MAXWELL AND DIRAC FIELD WITH THREE-DIMENSIONAL TIME

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Abstract:

In this work we further examine possible mathematical structures and physical properties of Maxwell and Dirac field by formulating the two fields in a six-dimensional spatiotemporal manifold in which both space and time have three dimensions. We show that although Maxwell and Dirac field each can be formulated as a single field in a more symmetrical structure in terms of space and time in the six-dimensional spatiotemporal continuum, Maxwell and Dirac six-dimensional field can be decoupled into two separate fields that exist in the Minkowski pseudo-Euclidean spactime and the Euclidean three-dimensional temporal manifold, respectively. The coexistent temporal elliptic field to Maxwell field in the three-dimensional temporal manifold is a free field and the coexistent temporal elliptic field to Dirac field is massless. While Maxwell and Dirac field comply with the pseudo-Euclidean relativity, both coexistent fields conform to the temporal Euclidean relativity.

1. Introduction

We have shown in our previous work on the nature of Maxwell and Dirac field that the sign of the time derivative plays an important role in the determination of their mathematical structures and physical properties when they are formulated as a coupling of two subfields associated with the positive and negative time derivatives respectively [1]. Since the inception of the negative time derivative arises from the mathematical formulation of Maxwell and Dirac field, the subfield associated with the negative time derivative may be perceived as a physical field that travels backwards in time. Despite the concept of travelling backwards in time seems unconceivable due mainly to the fact that we are conceivably composed of matter that involve only in physical processes that progress forward in time, it has been shown that the concept of progressing backwards in time can be invoked to formulate physical theories such as the theory of positrons [2]. In a more general framework we have also shown in our work on temporal dynamics that it is possible to formulate physical theories in which the temporal continuum possesses a similar structure to the spatial space in the sense that it may also be considered as a three-dimensional manifold [3]. It could be that, except for one dimension, all other temporal dimensions are intrinsic and the dynamical properties of quantum particles that are associated with these intrinsic temporal dimensions are therefore also intrinsic. It should be mentioned here that intrinsic physical properties, such as intrinsic energy, can be observable even though their associated intrinsic dynamics cannot. For example, we have also shown in our work on a formulation of spin dynamics using Schrödinger wave equation that the intrinsic coordinates that represent the spin dynamics are two-dimensional therefore it is reasonable to suggest that the two-dimensional intrinsic coordinates are temporal because the spin dynamics does not depend on the nature of the coordinate system for its formulation [4]. From the perspective of time as a three-dimensional manifold, in this work we extend our discussion on temporal dynamics by formulating Maxwell and Dirac field with three-dimensional time. Essentially, the formulation is a description of the dynamics of Maxwell and Dirac field in a six-dimensional spatiotemporal manifold in which both space and time have three dimensions. Since we will also formulate Maxwell and Dirac field from a system of linear first order partial differential equations therefore for reference we give an outline of the mathematical framework that is required for the formulation. The system of linear first order partial differential equations that we need to use in this work is given as follows

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{r} \frac{\partial \psi_i}{\partial x_j} = k_1 \sum_{l=1}^{n} b_l^r \psi_l + k_2 c^r, \qquad r = 1, 2, \dots, n \tag{1}$$

Equation (1) can be rewritten in a matrix form as

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\psi = k_1 \sigma \psi + k_2 J \tag{2}$$

where $\psi = (\psi_1, \psi_2, \dots, \psi_n)^T$, $\partial\psi/\partial x_i = (\partial\psi_1/\partial x_i, \partial\psi_2/\partial x_i, \dots, \partial\psi_n/\partial x_i)^T$, A_i, σ and J are matrices representing the quantities a_{ij}^k, b_l^r and c^r, and k_1 and k_2 are undetermined constants. Now, if we apply the operator $\sum_{i=1}^{n} A_i\, \partial/\partial x_i$ on the left on both sides of Equation (2) then we obtain

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}\right)\psi = \left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)(k_1 \sigma \psi + k_2 J) \tag{3}$$

If we assume further that the coefficients a_{ij}^k and b_l^r are constants and $A_i \sigma = \sigma A_i$, then Equation (3) can be rewritten in the following form

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n}\sum_{j>i}^{n} \left(A_i A_j + A_j A_i\right) \frac{\partial^2}{\partial x_i \partial x_j}\right)\psi = k_1^2 \sigma^2 \psi + k_1 k_2 \sigma J + k_2 \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{4}$$

In order for the above system of partial differential equations to be applied to physical phenomena, the matrices A_i must be determined. For the case of Maxwell and Dirac field, the matrices A_i must take a form so that Equation (4) reduces to a wave equation

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2}\right)\psi = k_1^2 \sigma^2 \psi + k_1 k_2 \sigma J + k_2 \sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{5}$$

In fact, we will show later that in order to reduce Equation (4) to Equation (5) we will also need extra conditions on the components of the wavefuction ψ in the form of divergences and commutative relations between spatiotemporal cross derivatives. In this work we only discuss Maxwell and Dirac field therefore we will set $\sigma = 1$.

2. Maxwell field with three-dimensional time

In this section we formulate Maxwell field of electromagnetism in a six-dimensional spatiotemporal manifold in which space and time both have three dimensions. For the time coordinates the matrices A_i are given in terms of the theta matrices θ_i which are defined as follows

$$\theta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \theta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \theta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (6)$$

Then the following commutative relations can be obtained

$$\theta_1^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \theta_2^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \theta_3^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\theta_1\theta_2 + \theta_2\theta_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \theta_1\theta_3 + \theta_3\theta_1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\theta_2\theta_3 + \theta_3\theta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \qquad (7)$$

For the space coordinates the matrices A_i are given in terms of the gamma matrices γ_i which are defined as follows

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (8)$$

Then we obtain the following commutative relations

$$\gamma_1^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma_2^2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma_3^2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_1\gamma_2+\gamma_2\gamma_1=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\quad \gamma_1\gamma_3+\gamma_3\gamma_1=\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\quad \gamma_2\gamma_3+\gamma_3\gamma_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}\qquad(9)$$

We also obtain the following cross commutative relations between the theta and gamma matrices

$$\theta_1\,\gamma_1+\ \gamma_1\,\theta_1=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\qquad \theta_1\,\gamma_2+\ \gamma_2\,\theta_1=\begin{pmatrix}0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&-1&0&0\\0&0&0&0&0&0\end{pmatrix}$$

$$\theta_1\,\gamma_3+\ \gamma_3\,\theta_1=\begin{pmatrix}0&0&-1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&-1&0&0\end{pmatrix}\qquad \theta_2\,\gamma_1+\ \gamma_1\,\theta_2=\begin{pmatrix}0&1&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}$$

$$\theta_2\,\gamma_2+\ \gamma_2\,\theta_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\qquad \theta_2\,\gamma_3+\ \gamma_3\,\theta_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&-1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&-1&0\end{pmatrix}$$

$$\theta_3\,\gamma_1+\ \gamma_1\,\theta_3=\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\qquad \theta_3\,\gamma_2+\ \gamma_2\,\theta_3=\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\end{pmatrix}$$

$$\theta_3\,\gamma_3+\ \gamma_3\,\theta_3=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\qquad(10)$$

Using the theta and gamma matrices given in Equations (6) and (8), then from Equation (2) with $\psi=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6\ )^T$ and $J=(j_1,j_2,j_3,j_4,j_5,j_6)^T$ we obtain the following system of equations

$$\frac{\partial\psi_6}{\partial t_2}-\frac{\partial\psi_5}{\partial t_3}+\frac{\partial\psi_6}{\partial x_2}-\frac{\partial\psi_5}{\partial x_3}=k_1\psi_1+k_2j_1\qquad(11)$$

$$-\frac{\partial\psi_6}{\partial t_1}+\frac{\partial\psi_4}{\partial t_3}-\frac{\partial\psi_6}{\partial x_1}+\frac{\partial\psi_4}{\partial x_3}=k_1\psi_2+k_2j_2\qquad(12)$$

$$\frac{\partial\psi_5}{\partial t_1}-\frac{\partial\psi_4}{\partial t_2}+\frac{\partial\psi_5}{\partial x_1}-\frac{\partial\psi_4}{\partial x_2}=k_1\psi_3+k_2j_3\qquad(13)$$

$$-\frac{\partial \psi_3}{\partial t_2}+\frac{\partial \psi_2}{\partial t_3}+\frac{\partial \psi_3}{\partial x_2}-\frac{\partial \psi_2}{\partial x_3}=k_1\psi_4+k_2 j_4 \tag{14}$$

$$\frac{\partial \psi_3}{\partial t_1}-\frac{\partial \psi_1}{\partial t_3}-\frac{\partial \psi_3}{\partial x_1}+\frac{\partial \psi_1}{\partial x_3}=k_1\psi_5+k_2 j_5 \tag{15}$$

$$-\frac{\partial \psi_2}{\partial t_1}+\frac{\partial \psi_1}{\partial t_2}+\frac{\partial \psi_2}{\partial x_1}-\frac{\partial \psi_1}{\partial x_2}=k_1\psi_6+k_2 j_6 \tag{16}$$

Now, as shown in our previous work on Dirac negative mass and magnetic monopole that the magnetic field is associated with the Gaussian curvature in the temporal continuum and the electric field with the Gaussian curvature of the spatial continuum [5], therefore we identify $\mathbf{B}=(\psi_1,\psi_2,\psi_3)$, $\mathbf{E}=(\psi_4,\psi_5,\psi_6)$, $\mathbf{j}_1=(j_1,j_2,j_3)$ and $\mathbf{j}_2=(j_4,j_5,j_6)$, then the system of equations given in Equations (11-16) can be rewritten in a vector form as

$$-\nabla_t\times\mathbf{B}+\nabla_s\times\mathbf{B}=k_1\mathbf{E}+k_2\mathbf{J}_2 \tag{17}$$

$$\nabla_t\times\mathbf{E}+\nabla_s\times\mathbf{E}=k_1\mathbf{B}+k_2\mathbf{J}_1 \tag{18}$$

where ∇_t and ∇_s represent the temporal and spatial differential operators in the temporal and spatial Euclidean space, respectively. As we have mentioned in the introduction that the system of equations given in Equations (17) and (18) may contain intrinsic temporal coordinates that may not be observable in the three-dimensional spatial space. Therefore, in order to express Equations (17) and (18) in a form that is observable we would need to introduce a time that behaves as a one-dimensional continuum. For this purpose, we now show that the above system of equations can be rewritten as two systems of electromagnetic equations, one in the Minkowski pseudo-Euclidean spacetime and the other in a temporal Euclidean space. In general, the one-dimensional time that we can introduce is the temporal arclength defined in terms of the time coordinates (t_1,t_2,t_3) as $dt^2=dt_1^2+dt_2^2+dt_3^2$. For Maxwell field equations we also set $k_1=0$. With the one-dimensional time t that can be identified as the Minkowski time, we assume that the temporal curl of the field **B** and **E** in Equations (17) and (18) satisfy the following equations

$$\nabla_t\times\mathbf{B}-\frac{\partial\mathbf{E}}{\partial t}=0 \tag{19}$$

$$\nabla_t\times\mathbf{E}-\frac{\partial\mathbf{B}}{\partial t}=0 \tag{20}$$

As we have shown in our work on Maxwell field equations in Euclidean relativity that the free temporal electromagnetic field given in Equations (19) and (20) satisfy the elliptic equations [6]

$$\frac{\partial^2\mathbf{E}}{\partial t^2}+\nabla_t^2\mathbf{E}=0 \tag{21}$$

$$\frac{\partial^2\mathbf{B}}{\partial t^2}+\nabla_t^2\mathbf{B}=0 \tag{22}$$

Therefore the temporal electromagnetic field given in Equations (19-22) conform to the Euclidean relativity. With the assumed equations given in Equations (19) and (20) then we obtain Maxwell field equations for the electromagnetic field in the Minkowski pseudo-Euclidean spacetime as

$$\nabla_s \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = k_2 \mathbf{j}_1 \tag{23}$$

$$\nabla_s \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = k_2 \mathbf{j}_2 \tag{24}$$

The system of equations given in Equations (23) and (24) together with Gauss's laws that we will show later as extra conditions to reduce the equation given in Equation (4) to a wave equation will form the Maxwell field equations for the electromagnetic field in a six-dimensional spatiotemporal continuum.

Using the commutative relations that are obtained for the theta and gamma matrices given in Equations (7), (9) and (10) we also obtain the following system of second order partial differential equations from Equation (4)

$$\frac{\partial^2\psi_1}{\partial t_2^2}+\frac{\partial^2\psi_1}{\partial t_3^2}-\frac{\partial^2\psi_1}{\partial x_2^2}-\frac{\partial^2\psi_1}{\partial x_3^2}-\frac{\partial^2\psi_2}{\partial t_1\partial t_2}-\frac{\partial^2\psi_3}{\partial t_1\partial t_3}-\frac{\partial^2\psi_2}{\partial t_1\partial x_2}-\frac{\partial^2\psi_3}{\partial t_1\partial x_3}+\frac{\partial^2\psi_2}{\partial t_2\partial x_1}+\frac{\partial^2\psi_3}{\partial t_3\partial x_1}+\frac{\partial^2\psi_2}{\partial x_1\partial x_2}+\frac{\partial^2\psi_3}{\partial x_1\partial x_3}=k_1^2\psi_1+k_1k_2j_1+k_2\left(\frac{\partial j_6}{\partial t_2}-\frac{\partial j_5}{\partial t_3}+\frac{\partial j_6}{\partial x_2}-\frac{\partial j_5}{\partial x_3}\right) \tag{25}$$

$$\frac{\partial^2\psi_2}{\partial t_1^2}+\frac{\partial^2\psi_2}{\partial t_3^2}-\frac{\partial^2\psi_2}{\partial x_1^2}-\frac{\partial^2\psi_2}{\partial x_3^2}-\frac{\partial^2\psi_1}{\partial t_1\partial t_2}-\frac{\partial^2\psi_3}{\partial t_2\partial t_3}+\frac{\partial^2\psi_1}{\partial t_1\partial x_2}-\frac{\partial^2\psi_1}{\partial t_2\partial x_1}-\frac{\partial^2\psi_3}{\partial t_2\partial x_3}+\frac{\partial^2\psi_3}{\partial t_3\partial x_2}+\frac{\partial^2\psi_1}{\partial x_1\partial x_2}+\frac{\partial^2\psi_3}{\partial x_2\partial x_3}=k_1^2\psi_2+k_1k_2j_2+k_2\left(-\frac{\partial j_6}{\partial t_1}+\frac{\partial j_4}{\partial t_3}-\frac{\partial j_6}{\partial x_1}+\frac{\partial j_4}{\partial x_3}\right) \tag{26}$$

$$\frac{\partial^2\psi_3}{\partial t_1^2}+\frac{\partial^2\psi_3}{\partial t_2^2}-\frac{\partial^2\psi_3}{\partial x_1^2}-\frac{\partial^2\psi_3}{\partial x_2^2}-\frac{\partial^2\psi_1}{\partial t_1\partial t_3}-\frac{\partial^2\psi_2}{\partial t_2\partial t_3}+\frac{\partial^2\psi_1}{\partial t_1\partial x_3}+\frac{\partial^2\psi_2}{\partial t_2\partial x_3}-\frac{\partial^2\psi_1}{\partial t_3\partial x_1}-\frac{\partial^2\psi_2}{\partial t_3\partial x_2}+\frac{\partial^2\psi_1}{\partial x_1\partial x_3}+\frac{\partial^2\psi_2}{\partial x_2\partial x_3}=k_1^2\psi_3+k_1k_2j_3+k_2\left(\frac{\partial j_5}{\partial t_1}-\frac{\partial j_4}{\partial t_2}+\frac{\partial j_5}{\partial x_1}-\frac{\partial j_4}{\partial x_2}\right) \tag{27}$$

$$\frac{\partial^2\psi_4}{\partial t_2^2}+\frac{\partial^2\psi_4}{\partial t_3^2}-\frac{\partial^2\psi_4}{\partial x_2^2}-\frac{\partial^2\psi_4}{\partial x_3^2}-\frac{\partial^2\psi_5}{\partial t_1\partial t_2}-\frac{\partial^2\psi_6}{\partial t_1\partial t_3}+\frac{\partial^2\psi_5}{\partial t_1\partial x_2}+\frac{\partial^2\psi_6}{\partial t_1\partial x_3}-\frac{\partial^2\psi_5}{\partial t_2\partial x_1}-\frac{\partial^2\psi_6}{\partial t_3\partial x_1}+\frac{\partial^2\psi_5}{\partial x_1\partial x_2}+\frac{\partial^2\psi_6}{\partial x_1\partial x_3}=k_1^2\psi_4+k_1k_2j_4+k_2\left(-\frac{\partial j_3}{\partial t_2}+\frac{\partial j_2}{\partial t_3}+\frac{\partial j_3}{\partial x_2}-\frac{\partial j_2}{\partial x_3}\right) \tag{28}$$

$$\frac{\partial^2\psi_5}{\partial t_1^2}+\frac{\partial^2\psi_5}{\partial t_3^2}-\frac{\partial^2\psi_5}{\partial x_1^2}-\frac{\partial^2\psi_5}{\partial x_3^2}-\frac{\partial^2\psi_4}{\partial t_1\partial t_2}-\frac{\partial^2\psi_6}{\partial t_2\partial t_3}-\frac{\partial^2\psi_4}{\partial t_1\partial x_2}+\frac{\partial^2\psi_4}{\partial t_2\partial x_1}+\frac{\partial^2\psi_6}{\partial t_2\partial x_3}-\frac{\partial^2\psi_6}{\partial t_3\partial x_2}+\frac{\partial^2\psi_4}{\partial x_1\partial x_2}+\frac{\partial^2\psi_6}{\partial x_2\partial x_3}=k_1^2\psi_2+k_1k_2j_2+k_2\left(\frac{\partial j_3}{\partial t_1}-\frac{\partial j_1}{\partial t_3}-\frac{\partial j_3}{\partial x_1}+\frac{\partial j_1}{\partial x_3}\right) \tag{29}$$

$$\frac{\partial^2\psi_6}{\partial t_1^2}+\frac{\partial^2\psi_6}{\partial t_2^2}-\frac{\partial^2\psi_6}{\partial x_1^2}-\frac{\partial^2\psi_6}{\partial x_2^2}-\frac{\partial^2\psi_4}{\partial t_1\partial t_3}-\frac{\partial^2\psi_5}{\partial t_2\partial t_3}-\frac{\partial^2\psi_4}{\partial t_1\partial x_3}-\frac{\partial^2\psi_5}{\partial t_2\partial x_3}+\frac{\partial^2\psi_4}{\partial t_3\partial x_1}+\frac{\partial^2\psi_5}{\partial t_3\partial x_2}$$
$$+\frac{\partial^2\psi_4}{\partial x_1\partial x_3}+\frac{\partial^2\psi_5}{\partial x_2\partial x_3}=k_1^2\psi_3+k_1k_2j_3+k_2\left(-\frac{\partial j_2}{\partial t_1}+\frac{\partial j_1}{\partial t_2}+\frac{\partial j_2}{\partial x_1}-\frac{\partial j_1}{\partial x_2}\right)\quad(30)$$

In the system of equations given in Equations (25-30) there are cross derivatives that involve both space and time. However, these cross derivatives can be removed by imposing the following conditions

$$\frac{\partial^2\psi_\mu}{\partial t_i\partial x_j}=\frac{\partial^2\psi_\mu}{\partial t_j\partial x_i}\qquad\text{for all }\mu, i\text{ and }j\qquad(31)$$

$$\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x_2}+\frac{\partial\psi_3}{\partial x_3}=\rho_s^1\qquad\frac{\partial\psi_4}{\partial x_1}+\frac{\partial\psi_5}{\partial x_2}+\frac{\partial\psi_6}{\partial x_3}=\rho_s^2\qquad(32)$$

$$\frac{\partial\psi_1}{\partial t_1}+\frac{\partial\psi_2}{\partial t_2}+\frac{\partial\psi_3}{\partial t_3}=\rho_t^1\qquad\frac{\partial\psi_4}{\partial t_1}+\frac{\partial\psi_5}{\partial t_2}+\frac{\partial\psi_6}{\partial t_3}=\rho_t^2\qquad(33)$$

The divergence conditions given in Equations (32) and (33) can be identified with the Gauss's law in the classical electrodynamics. On the other hand, the condition given in Equation (31) simply states the equivalence between space and time. Using the conditions given in Equations (31-33) we then obtain

$$\frac{\partial^2\psi_1}{\partial t_1^2}+\frac{\partial^2\psi_1}{\partial t_2^2}+\frac{\partial^2\psi_1}{\partial t_3^2}-\frac{\partial^2\psi_1}{\partial x_1^2}-\frac{\partial^2\psi_1}{\partial x_2^2}-\frac{\partial^2\psi_1}{\partial x_3^2}$$
$$=k_1^2\psi_1+k_1k_2j_1+k_2\left(\frac{\partial j_6}{\partial t_2}-\frac{\partial j_5}{\partial t_3}+\frac{\partial j_6}{\partial x_2}-\frac{\partial j_5}{\partial x_3}\right)+\frac{\partial\rho_t^1}{\partial t_1}-\frac{\partial\rho_s^1}{\partial x_1}\qquad(34)$$

$$\frac{\partial^2\psi_2}{\partial t_1^2}+\frac{\partial^2\psi_2}{\partial t_2^2}+\frac{\partial^2\psi_2}{\partial t_3^2}-\frac{\partial^2\psi_2}{\partial x_1^2}-\frac{\partial^2\psi_2}{\partial x_2^2}-\frac{\partial^2\psi_2}{\partial x_3^2}$$
$$=k_1^2\psi_2+k_1k_2j_2+k_2\left(-\frac{\partial j_6}{\partial t_1}+\frac{\partial j_4}{\partial t_3}-\frac{\partial j_6}{\partial x_1}+\frac{\partial j_4}{\partial x_3}\right)+\frac{\partial\rho_t^1}{\partial t_2}-\frac{\partial\rho_s^1}{\partial x_2}\qquad(35)$$

$$\frac{\partial^2\psi_3}{\partial t_1^2}+\frac{\partial^2\psi_3}{\partial t_2^2}+\frac{\partial^2\psi_3}{\partial t_3^2}-\frac{\partial^2\psi_3}{\partial x_1^2}-\frac{\partial^2\psi_3}{\partial x_2^2}-\frac{\partial^2\psi_3}{\partial x_3^2}$$
$$=k_1^2\psi_3+k_1k_2j_3+k_2\left(\frac{\partial j_5}{\partial t_1}-\frac{\partial j_4}{\partial t_2}+\frac{\partial j_5}{\partial x_1}-\frac{\partial j_4}{\partial x_2}\right)+\frac{\partial\rho_t^1}{\partial t_3}-\frac{\partial\rho_s^1}{\partial x_3}\qquad(36)$$

$$\frac{\partial^2\psi_4}{\partial t_1^2}+\frac{\partial^2\psi_4}{\partial t_2^2}+\frac{\partial^2\psi_4}{\partial t_3^2}-\frac{\partial^2\psi_4}{\partial x_1^2}-\frac{\partial^2\psi_4}{\partial x_2^2}-\frac{\partial^2\psi_4}{\partial x_3^2}$$
$$=k_1^2\psi_4+k_1k_2j_4+k_2\left(-\frac{\partial j_3}{\partial t_2}+\frac{\partial j_2}{\partial t_3}+\frac{\partial j_3}{\partial x_2}-\frac{\partial j_2}{\partial x_3}\right)+\frac{\partial\rho_t^2}{\partial t_1}-\frac{\partial\rho_s^2}{\partial x_1}\qquad(37)$$

$$\frac{\partial^2\psi_5}{\partial t_1^2}+\frac{\partial^2\psi_5}{\partial t_2^2}+\frac{\partial^2\psi_5}{\partial t_3^2}-\frac{\partial^2\psi_5}{\partial x_1^2}-\frac{\partial^2\psi_5}{\partial x_2^2}-\frac{\partial^2\psi_5}{\partial x_3^2}$$
$$=k_1^2\psi_2+k_1k_2j_2+k_2\left(\frac{\partial j_3}{\partial t_1}-\frac{\partial j_1}{\partial t_3}-\frac{\partial j_3}{\partial x_1}+\frac{\partial j_1}{\partial x_3}\right)+\frac{\partial\rho_t^2}{\partial t_2}-\frac{\partial\rho_s^2}{\partial x_2} \quad (38)$$

$$\frac{\partial^2\psi_6}{\partial t_1^2}+\frac{\partial^2\psi_6}{\partial t_2^2}+\frac{\partial^2\psi_6}{\partial t_3^2}-\frac{\partial^2\psi_6}{\partial x_1^2}-\frac{\partial^2\psi_6}{\partial x_2^2}-\frac{\partial^2\psi_6}{\partial x_3^2}$$
$$=k_1^2\psi_3+k_1k_2j_3+k_2\left(-\frac{\partial j_2}{\partial t_1}+\frac{\partial j_1}{\partial t_2}+\frac{\partial j_2}{\partial x_1}-\frac{\partial j_1}{\partial x_2}\right)+\frac{\partial\rho_t^2}{\partial t_3}-\frac{\partial\rho_s^2}{\partial x_3} \quad (39)$$

For the case of the electromagnetic field with three-dimensional time we set $k_1=0$, then we obtain the following system of equations

$$\frac{\partial^2\psi_1}{\partial t_1^2}+\frac{\partial^2\psi_1}{\partial t_2^2}+\frac{\partial^2\psi_1}{\partial t_3^2}-\frac{\partial^2\psi_1}{\partial x_1^2}-\frac{\partial^2\psi_1}{\partial x_2^2}-\frac{\partial^2\psi_1}{\partial x_3^2}$$
$$=k_2\left(\frac{\partial j_6}{\partial t_2}-\frac{\partial j_5}{\partial t_3}+\frac{\partial j_6}{\partial x_2}-\frac{\partial j_5}{\partial x_3}\right)+\frac{\partial\rho_t^1}{\partial t_1}-\frac{\partial\rho_s^1}{\partial x_1} \quad (40)$$

$$\frac{\partial^2\psi_2}{\partial t_1^2}+\frac{\partial^2\psi_2}{\partial t_2^2}+\frac{\partial^2\psi_2}{\partial t_3^2}-\frac{\partial^2\psi_2}{\partial x_1^2}-\frac{\partial^2\psi_2}{\partial x_2^2}-\frac{\partial^2\psi_2}{\partial x_3^2}$$
$$=k_2\left(-\frac{\partial j_6}{\partial t_1}+\frac{\partial j_4}{\partial t_3}-\frac{\partial j_6}{\partial x_1}+\frac{\partial j_4}{\partial x_3}\right)+\frac{\partial\rho_t^1}{\partial t_2}-\frac{\partial\rho_s^1}{\partial x_2} \quad (41)$$

$$\frac{\partial^2\psi_3}{\partial t_1^2}+\frac{\partial^2\psi_3}{\partial t_2^2}+\frac{\partial^2\psi_3}{\partial t_3^2}-\frac{\partial^2\psi_3}{\partial x_1^2}-\frac{\partial^2\psi_3}{\partial x_2^2}-\frac{\partial^2\psi_3}{\partial x_3^2}$$
$$=k_2\left(\frac{\partial j_5}{\partial t_1}-\frac{\partial j_4}{\partial t_2}+\frac{\partial j_5}{\partial x_1}-\frac{\partial j_4}{\partial x_2}\right)+\frac{\partial\rho_t^1}{\partial t_3}-\frac{\partial\rho_s^1}{\partial x_3} \quad (42)$$

$$\frac{\partial^2\psi_4}{\partial t_1^2}+\frac{\partial^2\psi_4}{\partial t_2^2}+\frac{\partial^2\psi_4}{\partial t_3^2}-\frac{\partial^2\psi_4}{\partial x_1^2}-\frac{\partial^2\psi_4}{\partial x_2^2}-\frac{\partial^2\psi_4}{\partial x_3^2}$$
$$=k_2\left(-\frac{\partial j_3}{\partial t_2}+\frac{\partial j_2}{\partial t_3}+\frac{\partial j_3}{\partial x_2}-\frac{\partial j_2}{\partial x_3}\right)+\frac{\partial\rho_t^2}{\partial t_1}-\frac{\partial\rho_s^2}{\partial x_1} \quad (43)$$

$$\frac{\partial^2\psi_5}{\partial t_1^2}+\frac{\partial^2\psi_5}{\partial t_2^2}+\frac{\partial^2\psi_5}{\partial t_3^2}-\frac{\partial^2\psi_5}{\partial x_1^2}-\frac{\partial^2\psi_5}{\partial x_2^2}-\frac{\partial^2\psi_5}{\partial x_3^2}$$
$$=k_2\left(\frac{\partial j_3}{\partial t_1}-\frac{\partial j_1}{\partial t_3}-\frac{\partial j_3}{\partial x_1}+\frac{\partial j_1}{\partial x_3}\right)+\frac{\partial\rho_t^2}{\partial t_2}-\frac{\partial\rho_s^2}{\partial x_2} \quad (44)$$

$$\frac{\partial^2\psi_6}{\partial t_1^2}+\frac{\partial^2\psi_6}{\partial t_2^2}+\frac{\partial^2\psi_6}{\partial t_3^2}-\frac{\partial^2\psi_6}{\partial x_1^2}-\frac{\partial^2\psi_6}{\partial x_2^2}-\frac{\partial^2\psi_6}{\partial x_3^2}$$
$$=k_2\left(-\frac{\partial j_2}{\partial t_1}+\frac{\partial j_1}{\partial t_2}+\frac{\partial j_2}{\partial x_1}-\frac{\partial j_1}{\partial x_2}\right)+\frac{\partial\rho_t^2}{\partial t_3}-\frac{\partial\rho_s^2}{\partial x_3} \quad (45)$$

The system of equations given in Equations (40-45) together with the conditions given in Equations (32) and (33) can be rewritten in vector form as

$$\boldsymbol{\nabla}_s\mathbf{E} = \rho_s^1 \qquad \boldsymbol{\nabla}_s\mathbf{B} = \rho_s^2 \tag{46}$$

$$\boldsymbol{\nabla}_t\mathbf{E} = \rho_t^1 \qquad \boldsymbol{\nabla}_t\mathbf{B} = \rho_t^2 \tag{47}$$

$$\nabla_t^2\mathbf{E} - \nabla_s^2\mathbf{E} = \nabla_t(\rho_t^1) - \nabla_s(\rho_s^1) + k_2(\boldsymbol{\nabla}_t \times \mathbf{J}_2 + \boldsymbol{\nabla}_s \times \mathbf{J}_2) \tag{48}$$

$$\nabla_t^2\mathbf{B} - \nabla_s^2\mathbf{B} = \nabla_t(\rho_t^2) - \nabla_s(\rho_s^2) + k_2(\boldsymbol{\nabla}_t \times \mathbf{J}_1 + \boldsymbol{\nabla}_s \times \mathbf{J}_1) \tag{49}$$

Thus, we have shown that although Maxwell field can be formulated as a single field in a more symmetrical mathematical structure in terms of space and time in the six-dimensional spatiotemporal continuum, the six-dimensional field can be decoupled into two separate fields that exist in two spaces, one of which is the Minkowski pseudo-Euclidean spactime and the other is the three-dimensional Euclidean temporal manifold. It should be mentioned here that as in the case of the Minkowski pseudo-Euclidean spacetime, the three-dimensional Euclidean temporal manifold can be made into a four-dimensional Euclidean temporal manifold by considering the temporal arclength t as an independent coordinate. The coexistent temporal elliptic field to Maxwell field is a free field and conforms to the Euclidean relativity. In the next section we will examine Dirac field also with three-dimensional time and show that Dirac field can also be decoupled into two separate fields that exist the Minkowski pseudo-Euclidean spactime and the Euclidean three-dimensional temporal manifold. The coexistent temporal elliptic field to Dirac field is a massless field and also complies with the Euclidean relativity.

3. Dirac field with three-dimensional time

In this section we formulate Dirac field in a six-dimensional spatiotemporal continuum in which both space and time have three dimensions. For the time coordinates the matrices A_i are given in terms of the theta matrices θ_i which are defined as follows

$$\theta_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{50}$$

$$\theta_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \theta_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad \theta_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{51}$$

where σ_i are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{52}$$

The Pauli matrices σ_i satisfy the following relations

$$\sigma_i^2 = 1 \qquad \text{and} \qquad \sigma_i\sigma_j + \sigma_j\sigma_i = 0 \qquad \text{for} \qquad i, j = 0, 1, 2, 3 \tag{53}$$

The theta matrices θ_i given in Equation (51) satisfy the following relations

$$\theta_i^2 = 1 \quad \text{for} \quad i = 1,2,3 \qquad \text{and} \qquad \theta_i\theta_j + \theta_j\theta_i = 0 \quad \text{for} \quad i \neq j \tag{54}$$

For the space coordinates the matrices A_i are given in terms of the gamma matrices γ_i which are defined as follows

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{55}$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{56}$$

The gamma matrices γ_i satisfy the following relations

$$\gamma_i^2 = -1 \quad \text{for} \quad i = 1,2,3 \qquad \text{and} \qquad \gamma_i\gamma_j + \gamma_j\gamma_i = 0 \quad \text{for} \quad i \neq j \tag{57}$$

In addition to the relations given in Equations (54) and (57) for the matrices γ_i and θ_i we also obtain the following cross commutative relations

$$\theta_1\gamma_1 + \gamma_1\theta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \theta_1\gamma_2 + \gamma_2\theta_1 = 2\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad \theta_1\gamma_3 + \gamma_3\theta_1 = 2\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{58}$$

$$\theta_2\gamma_1 + \gamma_1\theta_2 = 2\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad \theta_2\gamma_2 + \gamma_2\theta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \theta_2\gamma_3 + \gamma_3\theta_2 = 2\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{59}$$

$$\theta_3\gamma_1 + \gamma_1\theta_3 = 2\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \theta_3\gamma_2 + \gamma_2\theta_3 = 2\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \theta_3\gamma_3 + \gamma_3\theta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{60}$$

Using the gamma and theta matrices given in Equations (51) and (56), from Equation (2) we obtain the following system of equations for the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$

$$\frac{\partial\psi_4}{\partial t_1} - i\frac{\partial\psi_4}{\partial t_2} + \frac{\partial\psi_3}{\partial t_3} + \frac{\partial\psi_4}{\partial x_1} - i\frac{\partial\psi_4}{\partial x_2} + \frac{\partial\psi_3}{\partial x_3} = k_1\psi_1 + k_2 j_1 \tag{61}$$

$$\frac{\partial\psi_3}{\partial t_1} + i\frac{\partial\psi_3}{\partial t_2} - \frac{\partial\psi_4}{\partial t_3} + \frac{\partial\psi_3}{\partial x_1} + i\frac{\partial\psi_3}{\partial x_2} - \frac{\partial\psi_4}{\partial x_3} = k_1\psi_2 + k_2 j_2 \tag{62}$$

$$\frac{\partial\psi_2}{\partial t_1} - i\frac{\partial\psi_2}{\partial t_2} + \frac{\partial\psi_1}{\partial t_3} - \frac{\partial\psi_2}{\partial x_1} + i\frac{\partial\psi_2}{\partial x_2} - \frac{\partial\psi_1}{\partial x_3} = k_1\psi_3 + k_2 j_3 \tag{63}$$

$$\frac{\partial\psi_1}{\partial t_1} + i\frac{\partial\psi_1}{\partial t_2} - \frac{\partial\psi_2}{\partial t_3} - \frac{\partial\psi_1}{\partial x_1} - i\frac{\partial\psi_1}{\partial x_2} + \frac{\partial\psi_2}{\partial x_3} = k_1\psi_4 + k_2 j_4 \tag{64}$$

Similar to the case of Maxwell field, Dirac field in a six-dimensional spatiotemporal manifold can also be rewritten as a coupling of two fields in the Minkowski pseudo-Euclidean spactime and the Euclidean four-dimensional temporal manifold respectively. In the following we let $k_2 = 0$ and $k_1 = -im$. Using the terms that involve the time derivatives in

Equations (61-64), the temporal field in the Euclidean three-dimensional temporal manifold is assumed to take the form

$$-\frac{\partial\psi_1}{\partial t}+\frac{\partial\psi_4}{\partial t_1}-i\frac{\partial\psi_4}{\partial t_2}+\frac{\partial\psi_3}{\partial t_3}=0 \tag{65}$$

$$-\frac{\partial\psi_2}{\partial t}+\frac{\partial\psi_3}{\partial t_1}+i\frac{\partial\psi_3}{\partial t_2}-\frac{\partial\psi_4}{\partial t_3}= \tag{66}$$

$$\frac{\partial\psi_3}{\partial t}+\frac{\partial\psi_2}{\partial t_1}-i\frac{\partial\psi_2}{\partial t_2}+\frac{\partial\psi_1}{\partial t_3}=0 \tag{67}$$

$$\frac{\partial\psi_4}{\partial t}+\frac{\partial\psi_1}{\partial t_1}+i\frac{\partial\psi_1}{\partial t_2}-\frac{\partial\psi_2}{\partial t_3}=0 \tag{68}$$

Using Equations (65-68), Equations (61-64) can be rewritten as Dirac equation

$$\frac{\partial\psi_1}{\partial t}+\frac{\partial\psi_4}{\partial x_1}-i\frac{\partial\psi_4}{\partial x_2}+\frac{\partial\psi_3}{\partial x_3}=-im\psi_1 \tag{69}$$

$$\frac{\partial\psi_2}{\partial t}+\frac{\partial\psi_3}{\partial x_1}+i\frac{\partial\psi_3}{\partial x_2}-\frac{\partial\psi_4}{\partial x_3}=-im\psi_2 \tag{70}$$

$$-\frac{\partial\psi_3}{\partial t}-\frac{\partial\psi_2}{\partial x_1}+i\frac{\partial\psi_2}{\partial x_2}-\frac{\partial\psi_1}{\partial x_3}=-im\psi_3 \tag{71}$$

$$-\frac{\partial\psi_4}{\partial t}-\frac{\partial\psi_1}{\partial x_1}-i\frac{\partial\psi_1}{\partial x_2}+\frac{\partial\psi_2}{\partial x_3}=-im\psi_4 \tag{72}$$

Equations (69-72) can be rewritten as

$$\left(i\gamma^\mu\partial_\mu-m\right)\psi=0 \tag{73}$$

where the matrix γ^0 is given as

$$\gamma^0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \tag{74}$$

Equations (65-68) can also be rewritten in the following form

$$\theta^\mu\partial_\mu\psi=0 \tag{75}$$

where the matrix θ^0 is given as

$$\theta^0=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{76}$$

It is seen that since $\theta_i^2 = 1$ for $i = 0, 1, 2, 3$ the temporal Dirac equation given in Equation (75) for a massless particle also complies with Euclidean relativity.

Now, using the commutative relations for the gamma and theta matrices given in Equations (54), (57) and (58-60) we obtain the following system of second order partial differential equations for the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$

$$\frac{\partial^2\psi_1}{\partial t_1^2}+\frac{\partial^2\psi_1}{\partial t_2^2}+\frac{\partial^2\psi_1}{\partial t_3^2}-\frac{\partial^2\psi_1}{\partial x_1^2}-\frac{\partial^2\psi_1}{\partial x_2^2}-\frac{\partial^2\psi_1}{\partial x_3^2}-2i\frac{\partial^2\psi_1}{\partial t_1\partial x_2}+2\frac{\partial^2\psi_2}{\partial t_1\partial x_3}+2i\frac{\partial^2\psi_1}{\partial t_2\partial x_1}$$
$$-2i\frac{\partial^2\psi_2}{\partial t_2\partial x_3}-2\frac{\partial^2\psi_2}{\partial t_3\partial x_1}+2i\frac{\partial^2\psi_2}{\partial t_3\partial x_2}$$
$$=k_1^2\psi_1+k_1k_2j_1+k_2\left(\frac{\partial j_4}{\partial t_1}-i\frac{\partial j_4}{\partial t_2}+\frac{\partial j_3}{\partial t_3}+\frac{\partial j_4}{\partial x_1}-i\frac{\partial j_4}{\partial x_2}+\frac{\partial j_3}{\partial x_3}\right) \tag{77}$$

$$\frac{\partial^2\psi_2}{\partial t_1^2}+\frac{\partial^2\psi_2}{\partial t_2^2}+\frac{\partial^2\psi_2}{\partial t_3^2}-\frac{\partial^2\psi_2}{\partial x_1^2}-\frac{\partial^2\psi_2}{\partial x_2^2}-\frac{\partial^2\psi_2}{\partial x_3^2}+2i\frac{\partial^2\psi_2}{\partial t_1\partial x_2}-2\frac{\partial^2\psi_1}{\partial t_1\partial x_3}-2i\frac{\partial^2\psi_2}{\partial t_2\partial x_1}$$
$$-2i\frac{\partial^2\psi_1}{\partial t_2\partial x_3}+2\frac{\partial^2\psi_1}{\partial t_3\partial x_1}+2i\frac{\partial^2\psi_1}{\partial t_3\partial x_2}$$
$$=k_1^2\psi_2+k_1k_2j_2+k_2\left(\frac{\partial j_3}{\partial t_1}+i\frac{\partial j_3}{\partial t_2}-\frac{\partial j_4}{\partial t_3}+\frac{\partial j_3}{\partial x_1}+i\frac{\partial j_3}{\partial x_2}-\frac{\partial j_4}{\partial x_3}\right) \tag{78}$$

$$\frac{\partial^2\psi_3}{\partial t_1^2}+\frac{\partial^2\psi_3}{\partial t_2^2}+\frac{\partial^2\psi_3}{\partial t_3^2}-\frac{\partial^2\psi_3}{\partial x_1^2}-\frac{\partial^2\psi_3}{\partial x_2^2}-\frac{\partial^2\psi_3}{\partial x_3^2}+2i\frac{\partial^2\psi_3}{\partial t_1\partial x_2}-2\frac{\partial^2\psi_4}{\partial t_1\partial x_3}-2i\frac{\partial^2\psi_3}{\partial t_2\partial x_1}$$
$$+2i\frac{\partial^2\psi_4}{\partial t_2\partial x_3}+2\frac{\partial^2\psi_4}{\partial t_3\partial x_1}-2i\frac{\partial^2\psi_4}{\partial t_3\partial x_2}$$
$$=k_1^2\psi_4+k_1k_2j_4+k_2\left(\frac{\partial j_2}{\partial t_1}-i\frac{\partial j_2}{\partial t_2}+\frac{\partial j_1}{\partial t_3}-\frac{\partial j_2}{\partial x_1}+i\frac{\partial j_2}{\partial x_2}-\frac{\partial j_1}{\partial x_3}\right) \tag{79}$$

$$\frac{\partial^2\psi_4}{\partial t_1^2}+\frac{\partial^2\psi_4}{\partial t_2^2}+\frac{\partial^2\psi_4}{\partial t_3^2}-\frac{\partial^2\psi_4}{\partial x_1^2}-\frac{\partial^2\psi_4}{\partial x_2^2}-\frac{\partial^2\psi_4}{\partial x_3^2}-2i\frac{\partial^2\psi_4}{\partial t_1\partial x_2}+2\frac{\partial^2\psi_3}{\partial t_1\partial x_3}+2i\frac{\partial^2\psi_4}{\partial t_2\partial x_1}$$
$$+2i\frac{\partial^2\psi_3}{\partial t_2\partial x_3}-2\frac{\partial^2\psi_3}{\partial t_3\partial x_1}-2i\frac{\partial^2\psi_3}{\partial t_3\partial x_2}$$
$$=k_1^2\psi_4+k_1k_2j_4+k_2\left(\frac{\partial j_1}{\partial t_1}+i\frac{\partial j_1}{\partial t_2}-\frac{\partial j_2}{\partial t_3}-\frac{\partial j_1}{\partial x_1}-i\frac{\partial j_1}{\partial x_2}+\frac{\partial j_2}{\partial x_3}\right) \tag{80}$$

In the above systems of equations there are cross derivatives that involve both space and time. However, in this case the cross derivatives can be removed by simply assuming the following conditions

$$\frac{\partial^2\psi_\mu}{\partial t_i\partial x_j}=\frac{\partial^2\psi_\mu}{\partial t_j\partial x_i} \qquad \text{for all } \mu, i \text{ and } j \tag{81}$$

Then the system of equations given in Equations (77-80) can be reduced to the following system of equations in the six-dimensional spatiotemporal manifold

$$\frac{\partial^2\psi_1}{\partial t_1^2}+\frac{\partial^2\psi_1}{\partial t_2^2}+\frac{\partial^2\psi_1}{\partial t_3^2}-\frac{\partial^2\psi_1}{\partial x_1^2}-\frac{\partial^2\psi_1}{\partial x_2^2}-\frac{\partial^2\psi_1}{\partial x_3^2}$$
$$= k_1^2\psi_1 + k_1k_2j_1 + k_2\left(\frac{\partial j_4}{\partial t_1}-i\frac{\partial j_4}{\partial t_2}+\frac{\partial j_3}{\partial t_3}+\frac{\partial j_4}{\partial x_1}-i\frac{\partial j_4}{\partial x_2}+\frac{\partial j_3}{\partial x_3}\right) \tag{82}$$

$$\frac{\partial^2\psi_2}{\partial t_1^2}+\frac{\partial^2\psi_2}{\partial t_2^2}+\frac{\partial^2\psi_2}{\partial t_3^2}-\frac{\partial^2\psi_2}{\partial x_1^2}-\frac{\partial^2\psi_2}{\partial x_2^2}-\frac{\partial^2\psi_2}{\partial x_3^2}$$
$$= k_1^2\psi_2 + k_1k_2j_2 + k_2\left(\frac{\partial j_3}{\partial t_1}+i\frac{\partial j_3}{\partial t_2}-\frac{\partial j_4}{\partial t_3}+\frac{\partial j_3}{\partial x_1}+i\frac{\partial j_3}{\partial x_2}-\frac{\partial j_4}{\partial x_3}\right) \tag{83}$$

$$\frac{\partial^2\psi_3}{\partial t_1^2}+\frac{\partial^2\psi_3}{\partial t_2^2}+\frac{\partial^2\psi_3}{\partial t_3^2}-\frac{\partial^2\psi_3}{\partial x_1^2}-\frac{\partial^2\psi_3}{\partial x_2^2}-\frac{\partial^2\psi_3}{\partial x_3^2}$$
$$= k_1^2\psi_4 + k_1k_2j_4 + k_2\left(\frac{\partial j_2}{\partial t_1}-i\frac{\partial j_2}{\partial t_2}+\frac{\partial j_1}{\partial t_3}-\frac{\partial j_2}{\partial x_1}+i\frac{\partial j_2}{\partial x_2}-\frac{\partial j_1}{\partial x_3}\right) \tag{84}$$

$$\frac{\partial^2\psi_4}{\partial t_1^2}+\frac{\partial^2\psi_4}{\partial t_2^2}+\frac{\partial^2\psi_4}{\partial t_3^2}-\frac{\partial^2\psi_4}{\partial x_1^2}-\frac{\partial^2\psi_4}{\partial x_2^2}-\frac{\partial^2\psi_4}{\partial x_3^2}$$
$$= k_1^2\psi_4 + k_1k_2j_4 + k_2\left(\frac{\partial j_1}{\partial t_1}+i\frac{\partial j_1}{\partial t_2}-\frac{\partial j_2}{\partial t_3}-\frac{\partial j_1}{\partial x_1}-i\frac{\partial j_1}{\partial x_2}+\frac{\partial j_2}{\partial x_3}\right) \tag{85}$$

In order to obtain Dirac equation for a free particle we set $k_2 = 0$. Then the above system of equations can be rewritten for the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ as a six-dimensional spatiotemporal Klein-Gordon equation as

$$\frac{\partial^2\psi_\mu}{\partial t_1^2}+\frac{\partial^2\psi_\mu}{\partial t_2^2}+\frac{\partial^2\psi_\mu}{\partial t_3^2}-\frac{\partial^2\psi_\mu}{\partial x_1^2}-\frac{\partial^2\psi_\mu}{\partial x_2^2}-\frac{\partial^2\psi_\mu}{\partial x_3^2} = -m^2\psi_\mu \tag{86}$$

A plane wave solution can be found in the form

$$\psi_\mu = e^{i(\omega_\alpha t^\alpha + k_\beta x^\beta)} \tag{87}$$

where the quantities ω_α and k_β satisfy the following relation

$$\omega_1^2+\omega_2^2+\omega_3^2-k_1^2-k_2^2-k_3^2 = m^2 \tag{88}$$

It is seen that as in the case of Maxwell field, Dirac field can also be formulated as a single field in a more symmetrical mathematical structure in terms of space and time in the six-dimensional spatiotemporal continuum, and the six-dimensional field can also be decoupled into two separate fields that exist in two spaces, one of which is the Minkowski pseudo-Euclidean spactime and the other is a Euclidean three-dimensional temporal manifold. The coexistent temporal elliptic field to Dirac field is a massless field and also complies with the Euclidean relativity.

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