the amplitudes φ_0 of these equilibrium points.

The total energy $E = K + V$ is

$$\frac{2E}{ml^2} = \dot{\lambda}^2 \sin^2\varphi + \dot{\varphi}^2 + \frac{2g}{l}(1 - \cos\varphi) = \frac{2g}{l}\left(\frac{\Lambda^2}{\sin^2\varphi} + \varphi'^2 + 1 - \cos\varphi\right). \tag{A3}$$

It is constant in time and known from the initial conditions. The equilibrium angles φ_0 are found by setting $\varphi' = 0$ and solving the emerging cubic equation for $\cos\varphi_0$.

Appendix B: Antipodal Equilibrium Point

1. Long Division

If φ_0 and therefore $\cos\varphi_0$ are known, the other roots are found by long division of the cubic polynomial (41) through $\cos\varphi - \cos\varphi_0$, which gives the quadratic equation

$$\cos^2\varphi + [\cos\varphi_0 - D_0]\cos\varphi - \sin^2(\varphi_0) - \cos(\varphi_0)D_0 = 0 \tag{B1}$$

According to (40), $\cos\varphi_0 - D_0 > 0$, so we mainly search for for the positive upper sign in

$$\cos\varphi = -\frac{\cos\varphi_0 - D_0}{2} \pm \frac{1}{2}\sqrt{[\cos\varphi_0 - D_0^2 + 4\sin^2\varphi_0 + 4cos\varphi_0 D_0} \tag{B2}$$

This establishes the roots $\cos\varphi_u$, $\cos\varphi_l$ and a spurious solution $\cos\varphi_s < -1$, related by $\cos\varphi_u + \cos\varphi_l + \cos\varphi_s = D_0$.

2. Regularization

Removal of the quadratic term with the standard binomial compensation formula translates Equation (41) of the equilibrium angles into

$$\left[\cos\varphi - \frac{D_0}{3}\right]^3 - \left[1 + \frac{D_0^2}{3}\right]\left[\cos\varphi - \frac{D_0}{3}\right] + \Lambda^2 + \frac{2}{3}D_0 = 0. \tag{B3}$$

The coefficient in front of the remaining linear term is set to unity by dividing all therms through the $3/2$th power of the coefficient of the linear term — which is well defined because positive:

$$\left[\frac{\cos\varphi - \frac{D_0}{3}}{\sqrt{1 + D_0^2/3}}\right]^3 - \frac{\cos\varphi - \frac{D_0}{3}}{\sqrt{1 + D_0^2/3}} + \frac{\Lambda^2 + \frac{2}{3}D_0}{(1 + D_0^2/3)^{3/2}} = 0. \tag{B4}$$

This is a cubic equation of the form (41) at a scaled-shifted unknown $(\cos\varphi - D_0/3)/\sqrt{1+D_0^2/3}$ as if $D_0 = 0$, as if searching for the roots of the discriminant $D(\varphi)$ itself, and we only need to treat

$$\cos^3\varphi - \cos\varphi + \Lambda^2 = 0, \tag{B5}$$

a cubic equation for $\cos\varphi$. The roots can be written in terms of cubic roots of functions of Λ [7, 3.8.2][4, §2.1.6.2]. All three roots $\cos\varphi^{\uparrow}$, $\cos\varphi^{\downarrow}$ and $\cos\varphi^{\dagger}$ are real if (39) is satisfied; the are related by Vieta's formula: $\cos\varphi^{\dagger} = -\cos\varphi^{\uparrow} - \cos\varphi^{\downarrow} < 0$. We sort the others by $\varphi^{\downarrow} < \varphi^{\uparrow}$. The larger of these roots of D can be written as a Gaussian Hypergeometric Function [10]

$$\begin{aligned}\cos\varphi^{\uparrow} &= \Lambda^2 + \Lambda^6 + 3\Lambda^{10} + 12\Lambda^{14} + 55\Lambda^{18} + 273\Lambda^{22} + \cdots \\ &= \Lambda^2 \sum_{i\ge 0} \alpha_i^{\uparrow}\Lambda^{4i} = \Lambda^2 {}_2F_1(2/3, 1/3; 3/2; 27\Lambda^4/4)\end{aligned} \tag{B6}$$

where recursively

$$2i(2i+1)\alpha_i^{\uparrow} = 3(3i-1)(3i-2)\alpha_{i-1}^{\uparrow}. \tag{B7}$$

In the limit $\Lambda^2 \to 2/3^{3/2}$ we have $27\Lambda^4/4 \to 1$ and numerical implementation would use one of the quadratic transformations to accelerate convergence.

$${}_2F_1(2/3, 1/2; 3/2; 27\Lambda^4/4) \overset{\Lambda^2\to 2/3^{3/2}}{\longrightarrow} 3/2. \tag{B8}$$

$$\cos\varphi^{\uparrow} \overset{\Lambda^2\to 2/3^{3/2}}{\longrightarrow} \frac{1}{\sqrt{3}}. \tag{B9}$$

The smaller of these roots of D is at [11]

$$\begin{aligned}\cos\varphi^{\downarrow} &= -\frac{1}{2}\cos\varphi^{\uparrow} + \sqrt{1 - \frac{3}{4}\cos^2\varphi^{\uparrow}} \\ &= 1 - \frac{1}{2}\Lambda^2 - \frac{3}{8}\Lambda^4 - \frac{1}{2}\Lambda^6 - \frac{105}{128}\Lambda^8 - \frac{3}{2}\Lambda^{10} - \frac{3003}{1024}\Lambda^{12} + \cdots \\ &= {}_2F_1(1/6, -1/6; 1/2; 27\Lambda^4/4) - \frac{1}{2}\Lambda^2 {}_2F_1(1/3, 2/3; 3/2; 27/4\Lambda^4).\end{aligned} \tag{B10}$$

By setting $\varphi^{\uparrow} = \varphi^{\downarrow} = \hat{\varphi}$, i.e. solving cojointly the Equations (35) and

$$D(\hat{\varphi}) = 0 \Rightarrow \sin^2\varphi\cos\varphi = \Lambda^2 \tag{B11}$$

we arrive at (39).

Appendix C: Free Ballistic Fall

The model of the mass motion described in this manuscript keeps the mass at constant distance l from the point of suspension. It describes a mass rolling frictionless inside a sphere shell of radius l. This is not strictly equivalent to a pendulum realized with a cord, because that cord can pull the mass towards the pole, but cannot push it away. The vertical acceleration of the model mass is obtained by Eqs. (10), (23) and (28):

$$\ddot{z} = l\ddot{\varphi}\sin\varphi + l\cos\varphi\dot{\varphi}^2 = -g + 3g\cos^2\varphi - 2gD_0\cos\varphi. \tag{C1}$$

Free fall sets in, and the mass 'violates' the spherical constraint set forth by the Lagrangian, if it is at an 'overhead' position $\varphi > \pi/2$ — which requires $D_0 < 0$ — and if $\ddot{z} > -g$, that means if

$$D_0 > -\frac{3}{2}|\cos\varphi|. \tag{C2}$$

Eq. (40) confirms that this inequality is fulfilled at the farther equilibrium point if $\cos\varphi_u < 0$ and $\Lambda^2 < \frac{1}{2}|\cos\varphi_u|\sin^2\varphi_u$. So a pendulum with a real cord would leave the sphere shell under conditions of that kind.

---

[1] E. Horozov, J. reine angew. Math. **408**, 114 (1990).

[2] E. Horozov, Phys. Lett. A **173**, 279 (1993).
[3] P. H. Richter, H. R. Dullin, H. Waalkens, and J. Wiersig, J. Phys. Chem. **100**, 19124 (1996).
[4] I. N. Bronstein and K. A. Semendjajew, *Teubner's Taschenbuch der Mathematik*, edited by E. Zeidler (Teubner, 1996).
[5] A. Weinstein, Am. Math. Monthly **49**, 521 (1942).
[6] M. S. Albert and J. E. Marsden, Fields Inst. Commun. **8** (1995).
[7] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, 9th ed. (Dover Publications, New York, 1972).
[8] I. Gradstein and I. Ryshik, *Summen-, Produkt- und Integraltafeln*, 1st ed. (Harri Deutsch, Thun, 1981).
[9] P. F. Byrd and M. D. Friedman, *Handbook of Elliptical Integrals for Engineers and Physicists*, 2nd ed., Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Vol. 67 (Springer, Berlin, Göttingen, 1971).
[10] M. L. Glasser, J. Comp. Appl. Math. **118**, 169 (2000).
[11] H. F. Sandham, Quart. J. Math. **7**, 153 (1956).